Math 1060

Lecture 2
Inverse Functions & Logarithms

Outline

Summary of last lecture

Inverse Functions

- Domain, codomain, and range
- One-to-one functions
- Inverse functions

Inverse trig functions

Logarithms

- Definition
- Properties
- The natural log

Summary of last lecture

- Described coordinated courses, exams, homeworks, quizzes, and grading.
- Defined four common sets of numbers: $\mathbb{N}$ (the natural numbers), $\mathbb{Z}$ (the integers), $\mathbb{Q}$ (the rational numbers), and $\mathbb{R}$ (the real numbers).
- Described exponential functions, working our way up from a^n when $n \in \mathbb{N}$ through a^x when $x \in \mathbb{R}$.
- Described the graphs of the functions a^x and a^{-x}, and noticed some commonalities in these graphs.
- Defined the number e as the unique number so that the tangent line to the graph $y = e^x$ at $(0, 1)$ has slope 1.
- **Homework Due Monday**: Read Ch. 1 of Stewart, do problems §1.5: 2, 4, 7, 15 and §1.6: 5 - 8, 29, 30.

Domain and codomain

Recall that a *function* is a rule for associating a single output in a set C to each input in a set D. (This is the *vertical line test*.)

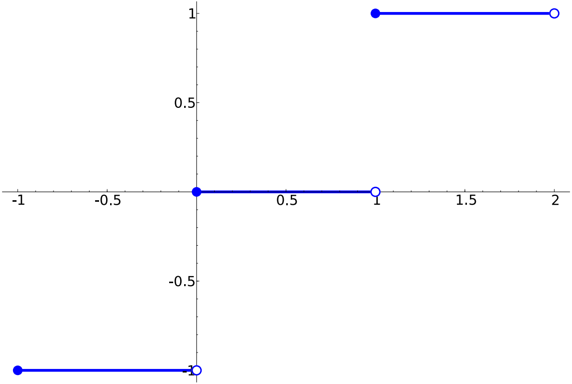
The set D is called the *domain* of the function, and the set C is called the *codomain* of the function.

To denote that f is a function with domain D and codomain C we write $f : D \to C$.

Example

The *greatest integer function*, denoted $[\![x]\!]$, returns the larger integer less than or equal to x. For instance $[\![2.34]\!] = 2$, and $[\![-3.78]\!] = -4$.

The domain of this function is $\mathbb{R}$, but because we can only ever hope to get integers out of the function, the codomain is $\mathbb{Z}$.



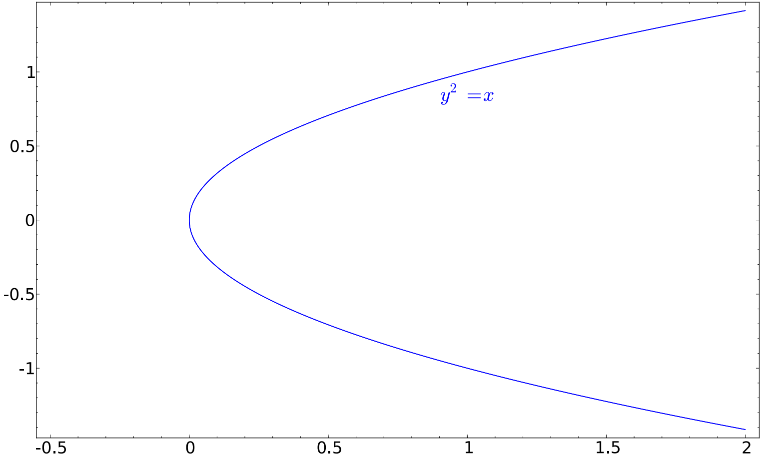
Examples

Function	Domain	Codomain	Notation
$f(x) = [\![x]\!]$	$\mathbb{R}$	$\mathbb{Z}$	$f : \mathbb{R} \to \mathbb{Z}$
$g(x) = x^2$	$\mathbb{R}$	$\mathbb{R}$	$g : \mathbb{R} \to \mathbb{R}$
$h(x) = \sqrt{x}$	$[0, \infty)$	$\mathbb{R}$	$h : [0, \infty) \to \mathbb{R}$
$k(x) = \frac{1}{\sqrt{x+2}}$	$(-2, \infty)$	$\mathbb{R}$	$k : (-2, \infty) \to \mathbb{R}$
$\ell(z) = (z + 2)^{-1/2}$	$(-2, \infty)$	$\mathbb{R}$	$\ell : (-2, \infty) \to \mathbb{R}$

Notice that the function is defined for everything in the domain, but does not necessarily hit every value in the codomain. E.g., $g(x) = x^2$ is never negative; $h(x) = \sqrt{x}$ is also never negative, by convention.

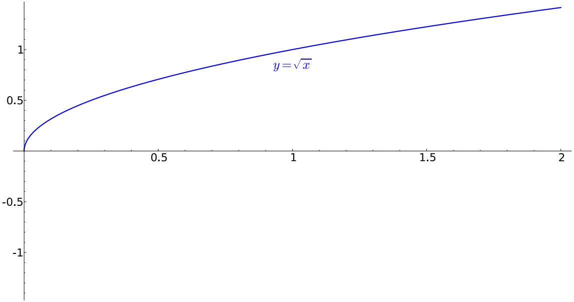
A convention

The function $h(x) = \sqrt{x}$ is always positive, but this is by convention. The square root of x, $\sqrt{x}$, should be the number y such that $y^2 = x$. If $x > 0$, then there are always two y values satisfying $y^2 = x$. For example, if $y^2 = 4$, then $y = \pm 2$.



A convention

Functions can only have one output value, though, so for $\sqrt{x}$ to be a function, we *must* restrict ourselves to either the positive or negative square roots. By convention, the symbol $\sqrt{x}$ will *always* mean the positive square root. If we want the negative root, we will explicitly write $-\sqrt{x}$.



Important Observation: $\sqrt{x^2} = |x|$.

The range of a function

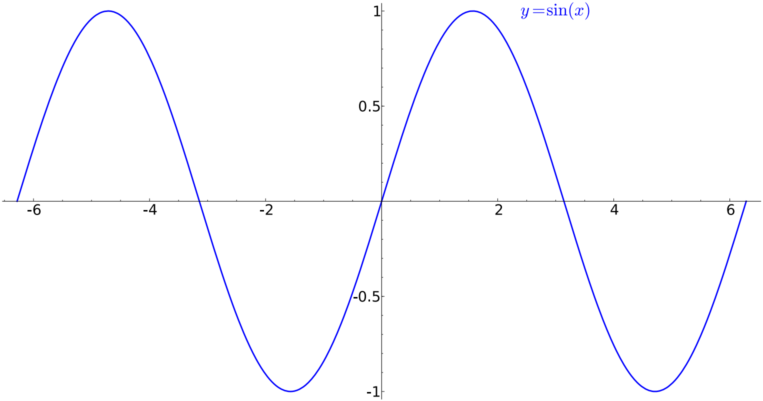
The *codomain* of a function $f(x)$ is the set of all possible outputs of the function.

The *range* of a function $f(x)$ is the set of all values in the codomain that are actually obtained. For example, the range of $g(x) = \sqrt{x}$ is $[0, \infty)$: these are the only values will ever get out of $\sqrt{x}$.

The vertical line test

The graph of a function $f(x)$ – i.e., the set of all (x, y)-pairs satisfying $y = f(x)$ – always satisfies the vertical line test because exactly one output is associated with each input.

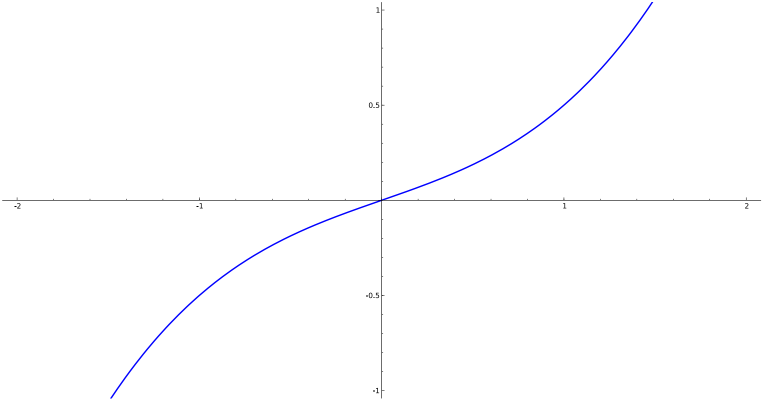
However, the same output could occur multiple times.



One-to-one

In the event that every value in the range occurs for exactly one input value, we say the function is *one-to-one* (sometimes denoted *1-1*). The graphs of one-to-one functions pass both the *vertical* and *horizontal line test*.

Example:



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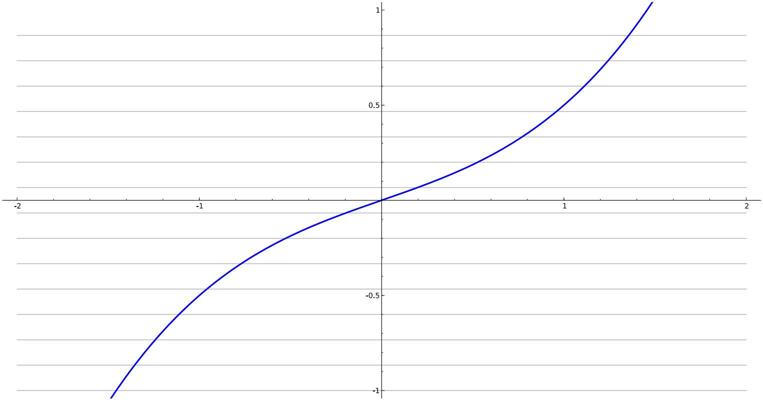

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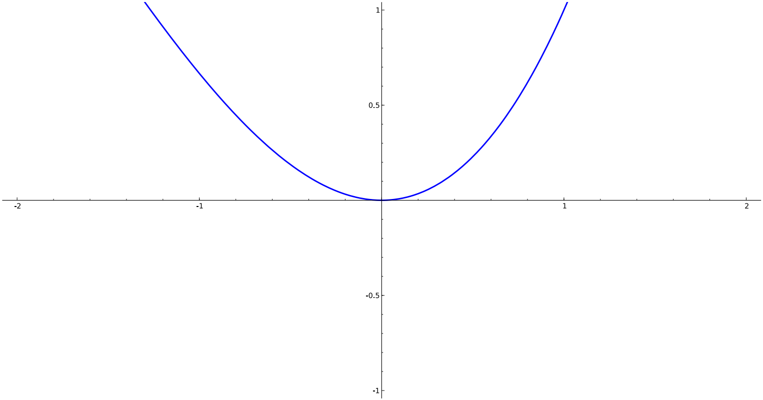
The graph of a function which is not one-to-one:



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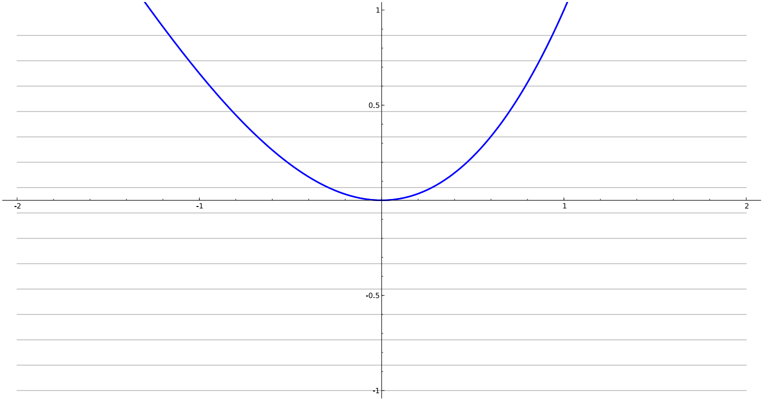


Inverse functions

One-to-one functions are special because they are *invertible*. That means, if $y = f(x)$, then we can find a function g that satisfies $x = g(y)$. I.e., inverse functions "undo" one another.

Example: If $f(x) = \sqrt[3]{x}$, then $g(x) = x^3$ is its inverse. E.g., $-3 = f(-27)$, and $-27 = g(-3)$: $f(g(-3)) = -3$ and $g(f(-27)) = -27$.

Example: If $f(x) = 3x + 2$, then $g(x) = \frac{x-2}{3}$ is its inverse. E.g., $8 = f(2)$, and $2 = g(8)$: $f(g(8)) = 8$ and $g(f(2)) = 2$.

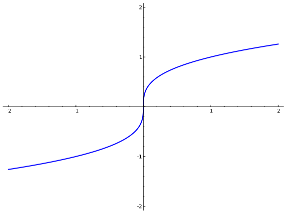
Inverse functions

If $f : D \to C$ is a one-to-one function with range R, then its inverse is usually denoted f^{-1} and is a function from R back to D, $f^{-1} : R \to D$.

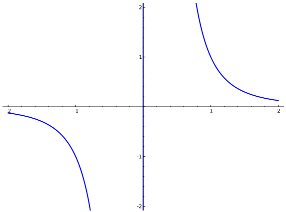
Warning: f^{-1} does not mean f raised to the -1! It is just a notational convention that means "the inverse of f." If you want to actually raise a function to -1, write it as $(f(x))^{-1}$.

$f^{-1}(x) \neq (f(x))^{-1}$

If $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$ while $(f(x))^{-1} = x^{-3}$.



$f^{-1}(x) = \sqrt[3]{x}$



$(f(x))^{-1} = x^{-3}$

Inverse functions

The defining property of the inverse function f^{-1} is that it "undoes" f. More precisely, f^{-1} is the unique function satisfying the following two equations:

$$f\left(f^{-1}(x)\right) = x$$

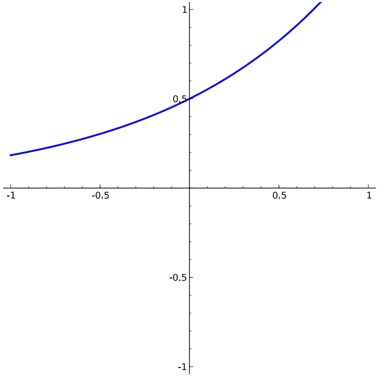
$$f^{-1}\left(f(x)\right) = x$$

Another way to say this is that if $y = f(x)$, then $x = f^{-1}(y)$. The inverse switches the roles of x and y.

Notice that the domain of f is the range of f^{-1}, and the domain of f^{-1} is the range of f.

Graphs of inverse functions

Since the inverse function switches the role of x and y, there is an easy graphical description of inverse functions: the graph of f^{-1} is the graph of f but reflected about the line $y = x$.



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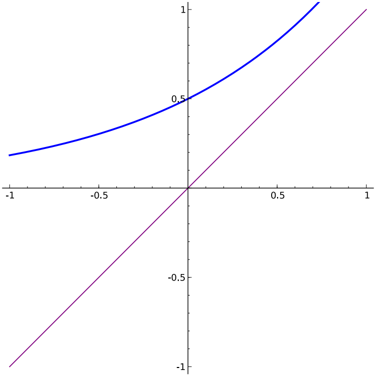

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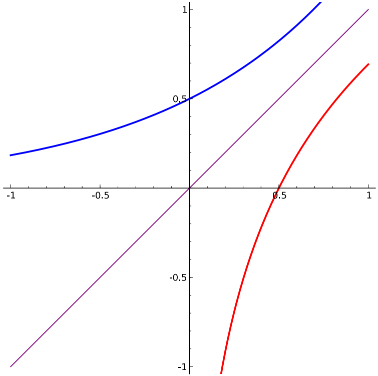
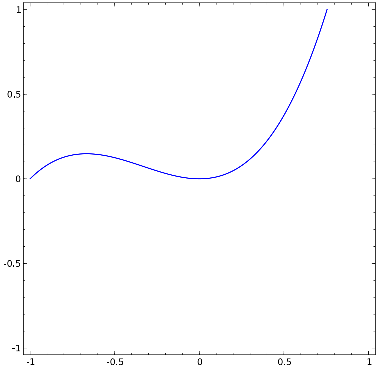


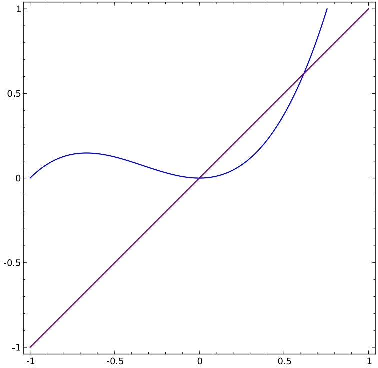
Trying to invert a non-invertible function

Notice that if a function f is not one-to-one, then its graph reflected about $y = x$ is not a function! This is another way of thinking about one-to-one and invertible functions: if a graph's reflection around $y = x$ *does not* pass the vertical line test (i.e., isn't the graph of a function), then the originally function is not one-to-one and so not invertible.



Trying to invert a non-invertible function

Notice that if a function f is not one-to-one, then its graph reflected about $y = x$ is not a function! This is another way of thinking about one-to-one functions: if a function's reflection around $y = x$ is *not* a function (i.e., fails the vertical line test).



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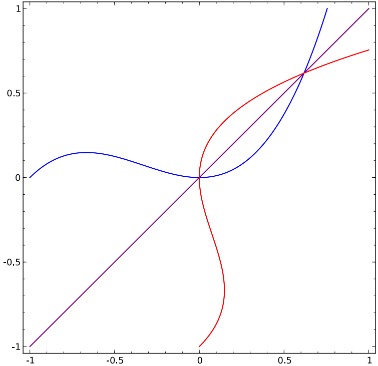


Inverse functions

In most situations you can solve for the inverse function using the following procedure:

1. Write down the equation $y = f(x)$
2. Solve for x, giving an equation $x =$ [some expression involving y].
3. Swap x and y to get an equation $y =$ [some expression involving x].
4. The expression on the right-hand side, involving x's, is the inverse function.

Example

Example: Calculate the inverse of $f(x) = x^3 - 5$.

Solution:

1. Write $y = f(x)$:
$$y = x^3 - 5.$$
2. Solve for x:
$$\begin{aligned} y &= x^3 - 5 \\ \Longrightarrow y + 5 &= x^3 \\ \Longrightarrow \sqrt[3]{y+5} &= x \\ \Longrightarrow x &= \sqrt[3]{y+5} \end{aligned}$$
3. Swap x and y:
$$y = \sqrt[3]{x+5}$$
4. The inverse is
$$f^{-1}(x) = \sqrt[3]{x+5}.$$

Calculating inverses

Why does this procedure work?

Keep in mind the defining property for an inverse function is $f^{-1}(f(x)) = x$.

If $y = f(x)$, we need $f^{-1}(y) = x$, which just means we have solved for x: we have x by itself on one side of the equation, and an expression involving y's on the other side.

The step where we swap x and y is simply putting the equation into the more familiar "$y =$ some function of x" notation.

Example

Example: Calculate the inverse of $g(x) = \frac{3x+1}{x-2}$.
Solution:

1. Write $y = f(x)$:
$$y = \frac{3x+1}{x-2}.$$

2. Solve for x:
$$\begin{aligned}
y &= \frac{3x+1}{x-2} \\
\implies y(x-2) &= 3x+1 \\
\implies xy - 2y &= 3x+1 \\
\implies xy - 2y - 3x &= 1 \\
\implies x(y-3) - 2y &= 1 \\
\implies x(y-3) &= 1 + 2y \\
\implies x &= \frac{1+2y}{y-3}.
\end{aligned}$$

Example (continued)

3. Swapping the x's and y's, we have that the inverse of

$$g(x) = \frac{3x+1}{x-2}$$

which is

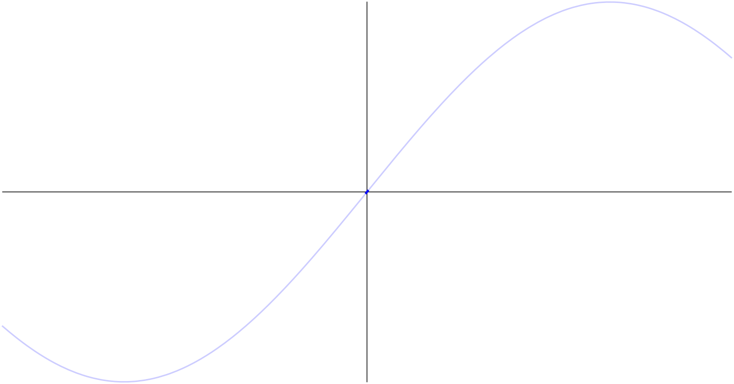
$$g^{-1}(x) = \frac{1+2x}{x-3}.$$

Inverse trig functions

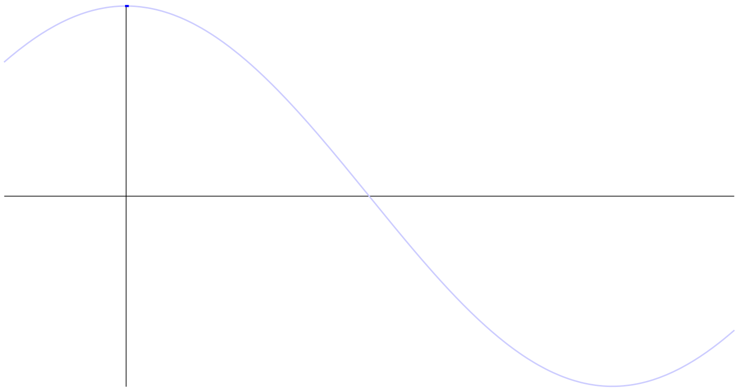
Notice that the six trig functions (sin(x), cos(x), tan(x), sec(x), csc(x), and tan(x)) are not one-to-one, and so are not invertible.

These functions *do* become invertible if we restrict their domains so that the graphs pass the horizontal line test. This is easiest to explain by example...

Restricting the domain to make $\sin(x)$ one-to-one



Restricting the domain to make cos(x) one-to-one



Domains and ranges of inverse trig functions

Function	Domain	Range
arcsin	$[-1, 1]$	$[-\pi/2, \pi/2]$
arccos	$[-1, 1]$	$[0, \pi]$
arctan	$\mathbb{R}$	$(-\pi/2, \pi/2)$
arcsec	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
arccsc	$[1, \infty)$	$(0, \pi/2]$
arccot	$(-\infty, \infty)$	$(0, \pi)$

Logarithms

If $a > 0$, then the function $f(x) = a^x$ is one-to-one, and so it must have an inverse. Like the trigonometric functions, this inverse does not have a nice, closed form.

The inverse of a^x is called the *logarithm with base a* and is denoted $\log_a(x)$.

Note this means the logarithm and exponential satisfy the following two equations:

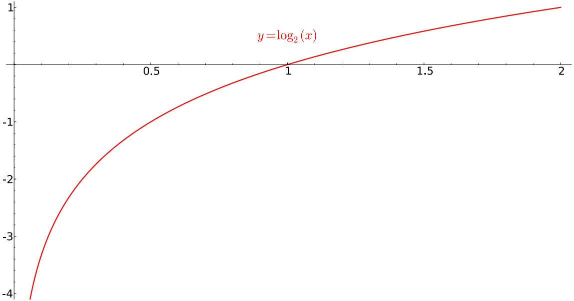
$$\log_a(a^x) = x$$

$$a^{\log_a(x)} = x.$$

Another way to say the same thing: if $y = \log_a(x)$, then $a^y = x$. For example, $\log_2(32) = 5$ because $2^5 = 32$. Similarly, $\log_7(49) = 2$ since $7^2 = 49$.

Graphs of logarithms

The graph of a logarithmic function is easy to determine if you know what the corresponding exponential function looks like.



Notice that $\log_a(x)$ is not defined if $x \leq 0$! If you see $\log_a(0)$ or $\log_a(-3)$ in one of your answers, then you've made a mistake somewhere!

Properties of exponentials

Recall that exponential functions satisfied five important properties:

(i) $a^0 = 1$

(ii) $a^x \cdot a^y = a^{x+y}$.

(iii) $\frac{a^x}{a^y} = a^{x-y}$

(iv) $(a^x)^y = a^{xy}$

(v) $(ab)^x = a^x \cdot b^x$.

Using these properties, we can show that $\log_a(x)$ must satisfy five similar properties. We will prove the first two in class, and leave the other three as an exercise.

Properties of logarithms

Theorem

For all positive real numbers $a > 0$ *and* $b > 0$*, and for every pair of real numbers* x *and* y*, the following five properties hold:*

(i) $\log_a(1) = 0$

(ii) $\log_a(xy) = \log_a(x) + \log_a(y)$

(iii) $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

(iv) $\log_a(x^y) = y \cdot \log_a(x)$

(v) $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$.

We'll show properties (i), (ii) and (v), and leave the others as exercises.

Properties of logarithms

Property (i): $\log_a(1) = 0$.

Recall that $\log_a(x)$ is the inverse of a^x. Thus $a^{\log_a(x)} = x$. So, $a^{\log_a(1)} = 1$, and $\log_a(1)$ must be the power we can raise a to to get 1. There is only possibility: $a^0 = 1$, and so $\log_a(1) = 0$.

Properties of logarithms

Property (ii) $\log_a(xy) = \log_a(x) + \log_a(y)$.

Using properties of exponents, we know

$$a^{\log_a(x)+\log_a(y)} = a^{\log_a(x)} \cdot a^{\log_a(y)}$$

$$= xy.$$

Taking $\log_a$ of both sides of the equation gives the result:

$$a^{\log_a(x)+\log_a(y)} = xy$$

$$\implies \log_a\left(a^{\log_a(x)+\log_a(y)}\right) = \log_a(xy)$$

$$\implies \log_a(x) + \log_a(y) = \log_a(xy).$$

Properties of logarithms

Property (v) $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$.

We know $b^{\log_b(x)} = x$. Taking log_a of both sides of the equation tells us $\log_a(b^{\log_b(x)}) = \log_a(x)$. By property (iv) (which we have not shown; try to prove it on your own), we have

$$\log_b(x) \cdot \log_a(b) = \log_a(x).$$

Solving for $\log_b(x)$ gives the result.

Example:

$$\log_3(243) = \frac{\log_{10}(243)}{\log_{10}(3)}$$

$$= 5$$

The natural log

As the function e^x comes up all the time in calculus, its inverse, $\log_e(x)$ comes up all the time as well. For this reason we give $\log_e(x)$ a special name and some special notation: $\log_e(x)$ is called the *natural logarithm* and is denoted $\ln(x)$.

So, $\ln(x)$ and e^x satisfy the following two equations:

$$\ln\left(e^x\right) = x$$

$$e^{\ln(x)} = x.$$

Homework

1. **Due Monday, 8/25**:
    - Read Ch. 1 of Stewart
    - Stewart §1.5: 2, 4, 7, 15
    - Stewart §1.6: 5 - 8, 29, 30