Math 1060

Lecture 4
Limit Laws

Outline

Summary of last lecture

- The basic idea of calculus: calculate hard quantities by approximating with easier ones.
- Tangent lines and instantaneous velocity.
- The idea of a limit.
- When limits DNE.
- One-sided limits.
- Infinite limits and vertical asymptotes
- Homework: Read §2.1 - 2.2 in Stewart. Exercises from 2.1: 2, 5, 7, 8. Exercises from 2.2: 1, 2, 4, 5, 11, 15.

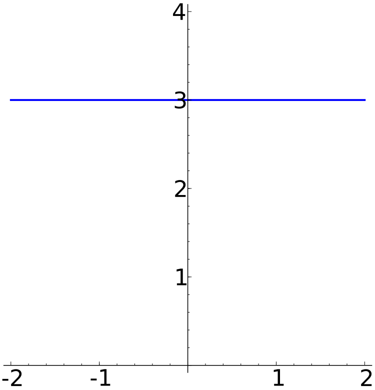
Motivation

Last time we described limits by looking at graphs. Now we want to calculate limits "algebraically," meaning we want to have some rules for manipulating equations to calculate limits.

The way we will do this is by describing the limits of some easy-to-understand functions, then giving rules for calculating the limits of new functions built from old ones (e.g., sums and products). By the end of the lecture we will know how to calculate limits for a large family of functions.

Two easy functions

To get started, we consider the limits of the two simplest types of functions: constants, and the identity.

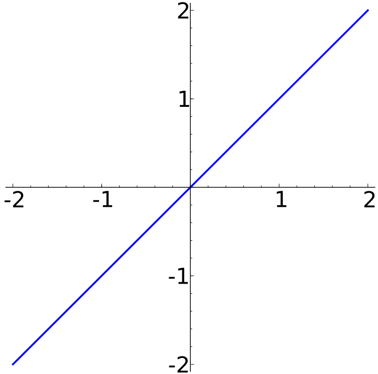


Let $c \in \mathbb{R}$ be any real number. Then for *every* a,

$$\lim_{x \to a} c = c.$$

The identity

The *identity function* is simply the function $f(x) = x$: this is the function that spits out whatever you give it.



For every $a \in \mathbb{R}$,

$$\lim_{x \to a} x = a.$$

Combining functions

We have two very simple types of functions at our desposal, but we can combine these two functions in various ways to get lots of other, more interesting functions.

Just like we can add, subtract, multiply, and divide numbers, we can add, subtract, multiply, and divide functions: we just do the operation (addition, subtraction, ...) *pointwise*. For example, to add two functions f and g together, we just define

$$(f+g)(x) = f(x) + g(x),$$

and similarly for other operations.

Notice functions like $3x^7 - 4x^3 + 2x$ and $\frac{5x+3}{x^3+9}$ can be built from adding, multiplying, and dividing the two functions above (constants and the identity) multiple times.

Limit laws

We have several *limit laws* which tell us how to calculate limits of functions built in this way (adding, subtracting, ...), and they are about as simple as you could hope for. We'll mention each one at a time and give an example.

Limit of a sum

Theorem

The limit of a sum is the sum of the limits, provided both limits exist. That is,

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x),$$

assuming both $\lim_{x \to a} f(x)$ *and* $\lim_{x \to a} g(x)$ *exist, and are not* $\pm\infty$*.*

Example:

$$\lim_{x \to 7} [4 + x]$$

$$= \lim_{x \to 7} 4 + \lim_{x \to 7} x$$

$$= 4 + 7$$

$$= 11$$

Limit of a difference

Theorem

The limit of a difference is the difference of the limits, provided both limits exist. That is,

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x),$$

assuming both $\lim_{x \to a} f(x)$ *and* $\lim_{x \to a} g(x)$ *exist and are not both* $\pm\infty$*.*

Example:

$$\lim_{x \to 3} [x - 5]$$

$$= \lim_{x \to 3} x - \lim_{x \to 3} 5$$

$$= 3 - 5$$

$$= -2$$

Limit of a product

Theorem

The limit of a product is the product of the limits, provided both limits exist. That is,

$$\lim_{x \to a} [f(x) \cdot g(x)] = \left(\lim_{x \to a} f(x) \right) \cdot \left(\lim_{x \to a} g(x) \right),$$

assuming both $\lim_{x \to a} f(x)$ *and* $\lim_{x \to a} g(x)$ *exist.*

Example:

$$\lim_{x \to -1} 2x$$

$$= \left(\lim_{x \to -1} 2 \right) \cdot \left(\lim_{x \to -1} x \right)$$

$$= 2 \cdot (-1)$$

$$= -2$$

Limits of polynomials

Notice the limit laws for products extends to powers of x:

$$\lim_{x \to a} x^n = \lim_{x \to a} \left(\underbrace{x \cdot x \cdots x}_{n \text{ times}} \right)$$

$$= \underbrace{\left(\lim_{x \to a} x \right) \cdot \left(\lim_{x \to a} x \right) \cdots \left(\lim_{x \to a} x \right)}_{n \text{ times}}$$

$$= \underbrace{a \cdot a \cdots a}_{n \text{ times}}$$

$$= a^n$$

Limits of polynomials

Now we can use the three limit laws above to show that the limit of a polynomial as x approaches a is just the polynomial evaluated at a:

Let $f(x) = 3x^2 - 7x + 2$.

$$
\begin{aligned}
\lim_{x\to 2} f(x) &= \lim_{x\to 2} \left(3x^2 - 7x + 2\right) \\
&= \lim_{x\to 2} \left(3x^2\right) - \lim_{x\to a} (7x) + \lim_{x\to a} 2 \\
&= \left(\lim_{x\to 2} 3\right) \cdot \left(\lim_{x\to 2} x^2\right) - \left(\lim_{x\to 2} 7\right) \cdot \left(\lim_{x\to 2} x\right) + \left(\lim_{x\to 2} 2\right) \\
&= 3 \cdot 2^2 - 7 \cdot 2 + 2 \\
&= 0 \\
&= f(2)
\end{aligned}
$$

Limits of polynomials

Theorem

If $f(x)$ is a polynomial (i.e., a function of the form

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + c_{n-2} x^{n-2} + \cdots + c_2 x^2 + c_1 x + c_0$$

where $c_n, c_{n-1}, ..., c_1, c_0 \in \mathbb{R}$ and n is a non-negative integer), then the limit of $f(x)$ as x goes to a exists and

$$\lim_{x \to a} f(x) = f(a).$$

Limits of quotients

Theorem

The limit of a quotient is the quotient of limits, provided both limits exist and the limit of the denominator is not zero. That is,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

if $\lim_{x \to a} f(x)$ *exists,* $\lim_{x \to a} g(x)$ *exists, and* $\lim_{x \to a} g(x) \neq 0$*.*

Example:

$$\lim_{x \to 2} \frac{3}{x} = \frac{\lim_{x \to 2} 3}{\lim_{x \to 2} x}$$

$$= \frac{3}{2}$$

Rational functions

A *rational function* is a ratio of polynomials, e.g.

$$\frac{17x^{12} - 9x^5 + 32x^3 - 21}{x^3 - 3x^2}.$$

The above laws tell us that to evaluate the limit of a rational function, all we need to do is evaluate the function.

Theorem

Let $f(x)$ be a rational function (i.e., $f(x) = \frac{g(x)}{h(x)}$ where g and h are polynomials). Then

$$\lim_{x \to a} f(x) = f(a).$$

Rational functions

Example: Let $f(x) = \frac{5x^3-6x^2+2}{x^4-1}$.

$$\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{5x^3 - 6x^2 + 2}{x^4 - 1}$$

$$= \frac{\lim_{x\to -3} \left(5x^3 - 6x^2 + 2\right)}{\lim_{x\to -3} (x^4 - 1)}$$

$$= \frac{5\cdot(-3)^3 - 6\cdot(-3)^2 + 2}{(-3)^4 - 1}$$

$$= \frac{5\cdot(-27) - 6\cdot 9 + 2}{81 - 1}$$

$$= \frac{187}{80} = f(-3)$$

Limits of roots

Theorem

The limit of the n-th root of a function is the n-th root of the limit of the function, assuming $n \in \mathbb{N}$, and the root "makes sense" (e.g., we don't wind up taking the square root of a negative number). That is,

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

Example: Let $f(x) = x^3 + 6x^2 - 10x + 5$.

$$\begin{aligned}
\lim_{x\to 4} \sqrt[3]{f(x)} &= \lim_{x\to 4} \sqrt[3]{x^3 + 6x^2 - 10x + 5} \\
&= \sqrt[3]{\lim_{x\to 4} (x^3 + 6x^2 - 10x + 5)} \\
&= \sqrt[3]{(4^3 + 6 \cdot 4^2 - 10 \cdot 4 + 5)} \\
&= \sqrt[3]{125} = \sqrt[3]{f(4)}
\end{aligned}$$

Notation

When taking the limit of a product, a quotient, or a root, you don't need to use any extra parentheses: just place $\lim\limits_{x\to a}$ next to the product/quotient/root.

When taking the limit or a sum or difference, however, you **must** include parentheses or brackets around the quantity you're taking the limit of: thinking of the limit as "distributing" over the sum/difference:

Right	**Wrong**
$\lim\limits_{x\to 3}\left(3x^2+2x\right)$	$\lim\limits_{x\to 3} 3x^2+2x$
$\lim\limits_{x\to 0}\left[9x^7-\sqrt[5]{4x^2+x}\right]$	$\lim\limits_{x\to 0} 9x^7-\sqrt[5]{4x^2+x}$

Equality away from a point

Before going any further, let's remind ourselves of something we saw in the last lecture: what a function does at $x = a$ is not important for calculating $\lim_{x \to a} f(x)$. What's important is what happens *near* $x = a$. The following theorem is the precise way of stating this.

Theorem

If $f(x) = g(x)$ near a, but $f(a) \neq g(a)$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(a),$$

assuming either limit (and hence both limits) exist.

Equality away from a point

One particular application of this theorem is our first example of limits yesterday.

Let $f(x) = \frac{x^2-4}{x-2}$ and $g(x) = x + 2$. Then $f(x) = g(x)$ except at $x = 2$: f is undefined at $x = 2$ because of division by zero. The above theorem tells us, however

$$\lim_{x\to 2} \underbrace{\frac{x^2-4}{x-2}}_{f(x)} = \lim_{x\to 2} \underbrace{(x+2)}_{g(x)} = 4.$$

It's important that you realize that we aren't simply factoring $x^2 - 4 = (x+2)(x-2)$ and cancelling the $x - 2$ factors in the numerator and denominator: we are comparing $f(x)$ to a function $g(x)$ which is equal everywhere except at $x = 2$. This is an important, though very subtle, distinction.

Some cautions

Now we have enough tools at our disposal to calculate limits of lots of different types of functions. However, there are some caveats we need to be aware of.

- Division by zero.
- Piecewise functions.
- Arithmetic with $\pm\infty$.

We can deal with each of these situations, but there are some non-obvious things you have to worry about.

Division by zero

As mentioned above, we can take a limit of quotients as the quotient of limits,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$$

provided both limits exist and $\lim\limits_{x \to a} g(x) \neq 0$.

However, sometimes the limit $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ still exists even if $\lim\limits_{x \to a} g(x) = 0$: we just can't use the limit law above to calculate the limit.

This issue will be very important next week when we talk about *derivatives*: derivatives are always expressed as limits of quotients where the denominator goes to zero.

Division by zero

Example: Calculate the following limit,

$$\lim_{x \to 0} \frac{(x+2)^3 - 8}{x}.$$

$$\begin{aligned}
&\lim_{x \to 0} \frac{(x+2)^3 - 8}{x} \\
&= \lim_{x \to 0} \frac{x^3 + 6x^2 + 12x + 8 - 8}{x} \quad \text{(binomial theorem)} \\
&= \lim_{x \to 0} \frac{x^3 + 6x^2 + 12x}{x} \\
&= \lim_{x \to 0} \left(x^2 + 6x + 12\right) \\
&= 12
\end{aligned}$$

Division by zero

Example: Calculate the following limit,

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}.$$

Notice that in this case the h is what's approaching 0: the x is not changing!

$$\begin{aligned}
\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} (2x + h) \\
&= 2x
\end{aligned}$$

One-sided limits

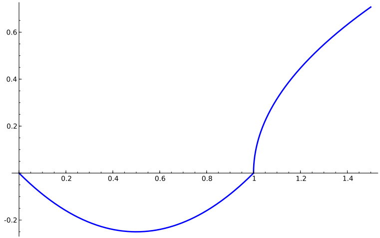
All of the limit laws above, which were originally stated in terms of two-sided limits, also hold for one-sided limits.

1. $\lim\limits_{x \to a^\pm} c = c$
2. $\lim\limits_{x \to a^\pm} x = a$
3. $\lim\limits_{x \to a^\pm} (f(x) + g(x)) = \lim\limits_{x \to a^\pm} f(x) + \lim\limits_{x \to a^\pm} g(x)$
4. $\lim\limits_{x \to a^\pm} (f(x) - g(x)) = \lim\limits_{x \to a^\pm} f(x) - \lim\limits_{x \to a^\pm} g(x)$
5. $\lim\limits_{x \to a^\pm} f(x) \cdot g(x) = \left(\lim\limits_{x \to a^\pm} f(x)\right) \cdot \left(\lim\limits_{x \to a^\pm} g(x)\right)$
6. $\lim\limits_{x \to a^\pm} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a^\pm} f(x)}{\lim_{x \to a^\pm} g(x)}$
7. $\lim\limits_{x \to a^\pm} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a^\pm} f(x)}$

Piecewise functions

We can now use these limit laws for one-sided limits to calculate the limits of piecewise functions. The idea is to calculator the left- and right-hand limits at the point which "join" two pieces of the function, and check whether or not they are the same thing.

For example, suppose want to calculate the limit of the following function at $x = 1$.



$$f(x) = \begin{cases} x^2 - x & \text{if } x < 1 \\ \sqrt{x - 1} & \text{if } x \geq 1 \end{cases}$$

Piecewise functions

We calculate the left- and right-hand limits at $x = 1$.
Left-hand limit:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left(x^2 - x\right)$$

$$= 0$$

Right-hand limit:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x - 1}$$

$$= 0$$

Because both one-sided limits agree, we know

$$\lim_{x \to 1} f(x) = 0.$$

Piecewise functions

Example: Calculate the limit as x goes to 3 of the following function:

$$f(x) = \begin{cases} x^2 & \text{if } x > 3 \\ -x & \text{if } x < 3. \end{cases}$$

Notice $\lim_{x \to 3^+} f(x) = 9$, while $\lim_{x \to 3^-} f(x) = -3$, and so the limit does not exist.

The squeeze theorem

The *squeeze theorem* (aka *sandwich theorem*) gives us a way to calculate limits of complicated functions by comparing them to simpler functions. Before stating the squeeze theorem, let's one simple observation.

Suppose that $f(x)$ and $g(x)$ are two functions defined near $x = a$ (a may or may not be in the domain of the functions). Suppose also that $f(x) \geq g(x)$ for all x values "near" a. Then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

If the outputs of $f(x)$ are always less-than-or-equal-to the outputs of $g(x)$, then it is impossible for the limit of f to be greater than the limit of g.

The squeeze theorem (aka *sandwich theorem*)

Theorem

Suppose $f(x)$, $g(x)$, and $h(x)$ are three functions defined near $x = a$, and suppose for all values of x near a the following inequalities hold:

$$f(x) \leq g(x) \leq h(x).$$

If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$ as well.

The limit of $g(x)$ is *sandwiched* inbetween the limits of $f(x)$ and $h(x)$, so if the limits of f and h are the same, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \leq \lim_{x \to a} h(x)$$

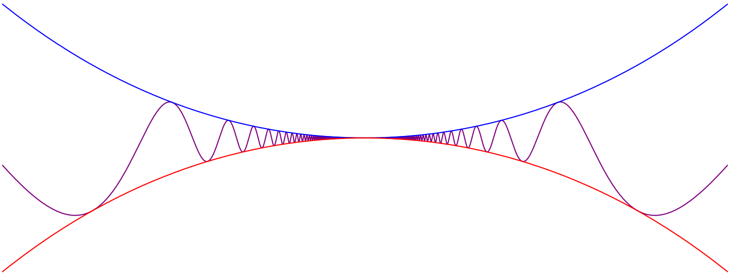
$$\implies L \leq \lim_{x \to a} g(x) \leq L$$

$$\implies \lim_{x \to a} g(x) = L.$$

The squeeze theorem

Example: Calculate the limit as x goes to 0 of the following function,

$$\cos\left(\frac{1}{x^2}\right) \cdot \left(x^4 + 6x^2\right).$$



The squeeze theorem

Notice that because $-1 \leq \cos(x) \leq 1$, for *any* functions $\alpha(x)$ and $\beta(x)$, with $\beta(x) > 0$, we have

$$-\beta(x) \leq \cos(\alpha(x)) \cdot \beta(x) \leq \beta(x).$$

$$\begin{aligned} & -1 \leq \cos(x) \leq 1 \\ \implies & -1 \leq \cos(\alpha(x)) \leq 1 \\ \implies & -\beta(x) \leq \cos(\alpha(x)) \cdot \beta(x) \leq \beta(x) \end{aligned}$$

This is a very nice observation, which also works with sine, that is sometimes helpful in finding the *sandwiching functions* we need to apply the sandwich theorem.

The squeeze theorem

Using the observation on the last slide and applying the sandwich theorem, we have the following:

$$-\left(x^4+6x^2\right) \leq \cos\left(\frac{1}{x^2}\right)\cdot\left(x^4+6x^2\right) \leq \left(x^4+6x^2\right)$$

$$\implies \lim_{x\to 0} -\left(x^4+6x^2\right) \leq \lim_{x\to 0} \cos\left(\frac{1}{x^2}\right)\cdot\left(x^4+6x^2\right) \leq \lim_{x\to 0}\left(x^4+6x^2\right)$$

$$\implies 0 \leq \lim_{x\to 0} \cos\left(\frac{1}{x^2}\right)\cdot\left(x^4+6x^2\right) \leq 0$$

And thus

$$\lim_{x\to 0} \cos\left(\frac{1}{x^2}\right)\cdot\left(x^4+6x^2\right) = 0.$$

Homework

Due Friday, 8/29 :

Complete the following exercises in Stewart:

- Read §2.1 and §2.2 in Stewart.
- In §2.1: 2, 5, 7, 8.
- In §2.2: 1, 2, 4, 5, 11, 15.

Due Monday, 9/1 :

- Read about the *binomial theorem* on Wikipedia; know how to expand quantities like $(a+b)^7$ *without* FOIL-ing.
- Read §2.3 and §2.4 in Stewart.

There will be an in-class quiz on Monday, 9/1.