CLEMSON UNIVERSITY

Math 1060

Lecture 6
Continuity

Outline

Summary of last lecture

- Described the ε-δ definition of limits:

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  We say $\lim_{x \to a} f(x) = L$ *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|f(x) - L| < \varepsilon$ *whenever* $0 < |x - a| < \delta$*.*

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- Described the ε-δ definitions for left- and right-hand limits.
- Described the ε-δ definition of an infinite limit.

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Last week we saw that some functions $f(x)$, like polynomials and rational functions, satisfied a very nice property:

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This property is nice because it makes calculating limits easy.

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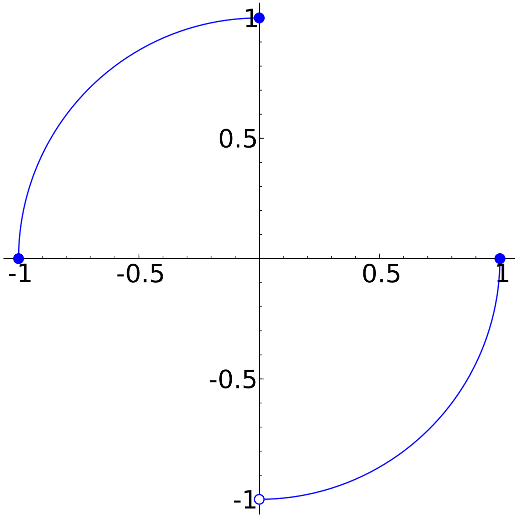
We have seen, however, that not all functions satisfy this nice property.



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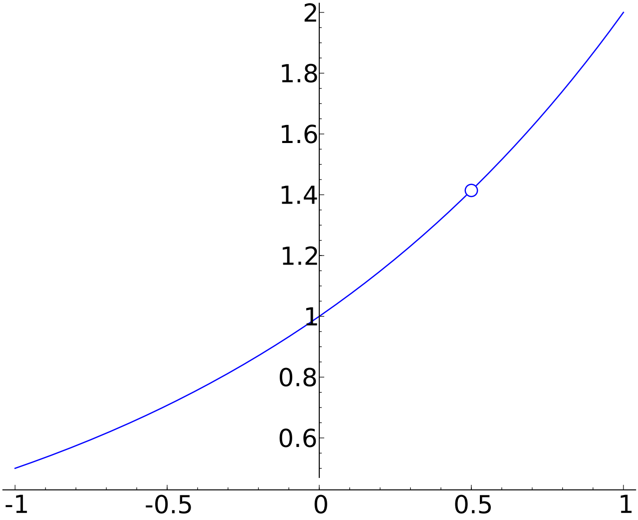

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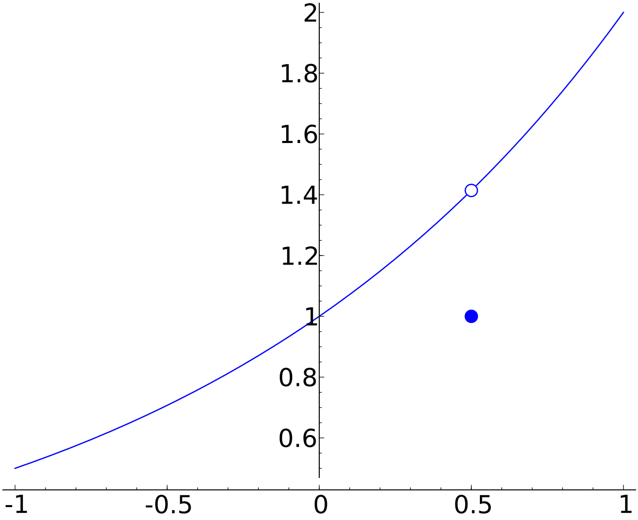

Continuity at a point

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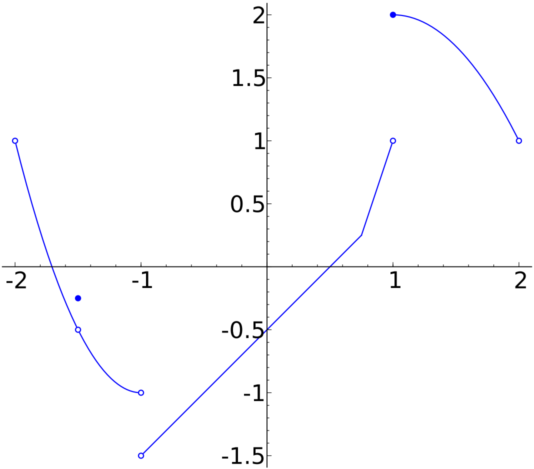
All that this definition is trying to convey is that $f(x)$ can't behave "wildly" very close to $x = a$: inputs very close to a give outputs very close to $f(a)$.

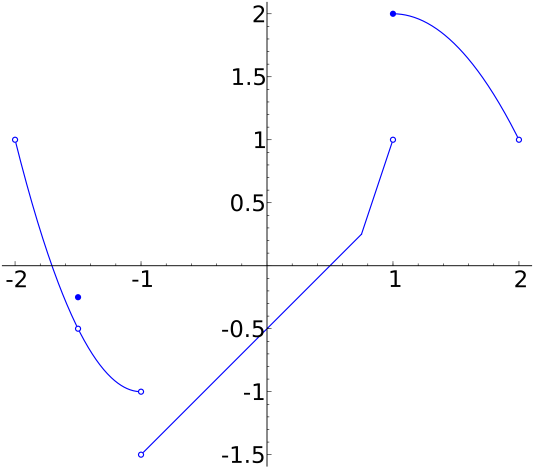
Continuity at a point

At which points below is the function continuous? Why is the function not continuous (aka *discontinuous*) at the other points?



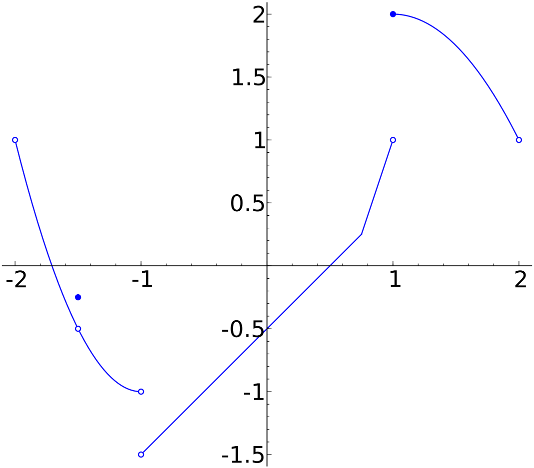
Continuity at a point

The domain of this function is $(-2,-1) \cup (-1,2)$.



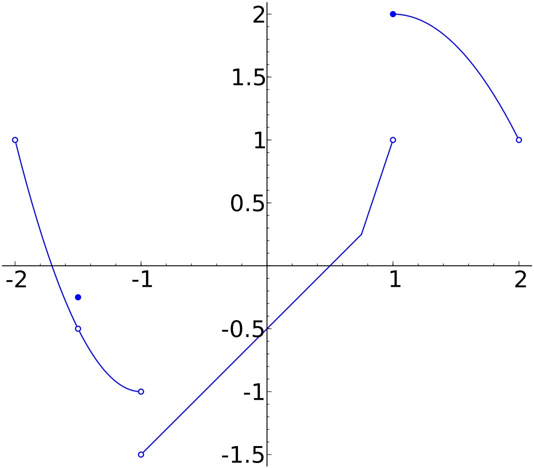
Continuity at a point

The function is not continuous at $x = -1$ because it is not defined there.



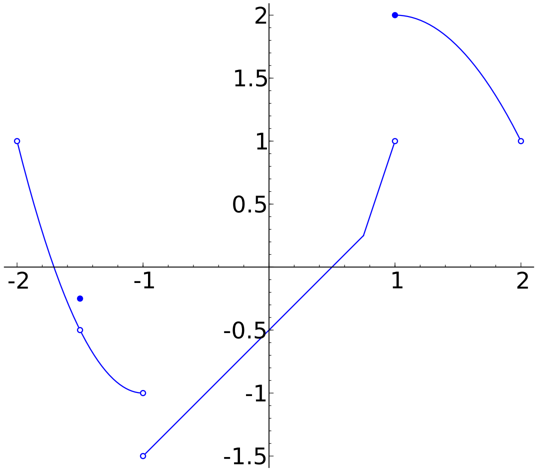
Continuity at a point

Similarly, the function is not continuous if $x \geq 2$ or $x \leq -2$ since it is not defined there.



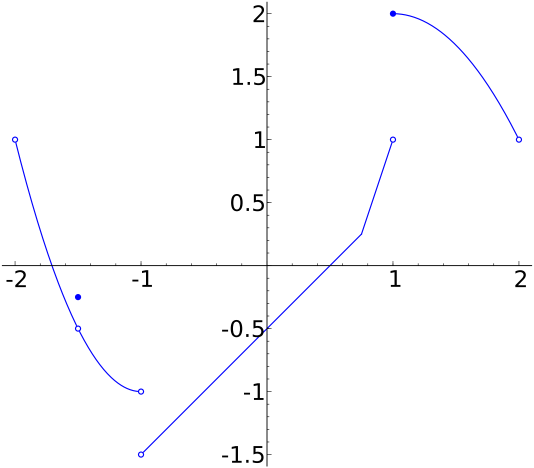
Continuity at a point

The function is defined at $x = 1$, but the limit does not exist; and so the function is not continuous at $x = 1$.



Continuity at a point

The function is defined at $x = -3/2$, and the limit $\lim_{x \to a} f(x)$ exists, but it does not equal $f(-3/2)$: the function is not continuous at $x = -3/2$.



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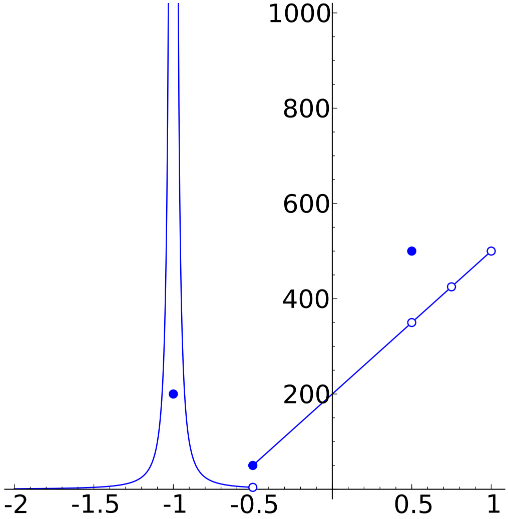
For the purposes of calculus, we like continuity at a point because it makes our lives a little easier. It is helpful, then, to know which points cause us trouble: sometimes we would like to know where a function is discontinuous.

Discontinuities

Example: Where is the following function discontinuous, and why is the function discontinuous at those points?



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This happens precisely when the denominator is zero. Factoring the denominator,

$$x^2 + 2x = x(x+2),$$

we see the function is discontinuous at $x = 0$ and at $x = -2$ because that is where the function is undefined.

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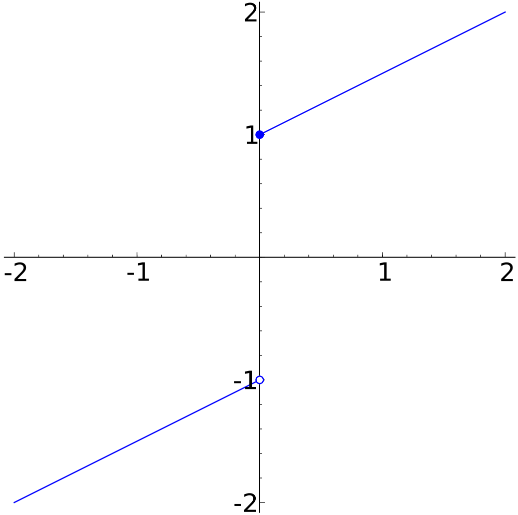
Discontinuities can be classified as falling into a few different categories:

- Jump discontinuities
- Infinite discontinuities
- Removable discontinuities

Each of these three types corresponds to one of the three conditions in the definition of continuity at a point failing.

Jump discontinuities

A *jump discontinuity* occurs when the graph of the function has a "break" in it.



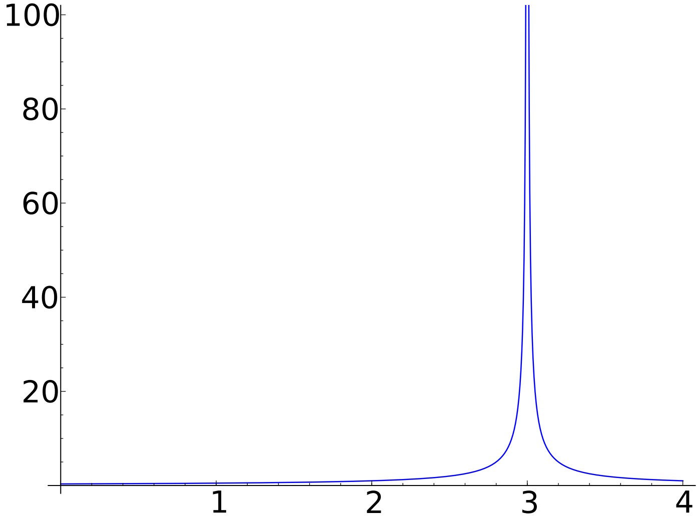
Jump discontinuities

More formally, we say that f has a *jump discontinuity* at $x = a$ if both the left- and right-hand limits of f at a exist and are finite, but are not equal.

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x).$$

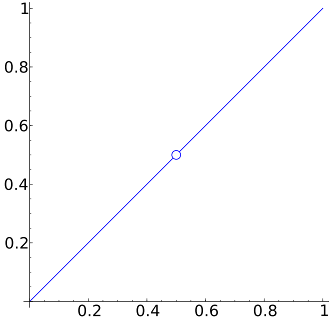
Infinite discontinuities

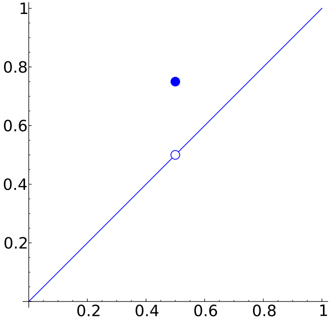
An *infinite discontinuity* occurs when the limit of the function at $x = a$ equals $\pm\infty$.



Removable discontinuities

We say f has a *removable* discontinuity at $x = a$ if $\lim_{x \to a} f(x)$ exists, but does not equal $f(a)$. This includes both the case when a is in the domain of f, and the case when it is not.





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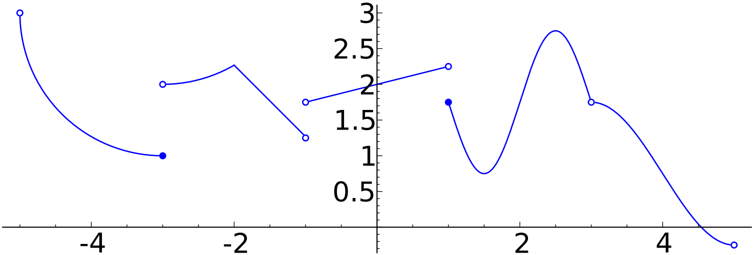
Continuity from the left or right

Example: Where is the function graphed below continuous from the left? Where is it continuous from the right? Where it is continuous?



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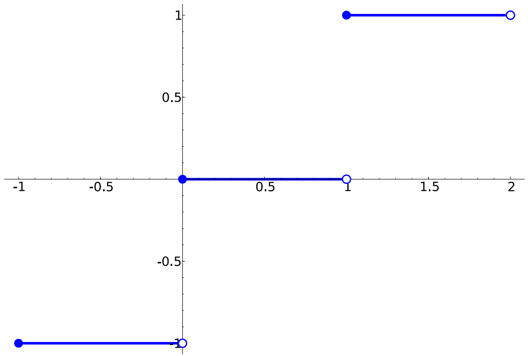
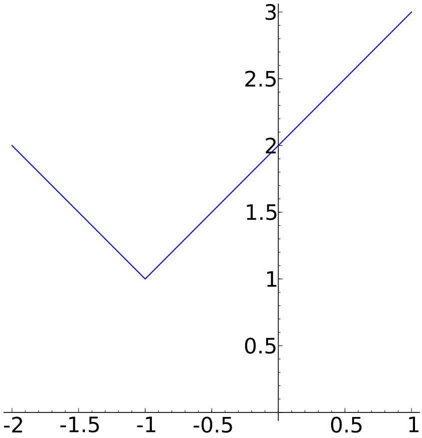


This function is left-continuous everywhere, but is not right-continuous at any integer. The integers are jump discontinuities of the function.

Continuity from the left or right

Notice that a function is continuous at $x = a$ if and only if it is both left- and right-continuous at $x = a$.



This function is continuous at $x = -1$ because it is both left- and right-continuous at $x = -1$.

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- **continuous at a point**: $\lim_{x \to a} f(x) = f(a)$
- **continuous**: the *continuous at a point* definition applies at every point where the function is defined.

The intuition behind continuity

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Throughout your calculus careers, and in any applications of calculus (physics, engineering, computer science, ...), continuity is a very desirable property, and sometimes essential assumption, of most of the functions you will come in contact with.

Continuity

Luckily, lots of commonly used functions are continuous:

Theorem

All of the following types of functions are continuous:

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Thus, for each of these types of functions,

$$\lim_{x \to a} f(x) = f(a)$$

as long as $f(a)$ is defined.

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Examples: All of the following functions are continuous because they are "built" from continuous functions:

1. $x^2 + \sin(x) - \frac{\sqrt{x}\cos(x)}{e^x}$
2. $\text{arcsec}(x) - \ln(x)$
3. $\frac{\sqrt[3]{x}+6x^3}{\tan(x)-4x^5+x}$

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Corollary

If f and g are continuous functions and a is in the domain of $f \circ g$ (so $g(a)$ is defined, and $f(g(a))$ is defined as well, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(g(a)).$$

Composition of continuous functions

More generally, suppose f is continuous at $x = b$ and suppose g is a function with the property that $\lim_{x \to a} g(x) = b$. Then

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Example: Determine the limit $\lim_{x\to 0} \ln\left(\frac{2\sin(x)}{x}\right)$.

$$\lim_{x\to 0} \ln\left(\frac{2\sin(x)}{x}\right) = \ln\left(\lim_{x\to 0} \frac{2\sin(x)}{x}\right)$$

Composition of continuous functions

More generally, suppose f is continuous at $x = b$ and suppose g is a function with the property that $\lim_{x \to a} g(x) = b$. Then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

Example: Determine the limit $\lim_{x \to 0} \ln\left(\frac{2\sin(x)}{x}\right)$.

$$\begin{aligned}\lim_{x \to 0} \ln\left(\frac{2\sin(x)}{x}\right) &= \ln\left(\lim_{x \to 0} \frac{2\sin(x)}{x}\right) \\ &= \ln\left(2 \cdot \lim_{x \to 0} \frac{\sin(x)}{x}\right)\end{aligned}$$

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The intermediate value theorem

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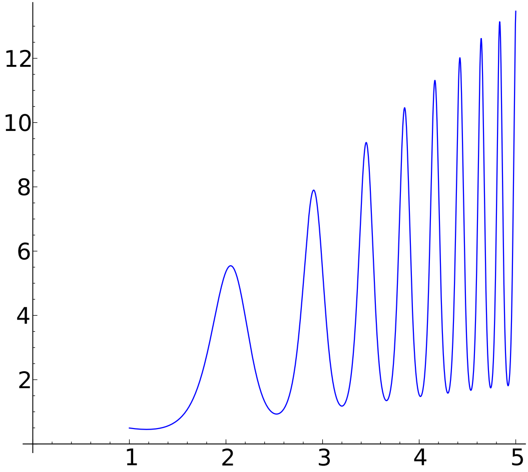
If $f(x)$ is defined for every $x \in [a, b]$, then the graph $y = f(x)$ has no holes or jumps in it: it makes one "continuous" curve.

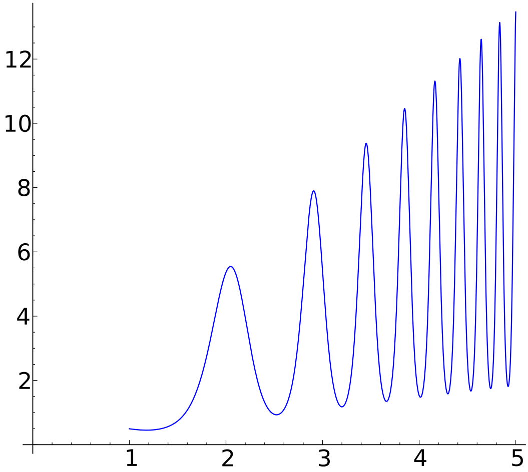
The intermediate value theorem

This means the graph hits every y-value between $f(a)$ and $f(b)$.



This is all that the intermediate value theorem says.

The intermediate value theorem



In this picture $f(1) = 0.5$ and $f(5) = 13.5$. The intermediate value theorem says that this function hits each value between 0.5 and 13.5 for some x between 1 and 5.

The intermediate value theorem

Theorem

Let f be a continuous function defined on the interval $[a, b]$. For every D between $f(a)$ and $f(b)$, there exists a $x \in [a, b]$ such that $f(x) = D$.

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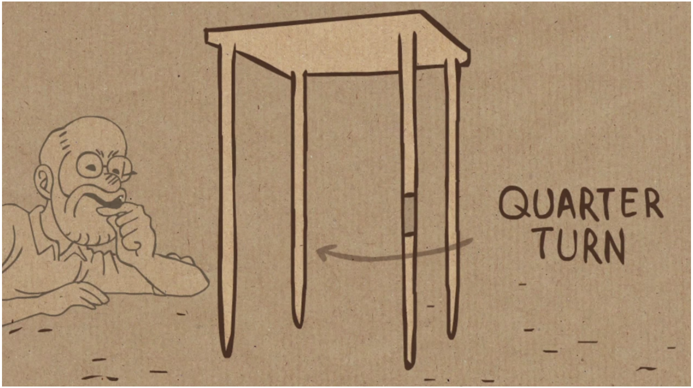
The intermediate value theorem seems like a simple observation, but it implies some very interesting, and non-obvious, facts.

The intermediate value theorem

Application: If you walk from the library bridge, around the reflection pond, and then over to Starbucks in the University Union, at some point along the path your elevation will be exactly the same as the elevation of your starting point on the library bridge.

The intermediate value theorem

Application: Suppose you have a table whose legs all have the same length, but the table is placed on uneven ground making the table wobbly. You can make the table stable by rotating it.



At most you only need to turn the table 90°.

Pathological examples

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Sometimes in mathematics examples that are very strange and atypical are called *pathological*; these are usually very counter-intuitive examples that show you how your intuition can sometimes lead you astray.

We will consider three such examples which should convince you there are some very strange functions out there that don't behave like anything you've ever seen before. (I.e., all of your intuition goes out the window when considering such functions.)

A nowhere continuous function

In the examples of functions we've seen thus far, the functions were continuous at "most" points: there were only finitely-many discontinuities.

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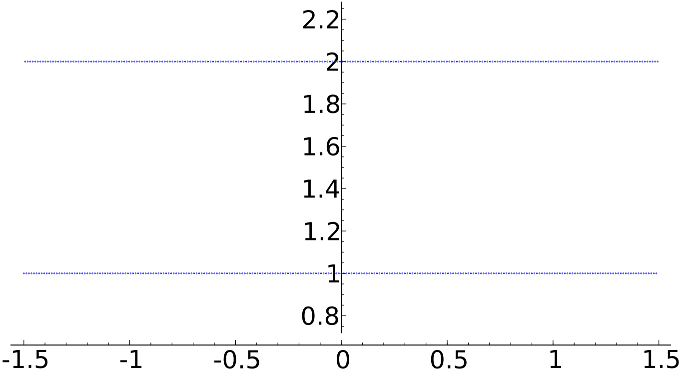
Here's one example of such a function:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 2 & \text{if } x \text{ is rational} \end{cases}$$

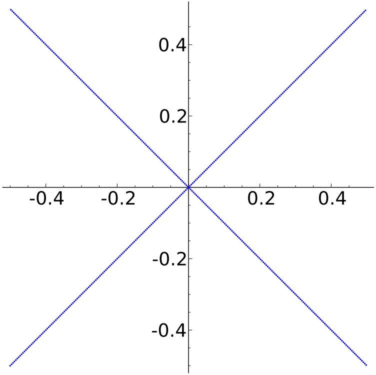
A nowhere continuous function

This function is not continuous because for every a, every L, every $0 < \varepsilon < 1$ and for every $\delta > 0$, there will be points satisfying $0 < |x - a| < \delta$ but $|f(x) - L| > \varepsilon$.



A function which is continuous at exactly one point

The following function is continuous at $x = 0$, but that is the only point where it's continuous:



$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ -x & \text{if } x \text{ is rational} \end{cases}$$

A continuous function with no smooth components

We've seen that the graph of a continuous function can have sharp edges and corners, but in the examples thus far these corners have been few and far between?

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A continuous function with no smooth components

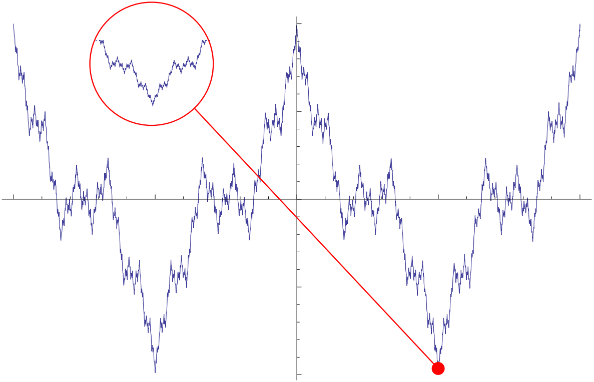
We've seen that the graph of a continuous function can have sharp edges and corners, but in the examples thus far these corners have been few and far between?

Can the graph of such a function have infinitely many such corners?

Can the graph of such a function consist *entirely* of corners?

A continuous function with no smooth components

The answer to both questions is yes: there are bizarre functions with weird properties like this. One example is the *Weierstrass function* plotted below.



Homework

Due Monday, September 8 :

- Read §2.5 in Stewart.
- Homework set listed on the website (will appear online late Wednesday / early Thursday)