CLEMSON UNIVERSITY

Math 1060

Lecture 5
The Precise Definition of a Limit

Outline

Summary of last lecture

The intuition behind limits

The precise definition of a limit
- The ε-δ definition of limits
- Treating the ε-δ definition as a game

Examples
- The δ for a particular ε
- The limit of a constant function
- The limit of the identity function
- The limit of x^2
- The limit of a sum of two functions

The precise definition of other types of limits
- The ε-δ definition of a right-hand limit
- The ε-δ definition of a left-hand limit
- The ε-δ definition of an infinite limit

Summary of last lecture

- Discussed the limits of two basic “building block” functions: the constant functions and the identity.
- Described the limit laws for sums, differences, products, quotients, and roots.
- Used the limit laws and our building block functions to show that calculating limits of polynomials and rational functions is easy to do: just evaluate the polynomial (or rational function) at the point where you’re taking the limit. *Caution: This does not work if the point is not in the domain of the function; e.g., if you have division by zero in a rational function.*
- Looked at a few examples where we could still calculate the limit of a rational function, even if we would have division by zero, by first doing some algebra.
- Talked about inequalities of functions and limits, and discussed the squeeze theorem.

The intuition behind limits

Earlier in the week we defined limits in very hand-wavy terms by considering graphs of functions. We now want to go back and make our definition of limit very precise. Let's first recall what the basic idea of a limit was:

We say $\lim_{x \to a} f(x) = L$ when the outputs of $f(x)$ get "very close" to L as the inputs x get "very close" to a.

The problem with this definition is that "very close" is not a precise mathematical idea. Our goal will be to replace "very close" with something more formal.

The intuition behind limits

By $f(x)$ *getting close to* L, what we really mean is that we can choose inputs x so that the value of $f(x)$ is within any prescribed distance of L.

That is, we can choose values of x so that $f(x)$ is within $1/1,000$ of L; or within $1/1,000,000$ of L; or within $1/1,000,000,000$ of L; etc.

You pick how close you want to get to L, and I can tell you how close to a the x's need to be in order to guarantee that the outputs, $f(x)$, are within your chosen distance of L.

For the limit to exist and equal L, this has to work for *every* (non-zero) distance you pick, no matter how small that distance is.

The precise definition of a limit

Here is the formal, precise definition:

We say *the limit as x goes to a of $f(x)$ equals L*, and write $\lim_{x \to a} f(x) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Let's take a moment to unwind this definition:

- You pick some distance describing how close you want to get to L; call that distance ε.
- I find some value, called δ, so that if x is within δ-distance of a, it is guaranteed that $f(x)$ is within ε-distance of L.
- I have to be able to do this for *every* $\varepsilon > 0$ that you choose.

One caution: the δ we choose can depend on ε (as you change ε you may need to change δ) and it can also depend on a. However, δ *can not depend on* x!

Another point of view

Another way to think about this definition of limit is the following:

- Consider graphing the function $y = f(x)$ in some little window centered at the point (a, L).
- You choose the height of the window to be whatever you want, and call that height 2ε.
- After you choose the height of the window, I can choose a width 2δ so that the curve $y = f(x)$ enters the window from the left-hand side and exits on the right-hand side, never leaving the window from the top or bottom.
- For *any* height 2ε you choose, I can find a width 2δ that makes this happen.

The ε-δ game

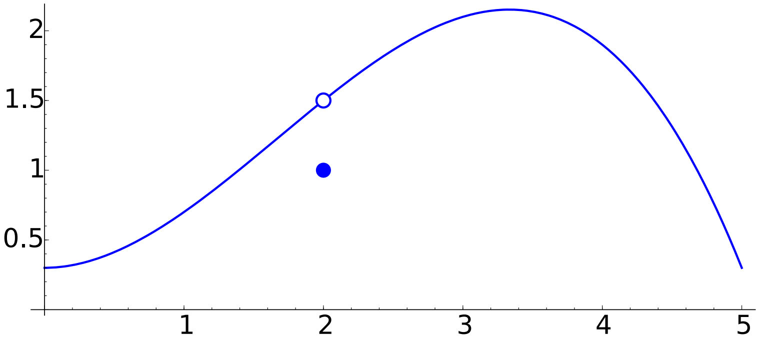
We can think about this whole "choosing a δ given an ε" as playing following game.

- You and I are opponents in this game.
- We play this game by drawing boxes centered at the point (a, L) where you choose the top and bottom of the box, and I choose the left- and right-hand sides.
- My objective is to convince you that $f(x)$ gets arbitrarily close to L, provided you pick x close enough to a.
- Your objective is to show $f(x)$ does not get arbitrarily close to L by forcing me to draw a box so that the graph $y = f(x)$ either misses the box completely, or exits the box on the top or the bottom.
- We play this game in rounds. Each round consists of you taking a turn, followed by me taking a turn.

The ε-δ game

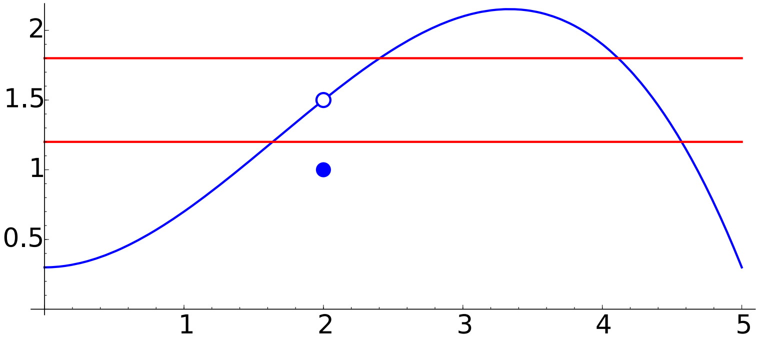
How the game is played:

1. At the start of the round we draw the graph $y = f(x)$.



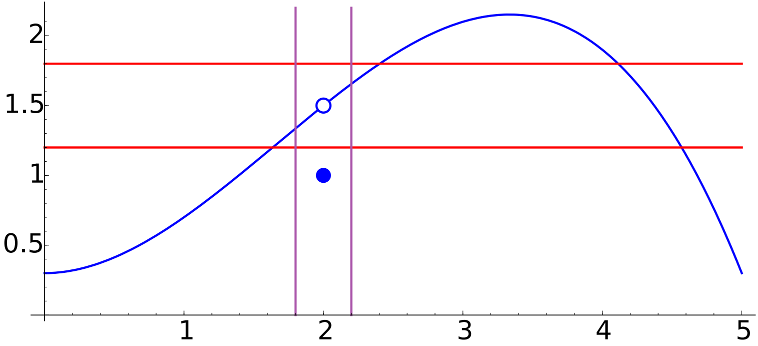
The ε-δ game

2. On your turn you pick an $\varepsilon > 0$ and draw the two horizontal lines $y = L \pm \varepsilon$.



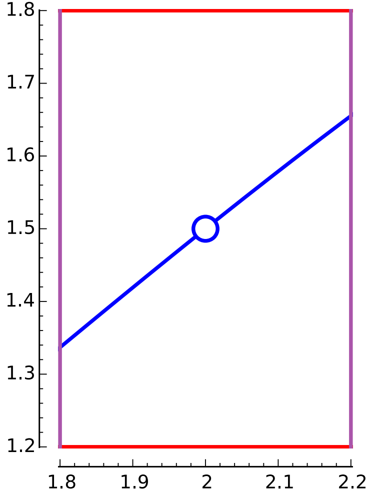
The ε-δ game

3. On my turn I pick a $\delta > 0$ and draw the two vertical lines $x = a \pm \delta$.



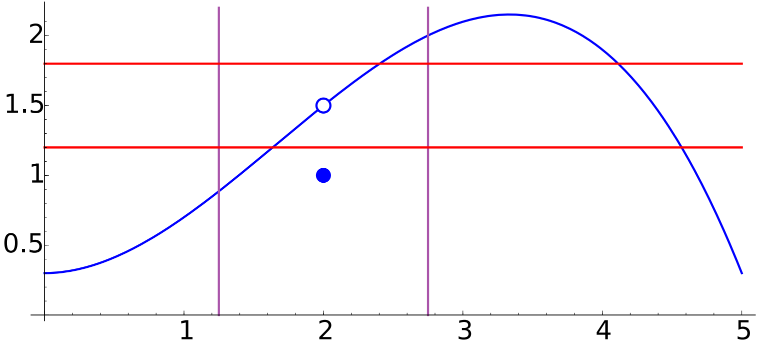
The ε-δ game

4. Now we look at the portion of the graph $y = f(x)$ that lives in the box we just made: you picked the top and bottom of the box, and then I picked the left- and right-hand sides.



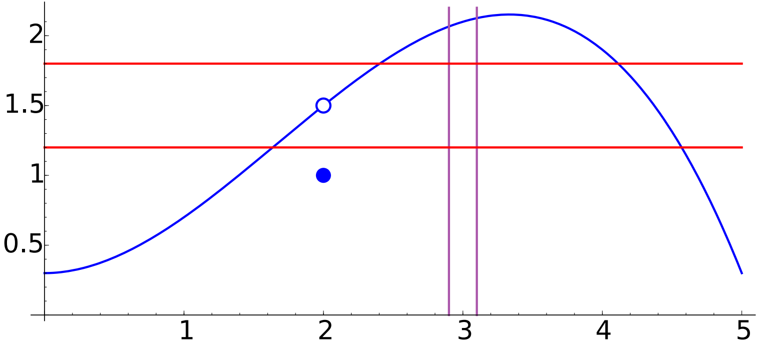
The ε-δ game

5. You win the game if any of the following conditions occur:
   (a) If the portion of $y = f(x)$ inside our box touches the top or the bottom, then you win.



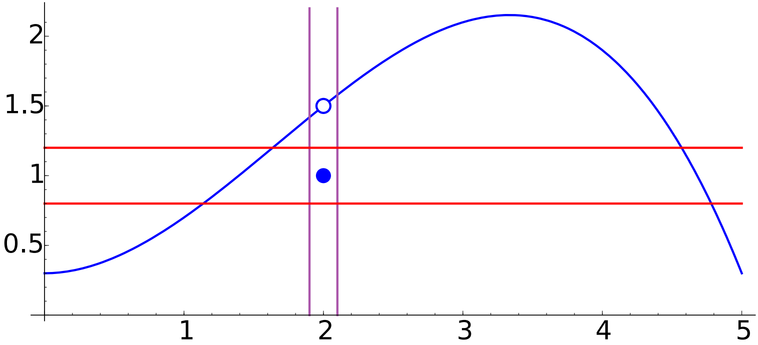
The ε-δ game

5. You win the game if any of the following conditions occur:
   (b) If the box we draw is empty – i.e., $y = f(x)$ never enters the box – then you win.



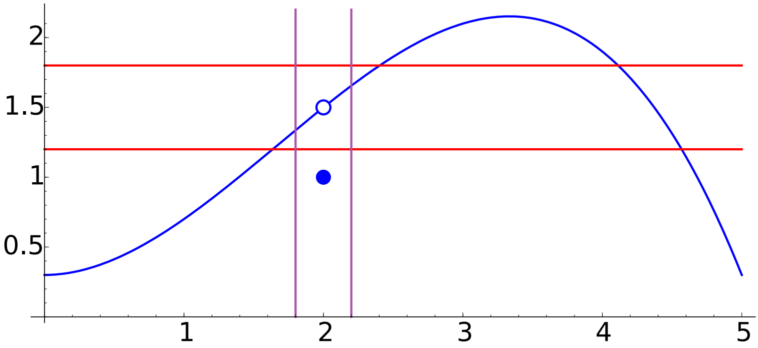
The ε-δ game

5. You win the game if any of the following conditions occur:
   (c) If the only point of $y = f(x)$ inside the box is the point (a, L), then you win. (This could happen with a piecewise function.)



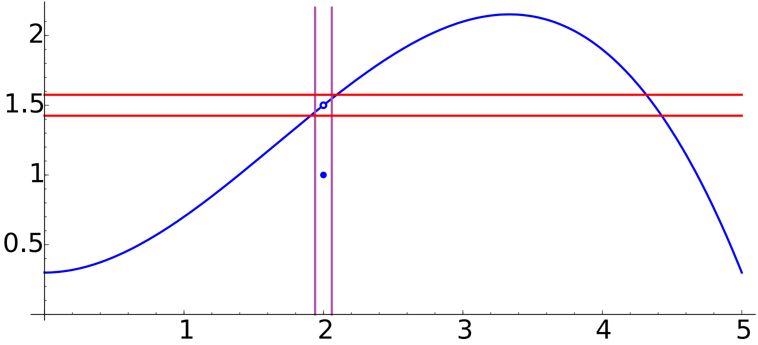
The ε-δ game

6. If the graph $y = f(x)$ enters the box, but never touches the top or the bottom, then we start a new round.



The ε-δ game

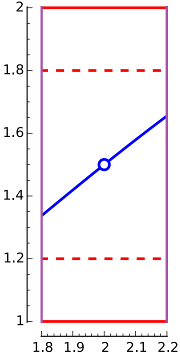
7. I win the game if for every $\varepsilon > 0$ that you choose for drawing the lines $y = L \pm \varepsilon$ (the top and bottom of the box), I can always force you to play another round: I can always find a $\delta > 0$ so that in the box with left- and right-hand sides $x = a \pm \delta$, the curve $y = f(x)$ enters the box from the left, exits on the right, and never touches the top or the bottom.



The ε-δ game

Notice that if I pick left- and right-hand sides so that the graph enters the box on the left, exits on the right, and never touches the top or the bottom, then on your next turn you should move the top and bottom closer together.

If you made the top and bottom further apart, I can just keep the same left and right-hand sides.



The ε-δ game

Your only strategy in this game is to force the box we draw to get smaller and smaller, zooming into the point (a, L).

If the limit of $f(x) = L$ though, I can always force you to play another round; I can always win the game if the $\lim_{x \to a} f(x) = L$.

All this is saying is that we force the outputs of $f(x)$ to be within ε-distance of L (whatever $\varepsilon > 0$ you want), by choosing x-values that are within δ-distance of a, and that is precisely what the formal definition of the limit says.

Again, notice that the δ may depend on ε (when you change ε, I get to change δ), and it may depend on the a where we're taking the limit (this is a value we pick before we start playing the game), but δ *can not* depend on x.

The ε-δ game

To summarize:

1. We play this game, first deciding on what $f(x)$, a, and L are going to be before we start.
2. You pick an $\varepsilon > 0$.
3. I pick a $\delta > 0$ and claim that if $0 < |x - a| < \delta$ (the distance from x to a is less-than δ), then $|f(x) - L| < \varepsilon$ (the distance from $f(x)$ to L is less-than ε).
4. If you don't believe me, pick whatever value of x you want between $a - \delta$ and $a + \delta$ except $x = a$ (because the function does not need to be defined there), plug that x-value into f, and see if $f(x)$ is within ε-distance of L.
5. I don't know what x you're going to pick, so the δ I tell you can't depend on x.

The δ for a particular ε

Let's make these ideas a bit more concrete by working a particular example from start to finish.

Example: Let $f(x) = 3x - 1$. What value of $\delta > 0$ will guarantee that $|f(x) - 5| < 1/10$ whenever $0 < |x - 2| < \delta$?

To answer this type of question, let's work backwards. We'll start with what we want to show, $|f(x) - 5| < 1/10$, and see if we can do some algebraic trickery to get an inequality of the form $|x - 2| < C$. If this happens, we'll take $\delta = C$.

After we do this and know what δ should be, we will then rewrite our work starting with $|x - 2| < \delta$ and showing this implies $|f(x) - 5| < 1/10$.

The δ for a particular ε

We start with $|f(x) - 5| < 1/10$ and try to work backwards to determine the appropriate δ.

$$
\begin{aligned}
&|f(x) - 5| < 1/10 \\
\implies &|3x - 1 - 5| < 1/10 \\
\implies &|3x - 6| < 1/10 \\
\implies &3|x - 2| < 1/10 \\
\implies &|x - 2| < 1/30
\end{aligned}
$$

At this point we feel confident that $1/30$ should be the correct choice for δ. To verify this let's run the argument in the other direction, showing $0 < |x - 2| < 1/30$ forces $|f(x) - 5| < 1/10$.

The δ for a particular ε

Claim: Let $f(x) = 3x - 1$. If $0 < |x - 2| < 1/30$, then $|f(x) - 5| < 1/10$.

Proof:

$$
\begin{aligned}
& 0 < |x - 2| < 1/30 \\
\implies & |x - 2| < 1/30 \\
\implies & 3|x - 2| < 3 \cdot 1/30 \\
\implies & 3|x - 2| < 1/10 \\
\implies & |3x - 6| < 1/10 \\
\implies & |3x - 1 - 5| < 1/10 \\
\implies & |f(x) - 5| < 1/10.
\end{aligned}
$$

The δ for a particular ε

What we have shown is that if $0 < |x - 2| < 1/30$, then it is *guaranteed* that $|f(x) - 5| < 1/10$.

In terms of our game, if our function was $f(x) = 3x - 1$, if $a = 2$, if $L = 5$, and if you chose the height of the window to be $1/5$ ($1/10$ up and $1/10$ down from $y = 5$), then I should choose the width of the window to be $1/15$ ($1/30$ to the left and $1/30$ to the right of of $x = 2$).

Exercise for you: Think about how to modify this argument if we replace $1/10$ with $1/500$. What if we replace $1/10$ with an arbitrary ε?

Some examples of this definition

Let's use this precise definition of limit to prove some of the things we claimed in the previous class:

- $\lim_{x \to a} c = c$,
- $\lim_{x \to a} x = a$, and
- $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

The limit of a constant function

The claim is that for any constant $c \in \mathbb{R}$, $\lim_{x \to a} c = c$; that is, for every $\varepsilon > 0$ there exists a $\delta > 0$ so that $|f(x) - c| < \varepsilon$ whenever $0 < |x - a| < \delta$.

1. You pick your favorite $\varepsilon > 0$.
2. I pick $\delta = 1$.
3. Now choose some x value so that $0 < |x - a| < 1$: is it guaranteed that $|f(x) - c| < \varepsilon$?
4. Yes: because $f(x) = c$, $|f(x) - c| = |c - c| = 0 < \varepsilon$.

This example *is not* representative of how these problems are usually solved. Normally the δ you choose will be a function of ε (if you change ε, you have to change δ too). The constant function is the easiest possible situation.

The limit of the identity

The claim is that if $f(x) = x$, then $\lim_{x \to a} x = a$; that is, for every $\varepsilon > 0$ there exists a $\delta > 0$ so that $|f(x) - a| < 0$ when $0 < |x - a| < \delta$.

Notice that $|f(x) - a| = |x - a|$. If we pick $\delta = \varepsilon$, then we're done:

$$0 < |x - a| < \delta = \varepsilon$$

$$\implies |x - a| < \varepsilon.$$

The limit of x^2

Before using the definition to justify the limit laws, let's consider the limit of another simple function: $f(x) = x^2$. We claim that $\lim_{x \to a} x^2 = a^2$. In terms of our definition, this means that for every $\varepsilon > 0$, there exists a $\delta > 0$ so that $\left|x^2 - a^2\right| < \varepsilon$ whenever $0 < |x - a| < \delta$.

This is another problem where it is helpful to work backwards. Start with what want to show, and then see if we can perform some manipulations to get a condition which will guarantee that what we want to happen will happen.

The limit of x^2

1. Let's first notice that $\left|x^2 - a^2\right| = |x + a|\,|x - a|$.
2. We want $\left|x^2 - a^2\right| < \varepsilon$, but this would be the same as requiring $|x + a|\,|x - a| < \varepsilon$.
3. It's tempting to now write $|x - a| < \frac{\varepsilon}{|x+a|}$, but this presents a slight problem: our δ can depend on ε and on a, but it *can not* depend on x!
4. But let's suppose that we knew there was some number, call it C, so that $|x + a| < C$. Then we could write

$$|x - a| < \frac{\varepsilon}{C}$$

   and take $\delta = \frac{\varepsilon}{C}$.
5. The question is how do we find this C?

The limit of x^2

6. The trick to finding C is to force δ to be smaller than some given number. For example, if we could somehow force $\delta \leq 1$, then we would have

$$
\begin{aligned}
& |x-a| < \delta \\
\implies & |x-a| < 1 \\
\implies & -1 < x-a < 1 \\
\implies & a-1 < x < a+1 \\
\implies & 2a-1 < x+a < 2a+1
\end{aligned}
$$

7. If $x+a$ is positive, then we have $|x+a| < 2a+1$. If $x+a$ is negative, then we have $|x+a| < 1-2a$. Let's set $C = \max\{2a+1, 1-2a\}$ so that we can write $|x+a| < C$.

The limit of x^2

8. Now we want to take δ to be ε/c, but remember that we had to suppose $\delta < 1$ first. To force both of these conditions to happen, we'll set $\delta = \min\{1, \varepsilon/c\}$.

Remember, our goal is to find a δ that guarantees that $|x^2 - a^2| < \varepsilon$ if $0 < |x - a| < \delta$. Our claim is that $\delta = \min\{1, \varepsilon/c\}$ is the right choice, but we need to verify that this is indeed the case.

The limit of x^2

Putting everything together, we have the following:

- We want to show that if $\varepsilon > 0$, there exists a $\delta > 0$ so that $|x^2 - a^2| < \varepsilon$ whenever $0 < |x - a| < \delta$.
- Pick your favorite $\varepsilon > 0$.
- Set $C = \max\{2a + 1, 1 - 2a\}$, and let $\delta = \min\{1, \varepsilon/c\}$.
- Notice that $|x - a| < \delta$ implies $|x - a| < 1$ and $|x - a| < \varepsilon/c$.

$$
\begin{aligned}
& 0 < |x - a| < \delta \\
\implies & -\delta < x - a < \delta \\
\implies & -\delta + 2a < x + a < \delta + 2a \\
\implies & -1 + 2a < x + a < 1 + 2a \\
\implies & |x + a| < C
\end{aligned}
$$

The limit of x^2

- Now that we know $|x - a| < \delta$ and $|x + a| < C$, we have

$$\begin{aligned} |x+a|\,|x-a| &< C\delta \\ \implies |x^2 - a^2| &< C\delta \\ \implies |x^2 - a^2| &< C \cdot (\varepsilon/C) \\ \implies |x^2 - a^2| &< \varepsilon \end{aligned}$$

We have now shown $\lim_{x \to a} x^2 = a^2$.

Sum of limits

Let's now use this precise definition of limit to prove one of our limit laws from earlier in the week:

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Our goal is to show that for every $\varepsilon > 0$ there exists a $\delta > 0$ so that $|f(x) + g(x) - (L + M)| < \varepsilon$ whenever $0 < |x - a| < \delta$.

First notice that because we are assuming $\lim_{x \to a} f(x) = L$, for every $\varepsilon > 0$, there exists a $\delta_L > 0$ so that $|f(x) - L| < \varepsilon/2$ whenever $0 < |x - a| < \delta_L$.

Similarly, because we assume $\lim_{x \to a} g(x) = M$, we know that there exists a $\delta_M > 0$ so that $|g(x) - M| < \varepsilon/2$ whenever $0 < |x - a| < \delta_M$.

Sum of limits

We will need one arithmetic observation for the proof:

For any pair of real numbers, A and B, $|A+B| \leq |A| + |B|$. (This is sometimes called the *triangle inequality*.)

We can use the triangle inequality to rewrite the $|f(x)+g(x)-(L+M)|$ that we care about:

$$\begin{aligned}
|f(x)+g(x)-(L+M)| &= |f(x)+g(x)-L-M| \\
&= |f(x)-L+g(x)-M| \\
&\leq |f(x)-L| + |g(x)-M|
\end{aligned}$$

Sum of limits

This is what we have so far:

1. You pick an $\varepsilon > 0$.
2. There are two numbers, δ_L and δ_M, which guarantee that $|f(x) - L| < \varepsilon/2$ whenever $0 < |x - a| < \delta_L$; and $|g(x) - M| < \varepsilon/2$ whenever $0 < |x - a| < \delta_M$.
3. We want to find a single $\delta > 0$ so that $|f(x) + g(x) - (L + M)| < \varepsilon$ whenever $0 < |x - a| < \delta$.
4. We know that

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M|.$$

5. So, what we need to do is choose a δ that is less-than-or-equal-to both δ_L and δ_M.
6. Let's just take $\delta = \min\{\delta_L, \delta_M\}$!

Sum of limits

Putting it all together:

1. Let $\varepsilon > 0$.
2. Let $\delta_L > 0$ be the value so that

$$0 < |x - a| < \delta_L \implies |f(x) - L| < \varepsilon/2$$

3. Let $\delta_M > 0$ be the value so that

$$0 < |x - a| < \delta_M \implies |g(x) - M| < \varepsilon/2$$

4. Take $\delta = \min\{\delta_L, \delta_M\}$.
5. If $0 < |x - a| < \delta$, then

$$|f(x)+g(x)-(L+M)| \leq |f(x)-L|+|g(x)-M| < \varepsilon/2+\varepsilon/2 = \varepsilon.$$

We have shown, using the formal definition of a limit, that

$$\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

The precise definition of a right-hand limit

The formal definition of a right-hand limit is very similar to the definition for the two-sided limit. The only thing that changes is that we only care about values of x that are larger than a:

We write $\lim_{x \to a^+} f(x) = L$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta$.

Example of the definition of a right-hand limit

Example: Show $\lim_{x \to -1^+} \sqrt{x+1} = 0$.

Solution: Let $\varepsilon > 0$ and let $\delta = \varepsilon^2$. Suppose $0 < x - (-1) < \delta$. Then

$$0 < x - (-1) < \delta$$

$$\implies x + 1 < \delta$$

$$\implies x + 1 < \varepsilon^2$$

$$\implies \sqrt{x+1} < \varepsilon$$

$$\implies |f(x) - 0| < \varepsilon$$

The precise definition of a left-hand limit

We write $\lim_{x \to a^-} f(x) = L$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < a - x < \delta$.

Exercise for you: Use the precise definition of the left-hand limit to show that $\lim_{x \to 2^-} \left(\sqrt{2-x} + 1\right) = 1$.

The precise definition of an infinite limit

The definition of an infinite is perhaps the easiest of these precise definitions to understand.

We write $\lim_{x \to a} f(x) = \infty$ if for every $N > 0$ there exists a $\delta > 0$ so that $f(x) > N$ whenever $0 < |x - a| < \delta$.

What we're saying is that you can make $f(x)$ as big as you want (make it bigger than $N = 100$; bigger than $N = 1,000$; bigger than $N = 1,000,000$; etc.) by choosing x to be close enough (within δ-distance) of a.

The precise definition of an infinite limit

Let's show that $\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$.

Let $N > 0$. Let $\delta = {}^1\!/\!\sqrt{N}$. We then have the following:

$$
\begin{aligned}
& 0 < |x-1| < \delta \\
\implies & (x-1)^2 < \delta^2 \\
\implies & \frac{1}{\delta^2} < \frac{1}{(x-1)^2} \\
\implies & \frac{1}{\left({}^1\!/\!\sqrt{N}\right)^2} < \frac{1}{(x-1)^2} \\
\implies & \frac{1}{{}^1\!/\!N} < \frac{1}{(x-1)^2} \\
\implies & N < \frac{1}{(x-1)^2}.
\end{aligned}
$$

The precise definition of an infinite limit

We have shown that $\frac{1}{(x-1)^2}$ gets arbitrarily large (bigger than any $N > 0$ you choose) if we choose x close enough to 1. I.e.,
$\lim_{x \to 1} \dfrac{1}{(x-1)^2} = \infty$.

Exercise for you: How can you modify the definition of $\lim_{x \to a} f(x) = \infty$ to define $\lim_{x \to a} f(x) = -\infty$?

Homework

Due Monday, 9/1 :

1. Read about the *binomial theorem* on Wikipedia; know how to expand quantities like $(a+b)^7$ *without* FOIL-ing.
2. Read §2.3 and §2.4 in Stewart.

Due Tuesday, 9/2 :

1. Let $f(x) = 3x - 1$. What value of $\delta > 0$ guarantees that $|f(x) - 5| < 1/500$ if $0 < |x - 2| < \delta$?
2. Let $f(x) = 3x - 1$. What value of $\delta > 0$ guarantees that $|f(x) - 5| < \varepsilon$ if $0 < |x - 2| < \delta$?
3. Use the precise definition of the left-hand limit to show that $\lim_{x \to 2^-} \left(\sqrt{2-x} + 1\right) = 1$.
4. Modify the definition of $\lim\limits_{x \to a} f(x) = \infty$ to give a precise definition of $\lim\limits_{x \to a} f(x) = -\infty$.