Math 1060

Lecture 3
Limits

Outline

Summary of last lecture

- Defined functions, domain, codomain, range, and introduced the $f : D \to C$ notation.
- Defined one-to-one functions and the horizontal line test; described the relationship between one-to-one functions and invertibility.
- If f is one-to-one with domain D and range R, then its inverse f^{-1} has domain R and range D (inverses switch the role of input and output).
- The two fundamental equations of invertible functions:

$$f\left(f^{-1}(x)\right) = x$$

$$f^{-1}\left(f(x)\right) = x$$

- Gave an algorithm for determining the inverse of a given one-to-one function.

Summary of last lecture

- The six trig functions are not one-to-one, and so not invertible, but they become invertible if we restrict their domains.
- Each exponential function $f(x) = a^x$, for $a > 0$ and $a \neq 1$, is one-to-one, and its inverse is denoted $f^{-1}(x) = \log_a(x)$.
- Five properties of logarithms, coming from the five properties of exponentials and the fundamental equations of inverses.
- $\ln(x) = \log_e(x)$ is the *natural log*.

The basic idea of calculus

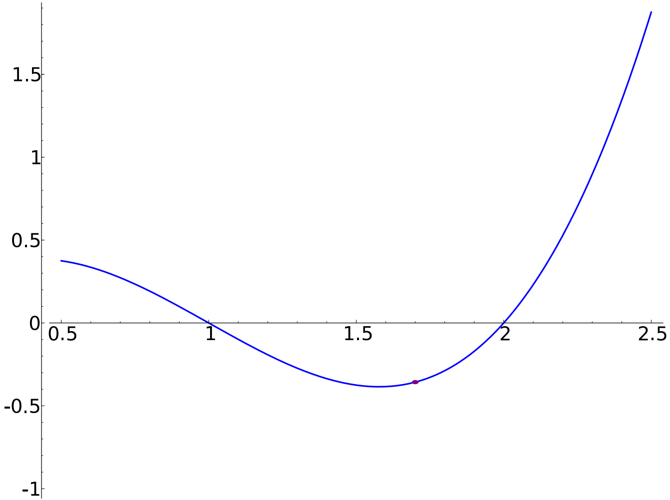
Everything in calculus boils down to the following simple idea:

> *If you want to calculate some quantity but don't know how, then approximate that quantity with something you do know how to calculate; then find a way to improve your approximation.*

This is a theme you will see over and over again in calculus, and even though it's easy to lose sight of this basic idea, it is lurking in the background of everything you do.

Tangent lines

Motivating Problem: Calculate the equation of the line tangent to the graph $y = f(x)$ at the point $(x_0, f(x_0))$.



Tangent lines

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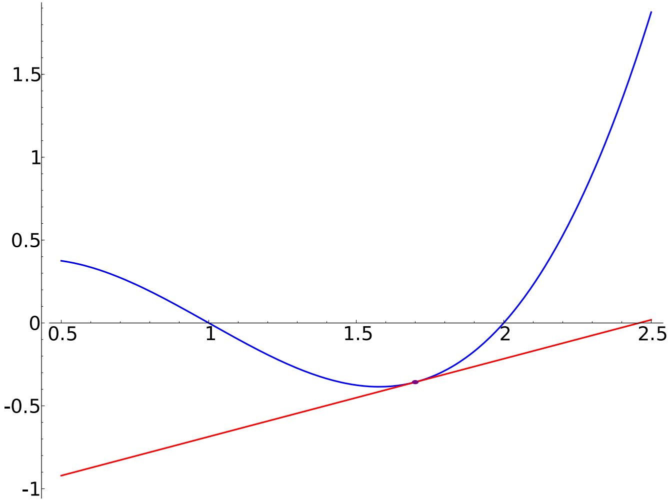
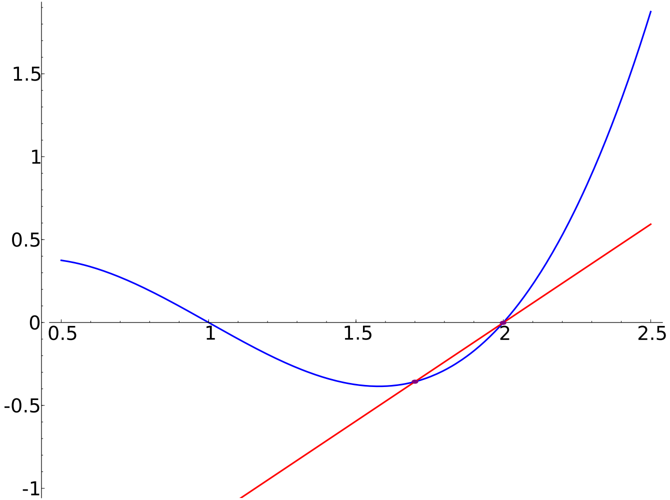


Tangent lines

Solution: Calculate the *secant line* through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ as an approximation. Then move x_1 closer to x_0.



Average velocity

Another motivating problem has to do with calculating velocities.

Suppose a car travels 110 miles over a period of 2 hours. What was the car's velocity?

If you said 55 miles-per-hour, you're only half-right. If you drove from Clemson to Charlotte over the course of two hours, you probably aren't driving 55 mph the whole time: you speed up and slow down and stop and start again, over and over.

The 55 miles-per-hour above is the *average velocity* of the car, and represents what your velocity would be if the velocity was constant for the entire drive.

Average velocity

In general, if a body moves a distance of D over a time T then the body's *average velocity* is

$$v_{\text{avg}} = \frac{D}{T}.$$

(Note: If you are given units for distance and time, then your answer to a velocity question should include the appropriate units as well.)

Average velocity

For our purposes, we will usually think of motion as occurring on the real line.

We will think of a body starting from position x_0 at time t_0 and moving to position x_1 at time t_1. The average velocity is then

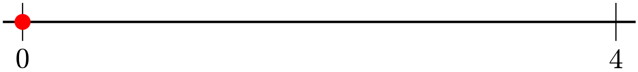
$$v_{\text{avg}} = \frac{x_1 - x_0}{t_1 - t_0}$$

$$= \frac{\Delta x}{\Delta t}.$$

We use the symbol Δ to mean the change in the variable.

Average velocity

In the animation below, the position of the particle at time t is given by $f(t) = {}^{t^2}\!/_{25}$, and the animation takes a total of 10 seconds.



The average velocity of the particle during this time is

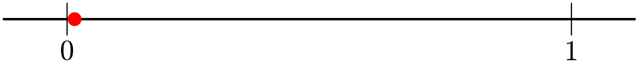
$$v_{\text{avg}} = \frac{x_1 - x_0}{t_1 - t_0}$$

$$= \frac{{}^{10^2}\!/_{25} - {}^{0^2}\!/_{25}}{10 - 0}$$

$$= \frac{4}{10} = \frac{2}{5}$$

Average velocity

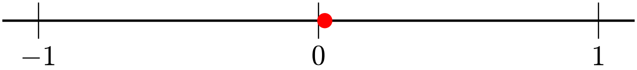
In the animation below, the position of the particle at time t is given by $f(t) = \sin\left(\pi t/7\right)$, and the animation takes a total of 7 seconds.



What's the average velocity of the particle?

Average velocity

In the animation below, the position of the particle at time t is given by $f(t) = \sin\left({}^{3\pi t}/_{14}\right)$, and the animation takes a total of 7 seconds.



The average velocity of the particle during this time is

$$v_{\text{avg}} = \frac{\sin\left({}^{3\pi}/_{2}\right) - \sin(0)}{7 - 0}$$

$$= \frac{-1}{7}$$

The problem with average velocity

Notice that average velocity ignores a lot of information about the motion of the particle. If the average velocity is zero, you don't know if the particle ever moved at all: it may have moved and come back to its starting point, or it may have stayed still.

One way of getting more information about how exactly the particle moves is to calculate its *instantaneous velocity* at each moment in time.

Now we have a slight problem: how can we go about calculating the velocity at an *instant* in time? Is this even an idea that makes sense?

Instantaneous velocity

Calculating instantaneous velocity seems like a hard problem at first glance, but we can actually do these calculations by applying the basic idea of calculus mentioned earlier.

We can approximate instantaneous velocity with something easier which we understand: average velocity, over very short intervals of time. To get better approximations, we look at average velocities over smaller and smaller portions of time.

The idea of a limit

The main idea in calculus is to approximate difficult-to-calculate quantities with easier-to-calculate quantities and then make the approximation "better."

But when do we stop? When can we stop improving our approximations and say we know what the true value is supposed to be?

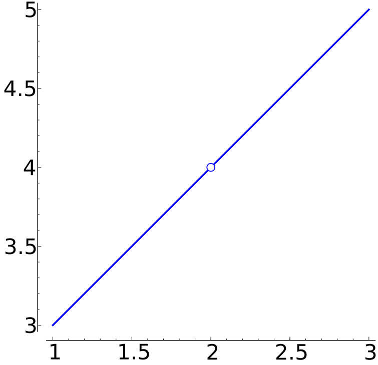
The main technical tool for doing this is called a *limit*, and limits make the notion of "one quantity getting closer to another quantity" precise.

We'll explain limits slowly, starting off by simply looking at some graphs and talking about limits in a naïve, hand-wavy way. Later we will make the notion of limit very precise, but first it will be good to get some intuition for what a limit is.

The idea of a limit

A *limit* tells us that the outputs of a function get very close to a specific value as the inputs get very close to some value. This idea is easiest to explain by looking at some specific examples.

We begin with a simple function, $f(x) = \frac{x^2-4}{x-2}$.



Limits

The numerator of this function may be factored, giving

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2}.$$

Notice this function is undefined at $x = 2$, but equals $x + 2$ everywhere else: everywhere except $x = 2$, the $x - 2$ in the numerator and denominator cancel.

The function is undefined at $x = 2$ because we have division by zero: $\frac{4 \cdot 0}{0} = \frac{0}{0}$ is undefined.

The domain of this function is $(-\infty, 2) \cup (2, \infty)$.

Limits

Because $f(x) = \frac{x^2-4}{x-2}$ equals $x+2$ **everywhere except at** $x = 2$, when we plug inputs very close to $x = 2$ into f, we get outputs which are very close 4.

Notice that we **never** actually plug in 2, and we **never** actually get out a 4, but we get outputs very, very, very close to 4 by plugging in inputs very, very, very close to 2.

We express this fact that inputs very close to 2 give outputs very close to 4 by saying "*the limit as* x *goes to* 2 *of* $f(x)$ *equals* 4."

Notationally this is written as

$$\lim_{x \to 2} f(x) = 4.$$

Limits

In general, we write

$$\lim_{x \to a} f(x) = L$$

if the outputs of the function $f(x)$ get *arbitrarily close* to L as the inputs x get *arbitrarily close* to a.

(We will make this idea of "arbitrarily close" precise later, but for right now, just think of that $f(x)$ gets very, very, very close to L when x is very, very, very close to a.)

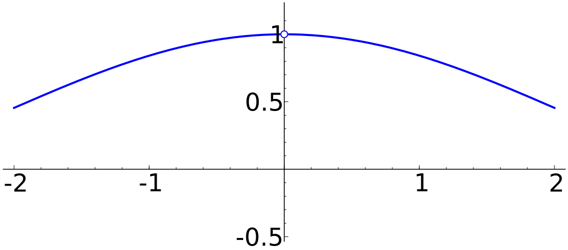
Limits from graphs

Many times when discussing limits, it is useful to have graphs to look at: we can usually guess what the limit should be based on the graph.

There is a technical way to actually calculate limits and we will talk about that later, but for the time being let's just look at some pictures to get some intuition for limits.

Example

What is the limit of $f(x) = \frac{\sin(x)}{x}$ as x goes to 0?

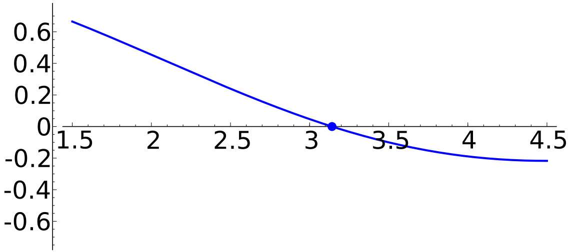


Even though this function is undefined at $x = 0$, the outputs of the function get very close to 1 (they never actually get to 1, however), as x gets very close to 1. So the limit as x goes to 0 of $\frac{\sin(x)}{x}$ is 1,

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Example

What is the limit of $f(x) = \frac{\sin(x)}{x}$ as x goes to π?



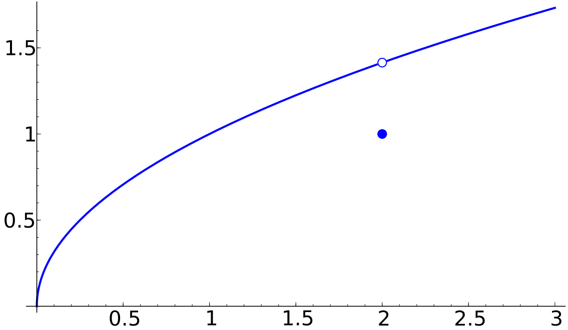
As x gets very close to π, $\frac{\sin(x)}{x}$ gets very close to 0. In this particular case, we actually hit 0 and can actually plug in $x = \pi$.

The limit as x goes to π of $\frac{\sin(x)}{x}$ is 0,

$$\lim_{x \to \pi} \frac{\sin(x)}{x} = 0.$$

Example

What is the limit of the following function?



$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

Example

In the case of the function on the previous slide,

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

the outputs of $f(x)$ get very close to $\sqrt{2}$ as x gets close to 2.

$$\lim_{x \to 2} f(x) = \sqrt{2}.$$

The fact that $f(2) = 1$ is irrelevant!

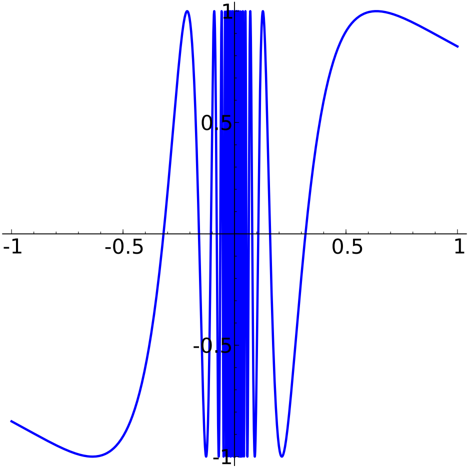
With limits, what is important is what the function does *near* a point, not what happens at the actual point.

Caution!

The value of $\lim_{x \to a} f(x)$ is *not necessarily* the same as $f(a)$. In particular, f does not need to be defined at $x = a$ for the limit to make sense, but even if $f(a)$ exists it may be different from $\lim_{x \to a} f(x)$!.

Non-existence

Consider the function $\sin\left(1/x\right)$. How should we make sense of $\lim_{x\to 0} \sin\left(1/x\right)$?



Non-existence

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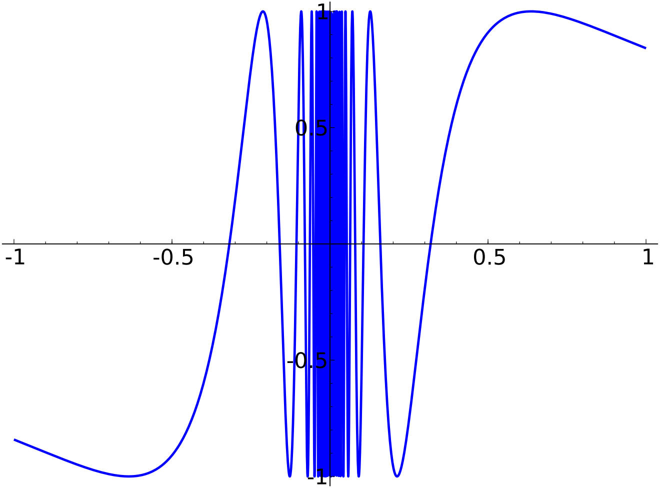


Non-existence

The problem with $\sin\left({}^{1}\!/\!{}_{x}\right)$ is that it oscillates back and forth between -1 and 1 infinitely-many times as x approaches 0. That is, the function is not getting close to any one particular value.

In situations such as this we say the limit *does not exist*, which we will usually abbreviate DNE. In this example we would write

$$\lim_{x \to 0} \sin\left({}^{1}\!/\!{}_{x}\right) \text{ DNE.}$$

Notational remark

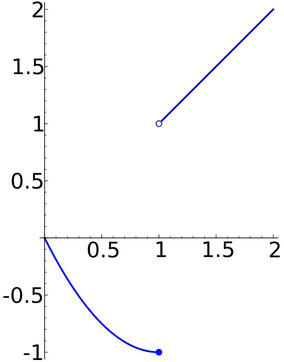
When the limit does not exist we simply write "DNE" next to the limit, and not "= DNE." Here DNE just means "does not exist," and is not a numerical quantity: writing "equals does not exist" does not make any sense.

Right	**Wrong**
$\lim_{x \to a} f(x)\,\mathrm{DNE}$	$\lim_{x \to a} f(x) = \mathrm{DNE}$

If you write "= DNE" on an assignment, you will lose points!

One-sided limits

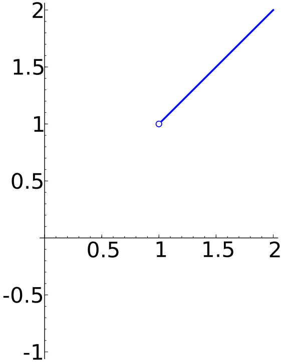
Consider the function below.



Notice the limit as x approaches 1 of $f(x)$ does not exist: the function does not get close to any one value.

One-sided limits

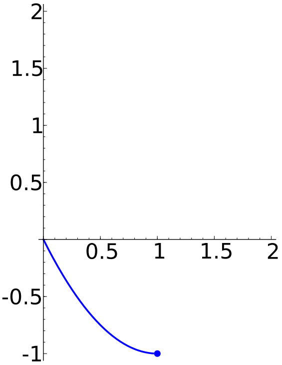
However, if we only saw the right-hand side of the graph,



then we would be comfortable saying the limit existed and equaled 1.

One-sided limits

Similarly, if we only saw the left-hand side of the graph,



we would be comfortable saying the limit existed and equaled -1.

One-sided limits

In situations such as these, we say the function has a *right-hand limit* and a *left-hand limit*.

The right-hand limit is denoted $\lim_{x \to a^+} f(x)$, and the left-hand limit is denoted $\lim_{x \to a^-} f(x)$.

In our previous example we have

$$\lim_{x \to 1^+} f(x) = 1, \quad \text{and} \quad \lim_{x \to 1^-} f(x) = -1.$$

One-sided limits

In general,

$$\lim_{x \to a^+} f(x) = L$$

if the outputs of the function $f(x)$ get *arbitrarily close* to L as the inputs x get *arbitrarily close* to a *from the right* (i.e., are larger than a).

Similarly,

$$\lim_{x \to a^-} f(x) = L$$

if the outputs of the function $f(x)$ get *arbitrarily close* to L as the inputs x get *arbitrarily close* to a *from the left* (i.e., are smaller than a).

Connecting one-sided and two-sided limits

There is a connection between the one-sided limits (i.e., the left- and right-hand limits) and the “usual” limit, which is sometimes called the *two-sided limit*.

Theorem

The two-sided limit of $f(x)$ at $x = a$ exists and equals L ($\lim_{x\to a} f(x) = L$) if and only if both one-sided limits exist at $x = a$ and equal L (i.e., $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$).

This factoid might seem a little silly right now, but will be useful several times in the semester. The will be helpful, in particular, for calculating limits of piecewise functions.

Connecting one-sided and two-sided limits

Another way of writing this theorem:

If $\lim_{x \to a} f(x) = L$, then

$$\lim_{x \to a^+} f(x) = L \text{ and } \lim_{x \to a^-} f(x) = L$$

If $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$, then

$$\lim_{x \to a} f(x) = L$$

This is what the *if and only if* (sometimes abbreviated *iff*) in the theorem means.

A iff B means that A implies B and B implies A.

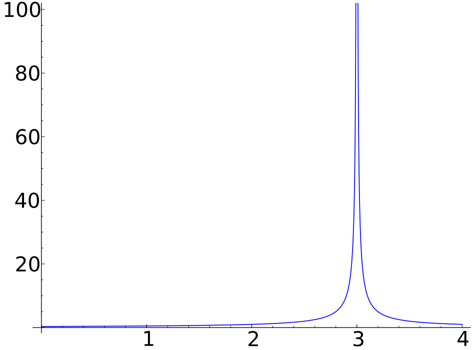
Connecting one-sided and two-sided limits

An important corollary of this theorem is the following: if $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, then $\lim_{x \to a} f(x)$ DNE.

This is one of the basic tricks for showing that a (two-sided) limit does not exist: show the one-sided limits are not the same.

Infinite limits

For some functions $f(x)$, as x approaches a the value of $f(x)$ gets arbitrarily large.

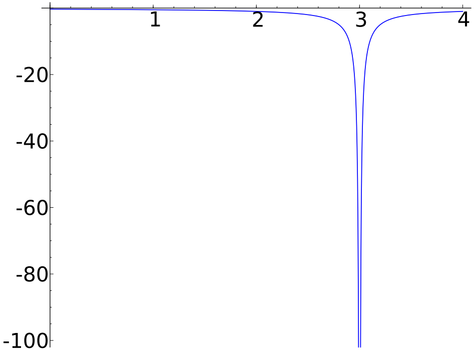


$$f(x) = \left| \frac{1}{x-3} \right|$$

When this happens we say the limit *goes to infinity* and write $\lim_{x \to a} f(x) = \infty$.

Infinite limits

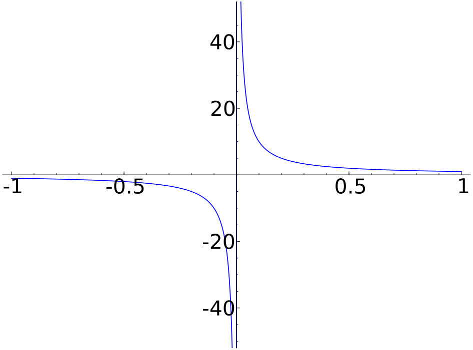
If instead $f(x)$ gets arbitrarily negative, we write $\lim_{x \to a} f(x) = -\infty$.



$$f(x) = \left| \frac{1}{x-3} \right|$$

Infinite limits

It could happen that the limit from one side approaches ∞ while the limit from the other side approaches $-\infty$.



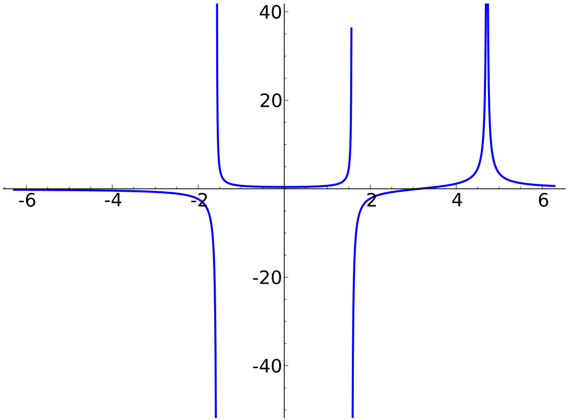
$$f(x) = {}^{1}\!/\!_{x}$$

Notice when in this case we have

$$\lim_{x \to 0^+} \frac{1}{x} = \infty, \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty, \qquad \text{and } \lim_{x \to 0} \frac{1}{x} \text{ DNE}$$

Vertical asymptotes

If any of the limits of $f(x)$ at $x = a$ (the two-sided limit, or either of the one-sided limits) equals $\pm\infty$, then we say $f(x)$ has a *vertical asymptote* at $x = a$.



The function graphed above has three vertical asymptotes.

Homework

1. **Due Friday, 8/29**: Complete the following exercises in Stewart:
   - Read §2.1 and §2.2 in Stewart.
   - In §2.1: 2, 5, 7, 8.
   - In §2.2: 1, 2, 4, 5, 11, 15.