Math 1060

Lecture 7
Limits at Infinity

Outline

Summary of last lecture

- Defined continuity at a point:
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  2. $\lim_{x \to a} f(x)$ exists, and
  3. $\lim_{x \to a} f(x) = f(a)$.

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- Basic idea of continuity: small changes in input result in small changes in output.
- Talked about three particular types of discontinuities:
  1. jump discontinuities,
  2. infinite discontinuities, and
  3. removable discontinuities.
- Listed several types of common continuous functions: polynomials, rational functions, roots, trig functions, inverse trig functions, exponentials, and logarithms.

Summary of last lecture

- Mentioned that if f and g were continuous, then so were $f+g$, $f-g$, fg, ${}^{f}/_{g}$, and $f \circ g$.

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- Described the intermediate value theorem and mentioned a few interesting applications.
- Looked at a few "pathological examples" to show that your intuition about what should and should not be called "continuous" can misled you.

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Now we want to discuss another notion of limit. Instead of considering values of x that get closer and closer to a, we will ask about what happens to the outputs $f(x)$ as x get arbitrarily large.

To denote that x gets larger and larger without bound we say that *x goes to infinity*, and the value that $f(x)$ approaches as x goes to infinity (if it approaches any particular value) is called the *limit of $f(x)$ at ∞* and denoted $\lim_{x \to \infty} f(x)$.

Limits at infinity

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Consider the function $f(x) = \frac{x}{x+1}$, and consider what happens as we plug in larger and larger values for x:

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Consider the function $f(x) = \frac{x}{x+1}$, and consider what happens as we plug in larger and larger values for x:

x	$f(x)$
1	$\frac{1}{2} = 0.5$
10	$\frac{10}{11} \approx 0.909090909...$
100	$\frac{100}{101} \approx 0.99099099...$
$1,000$	$\frac{1{,}000}{1{,}001} \approx 0.999099909...$
$10,000$	$\frac{10{,}000}{10{,}001} \approx 0.999909999...$
$\vdots$	$\vdots$

Limits at infinity

We see that as x gets larger and larger, $\frac{x}{x+1}$ gets closer and closer to 1. We will never be able to make $\frac{x}{x+1}$ equal 1, but we can make $\frac{x}{x+1}$ as close to 1 as we'd like by picking large enough values of x.

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If you wanted $\frac{x}{x+1}$ to be within distance $\frac{1}{1{,}000{,}000}$ of 1 (i.e., $\left|\frac{x}{x+1} - 1\right| < \frac{1}{1{,}000{,}000}$), then you need to choose $x > 999,999$.

Limits at infinity

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Limits at infinity

$$
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\Longrightarrow\ & x + 1 > 1,000,000 \\
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\implies &\frac{x+1-x}{x+1} < \frac{1}{1,000,000}
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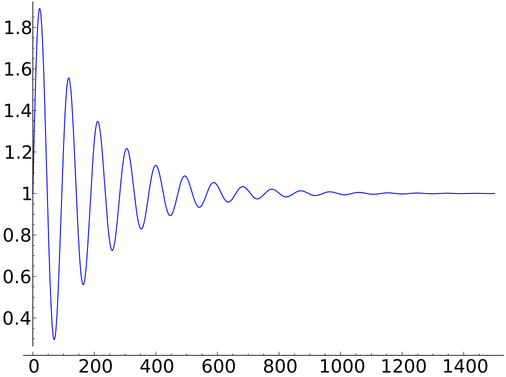
\end{aligned}
$$

(Notice that $|a - b| = |b - a|$.)

Limits at infinity

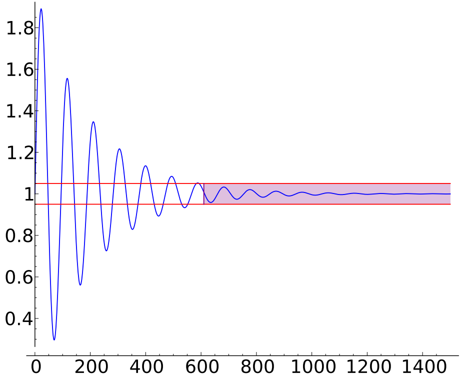
As another example, consider the function plotted below.



Here $\lim_{x\to\infty} f(x) = 1$. Notice the curve crosses the line $y = 1$ infinitely many times, jumping from a little above the line to a little below the line. The distance between these jumps decreases, though, and so the eventually stays within $^1/_{1,000}$ of $y = 1$; then within $^1/_{100,000}$ of $y = 1$, and so on.

Limits at infinity

The precise definition of the limit at infinity is as follows:
We say $\lim_{x \to \infty} f(x) = L$ if for every $\varepsilon > 0$ there exists an N such that $|f(x) - L| < \varepsilon$ whenever $x > N$.



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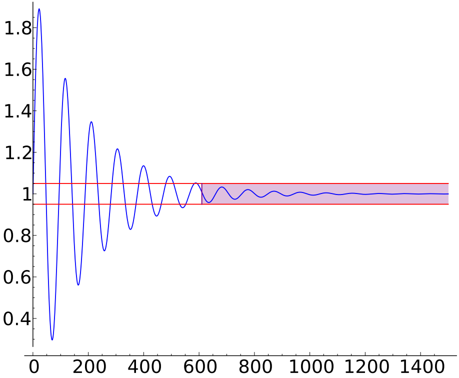
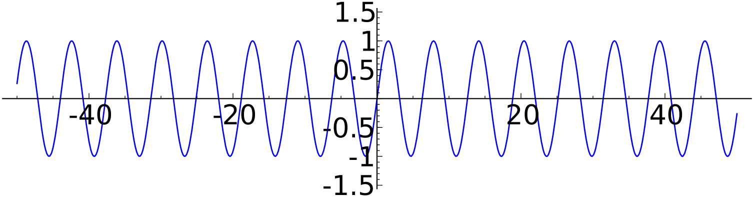


Implicit in this definition is the assumption that $f(x)$ is defined in the interval (N, ∞).

Limits at infinity

Of course, not all functions have limits at infinity.



As x gets larger and larger $\sin(x)$ just continues to oscillate between -1 and 1, never getting close to any one particular value.

Limits at infinity

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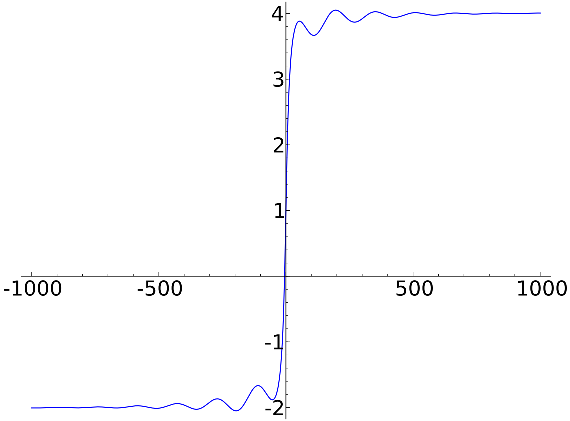
Of course, this only makes sense if $f(x)$ is defined in the interval $(-\infty, N)$.

Limits at infinity



$$\lim_{x \to -\infty} f(x) = -2 \qquad \lim_{x \to \infty} f(x) = 4$$

Limit laws

All of our previous limit laws for limits at a point also apply for limits at infinity.

Theorem

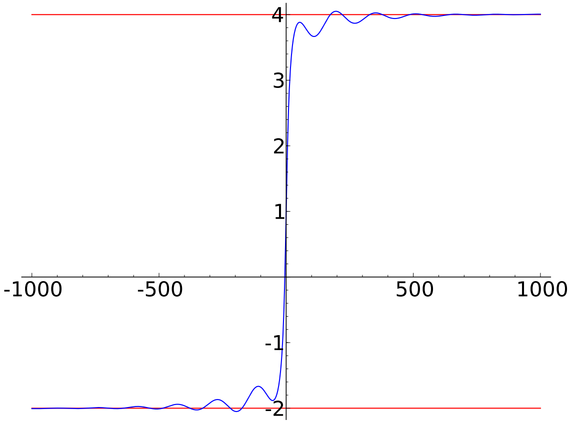
Suppose f and g are two functions with $\lim_{x\to\infty} f(x) = L$ *and* $\lim_{x\to\infty} g(x) = M$. *Then*

1. $\lim_{x\to\infty} (f(x) + g(x)) = L + M$
2. $\lim_{x\to\infty} (f(x) - g(x)) = L - M$
3. $\lim_{x\to\infty} (f(x) \cdot g(x)) = L \cdot M$
4. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$ *if* $M \neq 0$
5. $\lim_{x\to\infty} \sqrt[n]{f(x)} = \sqrt[n]{L}$

These laws still apply if we instead considered limits at $-\infty$.

Horizontal asymptotes

When $f(x)$ has a limit at ∞ or $-\infty$, we say that $f(x)$ has a horizontal asymptote.



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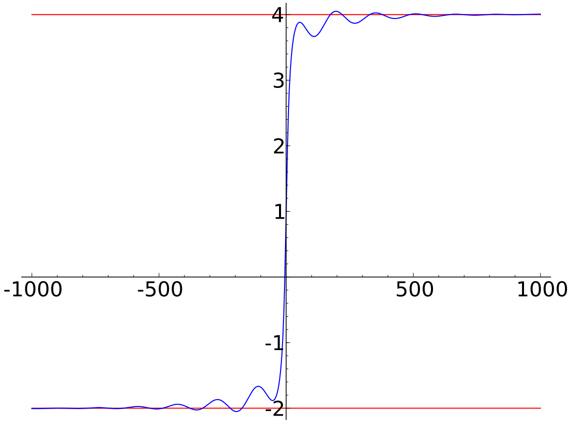
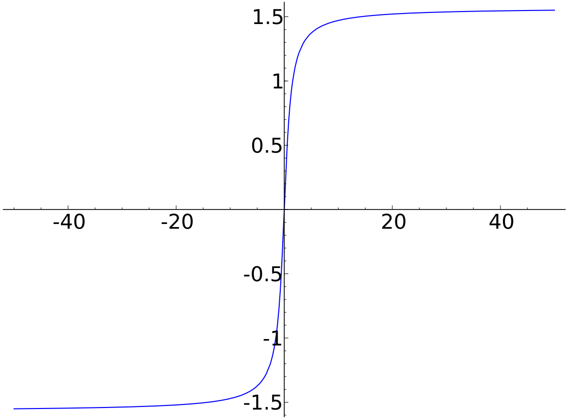


Caution: If someone told you that the graph $y = f(x)$ can never touch or cross its horizontal asymptote, then they are a liar and not to be trusted.

Horizontal asymptotes

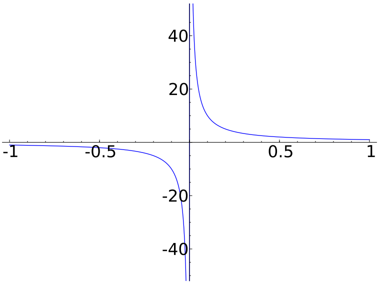
More precisely, we say that the line $y = L$ is a *horizontal asymptote* of $f(x)$ if $\lim_{x \to \pm\infty} f(x) = L$.



$$\lim_{x \to -\infty} \arctan(x) = -\pi/2 \qquad \lim_{x \to \infty} \arctan(x) = \pi/2$$

Horizontal asymptotes

An important example: $^1/_x$.



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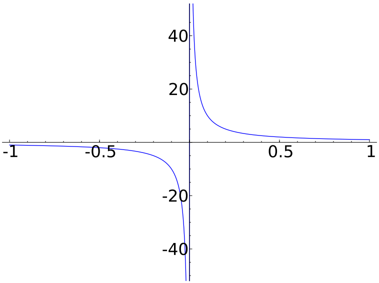

$$\lim_{x\to\infty}\frac{1}{x} = \lim_{x\to-\infty}\frac{1}{x} = 0.$$

This fact, $\lim_{x\to\pm\infty}\frac{1}{x} = 0$ is very helpful. We can't "plug in ∞" in when calculating limits at infinity, so it is helpful to know some basic examples we can compare more complicated functions to.

Useful observation

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If x is very large, then $\frac{1}{x^n}$ is very small for any natural number n. In particular,

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This simple observation gives us a tool we can use to rewrite limits of complicated functions as limits of simpler things which we understand.

Horizontal asymptotes

Example: Determine the horizontal asymptotes of

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Notice that any (non-zero) value divided by itself always equals 1, and multiplying any quantity by 1 does not change that quantity.

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This is a basic technique for solving these sorts of problems!

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$$\lim_{x\to\infty} \frac{3x^2 - 2x + 5}{x^2 - 4} = \lim_{x\to\infty} \frac{3x^2 - 2x + 5}{x^2 - 4} \cdot \frac{1/x^2}{1/x^2}$$

Horizontal asymptotes

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Horizontal asymptotes

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$$= \frac{3 - 0 + 0}{1 - 0}$$

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Notice the numerator and denominator of our original function both go th ∞, but the limit of their quotient is 3!

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Notice the numerator and denominator of our original function both go th ∞, but the limit of their quotient is 3!

Thus you can not write $\frac{\infty}{\infty} = 1$!

Horizontal asymptotes

In the previous problem we did some algebra to rewrite the function we were taking the limit of,

$$\frac{3x^2 - 2x + 5}{x^2 - 4} = \frac{3 - \frac{2}{x} + \frac{5}{x^2}}{1 - \frac{4}{x^2}}.$$

After doing this we could apply our limit laws to actually determine the limit.

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After doing this we could apply our limit laws to actually determine the limit.

Question: Why was multiplying by $\frac{1/x^2}{1/x^2}$ the right thing to do?

Answer: Because this made the denominator take the form $c_1 + c_2/x^n + \cdots + c_m/x^m$ and the limit of this is just c_1.

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$$= \lim_{x \to \pm\infty} \frac{6/x + 5/x^2 - 7/x^3}{11 - 1/x + 12/x^3 + 4/x^4}$$

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Example: Determine the horizontal asymptotes of

$$f(x) = \frac{6x^3 + 5x^2 - 7x}{11x^4 - x^3 + 12x + 4}$$

$$\lim_{x\to\pm\infty} \frac{6x^3 + 5x^2 - 7x}{11x^4 - x^3 + 12x + 4} = \lim_{x\to\pm\infty} \frac{6x^3 + 5x^2 - 7x}{11x^4 - x^3 + 12x + 4} \cdot \frac{1/x^4}{1/x^4}$$

$$= \lim_{x\to\pm\infty} \frac{6/x + 5/x^2 - 7/x^3}{11 - 1/x + 12/x^3 + 4/x^4}$$

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&= 0
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So the horizontal asymptotes are $y = 0$.

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Horizontal asymptotes

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Horizontal asymptotes

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The horizontal asymptote of $\frac{6x^2+4}{\sqrt[4]{5x^8-2}}$ is thus $y = {}^{6}\!/\!_{\sqrt[4]{5}}$.

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For example: if you say $\lim_{x\to\infty} \frac{3x^4-6x^2}{2x^4+5} = \frac{3}{2}$ without any justification, or simply saying this is the limit "because the numerator and denominator have the same degree," *then you will not receive any credit!*

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Instead, you must justify your answer by doing the algebra to rewrite the function as something simpler that you can actually take the limit of.

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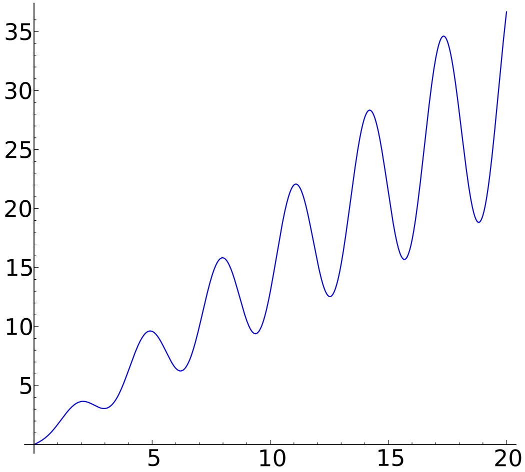
If you want to use these tricks to double-check your answers, then that's fine, but you can't use them as justification for your answer on a quiz or test.

Infinite limits at infinity

If $f(x)$ grows without bound as x goes to ∞, then we write $\lim_{x\to\infty} f(x) = \infty$.



Infinite limits at infinity

The precise definition of having an infinite limit at infinity is the following: we say that $\lim_{x \to \infty} f(x) = \infty$ if for every $M > 0$ there exists an N such that $f(x) > M$ whenever $x > N$.

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The definitions of $\lim_{x \to \infty} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = \infty$, and $\lim_{x \to -\infty} f(x) = -\infty$ are comparable.

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And so we can make $\ln(x)$ arbitrarily large.

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Example: Determine $\lim_{x\to\infty} \frac{3x^5+6x^2+2}{2x^3-7}$.

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Arithmetic with infinity

Finally, let's consider an example to show that you have to be very careful when dealing with infinite limits: ∞ does not obey the normal laws of arithmetic.

$$\lim_{x\to\infty}\left(\sqrt{x^2-2}-\sqrt{x^2+x}\right)$$

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$$=\lim_{x\to\infty}\frac{x^2-2-x^2-x}{\sqrt{x^2-2}+\sqrt{x^2+x}}$$

$$=\lim_{x\to\infty}\frac{-2-x}{\sqrt{x^2-2}+\sqrt{x^2+x}}$$

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Thus you can not write $\infty - \infty = 0$!

Homework

Due Monday, September 8 :

- Read §2.5 in Stewart.
- Homework set listed on the web site.

Due Wednesday, September 10 :

- Read §2.6 in Stewart.
- Complete 33% of ALEKS, or risk receiving nagging emails reminding you to work on ALEKS!

There will be a quiz on Wednesday, September 10. The quiz will focus on continuity, limits at infinity, and horizontal asymptotes; however, earlier material (e.g., the ε-δ definitions) may also make an appearance.