CLEMSON
UNIVERSITY

Math 1060

Lecture 7
Limits at Infinity

Outline

Summary of last lecture

- Defined continuity at a point:
  1. $f(a)$ is defined,
  2. $\lim_{x \to a} f(x)$ exists, and
  3. $\lim_{x \to a} f(x) = f(a)$.
- Defined continuity of functions: We say f is *continuous* if it is continuous at every point in its domain.
- Basic idea of continuity: small changes in input result in small changes in output.
- Talked about three particular types of discontinuities:
  1. jump discontinuities,
  2. infinite discontinuities, and
  3. removable discontinuities.
- Listed several types of common continuous functions: polynomials, rational functions, roots, trig functions, inverse trig functions, exponentials, and logarithms.

Summary of last lecture

- Mentioned that if f and g were continuous, then so were $f+g$, $f-g$, fg, ${}^{f}/_{g}$, and $f \circ g$.
- Described the intermediate value theorem and mentioned a few interesting applications.
- Looked at a few "pathological examples" to show that your intuition about what should and should not be called "continuous" can misled you.

Limits at infinity

Previously when we discussed limits we took the limit of a function $f(x)$ as x approached some particular value which we usually called a.

Now we want to discuss another notion of limit. Instead of considering values of x that get closer and closer to a, we will ask about what happens to the outputs $f(x)$ as x get arbitrarily large.

To denote that x gets larger and larger without bound we say that x *goes to infinity*, and the value that $f(x)$ approaches as x goes to infinity (if it approaches any particular value) is called the *limit of* $f(x)$ *at* ∞ and denoted $\lim\limits_{x\to\infty} f(x)$.

Limits at infinity

When we write $\lim_{x\to\infty} f(x) = L$, what we mean is that $f(x)$ gets *arbitrarily close* to L as x gets bigger and bigger.

Consider the function $f(x) = \frac{x}{x+1}$, and consider what happens as we plug in larger and larger values for x:

x	$f(x)$
1	$\frac{1}{2} = 0.5$
10	$\frac{10}{11} \approx 0.909090909...$
100	$\frac{100}{101} \approx 0.99099099...$
$1,000$	$\frac{1,000}{1,001} \approx 0.999099909...$
$10,000$	$\frac{10,000}{10,001} \approx 0.999909999...$
$\vdots$	$\vdots$

Limits at infinity

We see that as x gets larger and larger, $\frac{x}{x+1}$ gets closer and closer to 1. We will never be able to make $\frac{x}{x+1}$ equal 1, but we can make $\frac{x}{x+1}$ as close to 1 as we'd like by picking large enough values of x.

If you wanted $\frac{x}{x+1}$ to be within distance $\frac{1}{1{,}000{,}000}$ of 1 (i.e., $\left|\frac{x}{x+1} - 1\right| < \frac{1}{1{,}000{,}000}$), then you need to choose $x > 999{,}999$.

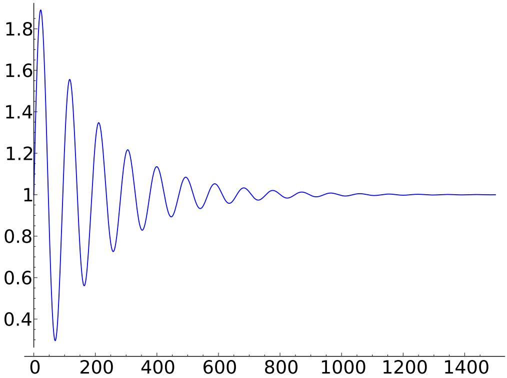
Limits at infinity

$$
\begin{aligned}
& x > 999,999 \\
\Longrightarrow\ & x + 1 > 1,000,000 \\
\Longrightarrow\ & \frac{1}{1,000,000} > \frac{1}{x+1} \\
\Longrightarrow\ & \frac{1}{x+1} < \frac{1}{1,000,000} \\
\Longrightarrow\ & \frac{x+1-x}{x+1} < \frac{1}{1,000,000} \\
\Longrightarrow\ & \frac{x+1}{x+1} - \frac{x}{x+1} < \frac{1}{1,000,000} \\
\Longrightarrow\ & \left|1 - \frac{x}{x+1}\right| < \frac{1}{1,000,000}
\end{aligned}
$$

(Notice that $|a - b| = |b - a|$.)

Limits at infinity

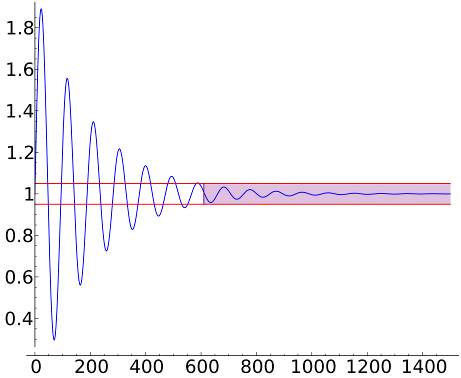
As another example, consider the function plotted below.



Here $\lim_{x\to\infty} f(x) = 1$. Notice the curve crosses the line $y = 1$ infinitely many times, jumping from a little above the line to a little below the line. The distance between these jumps decreases, though, and so the eventually stays within $1/1,000$ of $y = 1$; then within $1/100,000$ of $y = 1$, and so on.

Limits at infinity

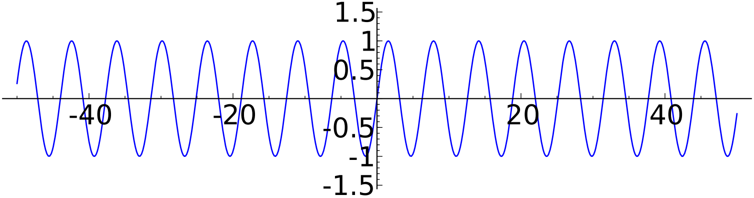
The precise definition of the limit at infinity is as follows:
We say $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists an N such that $|f(x) - L| < \varepsilon$ whenever $x > N$.



Implicit in this definition is the assumption that $f(x)$ is defined in the interval (N, ∞).

Limits at infinity

Of course, not all functions have limits at infinity.



As x gets larger and larger $\sin(x)$ just continues to oscillate between -1 and 1, never getting close to any one particular value.

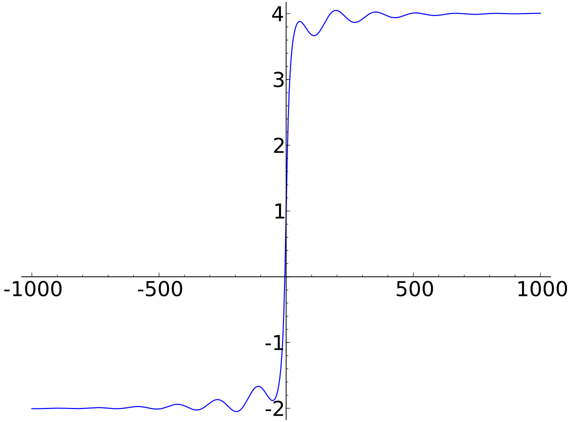
Limits at infinity

We can similarly talk about limits as x goes to $-\infty$, but the idea is exactly the same.

We say $\lim_{x \to -\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists a N such that $|f(x) - L| < \varepsilon$ whenever $x < N$.

Of course, this only makes sense if $f(x)$ is defined in the interval $(-\infty, N)$.

Limits at infinity



$$\lim_{x \to -\infty} f(x) = -2 \qquad \lim_{x \to \infty} f(x) = 4$$

Limit laws

All of our previous limit laws for limits at a point also apply for limits at infinity.

Theorem

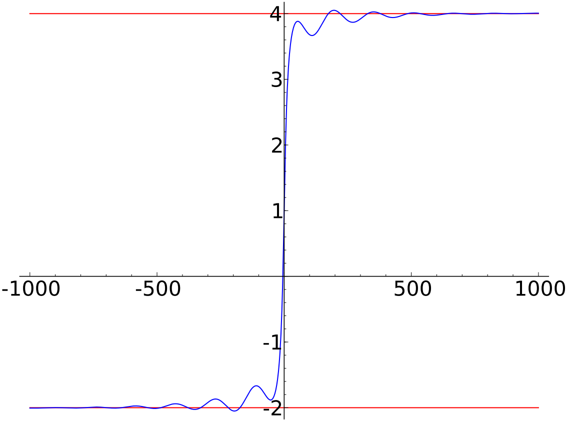
Suppose f and g are two functions with $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = M$. Then

1. $\lim_{x\to\infty} (f(x) + g(x)) = L + M$
2. $\lim_{x\to\infty} (f(x) - g(x)) = L - M$
3. $\lim_{x\to\infty} (f(x) \cdot g(x)) = L \cdot M$
4. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$ *if* $M \neq 0$
5. $\lim_{x\to\infty} \sqrt[n]{f(x)} = \sqrt[n]{L}$

These laws still apply if we instead considered limits at $-\infty$.

Horizontal asymptotes

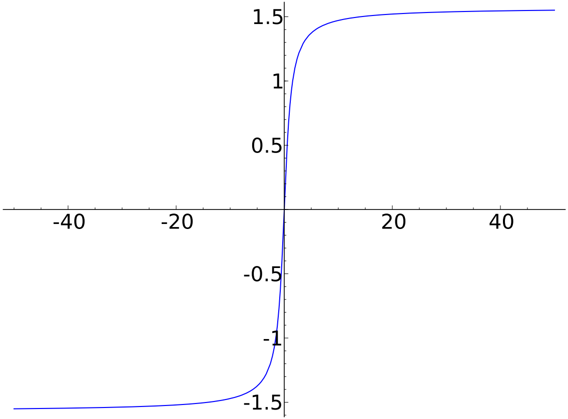
When $f(x)$ has a limit at ∞ or $-\infty$, we say that $f(x)$ has a horizontal asymptote.



Caution: If someone told you that the graph $y = f(x)$ can never touch or cross its horizontal asymptote, then they are a liar and not to be trusted.

Horizontal asymptotes

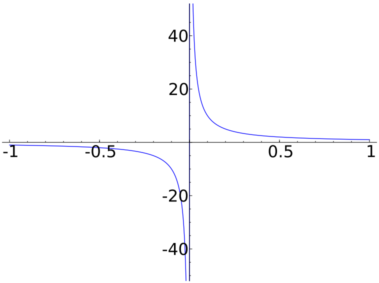
More precisely, we say that the line $y = L$ is a *horizontal asymptote* of $f(x)$ if $\lim_{x \to \pm\infty} f(x) = L$.



$$\lim_{x \to -\infty} \arctan(x) = -\pi/2 \qquad \lim_{x \to \infty} \arctan(x) = \pi/2$$

Horizontal asymptotes

An important example: $1/x$.



$$\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0.$$

This fact, $\lim_{x\to\pm\infty} \frac{1}{x} = 0$ is very helpful. We can't "plug in ∞" in when calculating limits at infinity, so it is helpful to know some basic examples we can compare more complicated functions to.

Useful observation

The following observation forms the basis for solving may horizontal asymptote / limit at infinity problems:

If x is very large, then $\frac{1}{x^n}$ is very small for any natural number n. In particular,

$$\lim_{x \to \pm\infty} \frac{1}{x^n} = 0.$$

This simple observation gives us a tool we can use to rewrite limits of complicated functions as limits of simpler things which we understand.

Horizontal asymptotes

Example: Determine the horizontal asymptotes of

$$f(x) = \frac{3x^2 - 2x + 5}{x^2 - 4}$$

We need to determine $\lim_{x\to\pm\infty} f(x)$. We can't plug in $\pm\infty$ for x, so we instead do some algebra to turn $f(x)$ into something we can understand.

Notice that any (non-zero) value divided by itself always equals 1, and multiplying any quantity by 1 does not change that quantity.

This is a basic technique for solving these sorts of problems!

Horizontal asymptotes

$$\lim_{x\to\infty} \frac{3x^2 - 2x + 5}{x^2 - 4} = \lim_{x\to\infty} \frac{3x^2 - 2x + 5}{x^2 - 4} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{3x^2-2x+5}{x^2}\right)}{\left(\frac{x^2-4}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{3 - \frac{2}{x} + \frac{5}{x^2}}{1 - \frac{4}{x^2}}$$

$$= \frac{\lim\limits_{x\to\infty} \left(3 - \frac{2}{x} + \frac{5}{x^2}\right)}{\lim\limits_{x\to\infty} \left(1 - \frac{4}{x^2}\right)}$$

Horizontal asymptotes

$$\ldots = \frac{\lim_{x\to\infty}\left(3 - \frac{2}{x} + \frac{5}{x^2}\right)}{\lim_{x\to\infty}\left(1 - \frac{4}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} \frac{2}{x} + \lim_{x\to\infty} \frac{5}{x^2}}{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{4}{x^2}}$$

$$= \frac{3 - 0 + 0}{1 - 0}$$

$$= 3$$

Notice the numerator and denominator of our original function both go th ∞, but the limit of their quotient is 3!

Thus you can not write $\frac{\infty}{\infty} = 1$!

Horizontal asymptotes

In the previous problem we did some algebra to rewrite the function we were taking the limit of,

$$\frac{3x^2 - 2x + 5}{x^2 - 4} = \frac{3 - \frac{2}{x} + \frac{5}{x^2}}{1 - \frac{4}{x^2}}.$$

After doing this we could apply our limit laws to actually determine the limit.

Question: Why was multiplying by $\frac{1/x^2}{1/x^2}$ the right thing to do?

Answer: Because this made the denominator take the form $c_1 + {c_2}/{x^n} + \cdots + {c_m}/{x^m}$ and the limit of this is just c_1.

Horizontal asymptotes

Example: Determine the horizontal asymptotes of

$$f(x) = \frac{6x^3 + 5x^2 - 7x}{11x^4 - x^3 + 12x + 4}$$

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{6x^3 + 5x^2 - 7x}{11x^4 - x^3 + 12x + 4} &= \lim_{x\to\pm\infty} \frac{6x^3 + 5x^2 - 7x}{11x^4 - x^3 + 12x + 4} \cdot \frac{1/x^4}{1/x^4} \\
&= \lim_{x\to\pm\infty} \frac{6/x + 5/x^2 - 7/x^3}{11 - 1/x + 12/x^3 + 4/x^4} \\
&= \frac{\lim_{x\to\pm\infty} \left(6/x + 5/x^2 - 7/x^3\right)}{\lim_{x\to\pm\infty} \left(11 - 1/x + 12/x^3 + 4/x^4\right)} \\
&= \frac{0 + 0 - 0}{11 - 0 + 0 + 0} \\
&= 0
\end{aligned}$$

So the horizontal asymptotes are $y = 0$.

Horizontal asymptotes

Example: Determine the horizontal asymptotes of

$$f(x) = \frac{6x^2+4}{\sqrt[4]{5x^8-2}}.$$

$$\lim_{x\to\pm\infty} \frac{6x^2+4}{\sqrt[4]{5x^8-2}} = \lim_{x\to\pm\infty} \frac{6x^2+4}{\sqrt[4]{5x^8-2}} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x\to\pm\infty} \frac{6x^2+4}{\sqrt[4]{5x^8-2}} \cdot \frac{1/x^2}{1/\sqrt[4]{x^8}}$$

$$= \lim_{x\to\pm\infty} \frac{6+4/x^2}{\sqrt[4]{(5x^8-2)\cdot 1/x^8}}$$

Horizontal asymptotes

$$\ldots = \lim_{x\to\pm\infty} \frac{6 + {}^{4}\!/\!_{x^2}}{\sqrt[4]{(5x^8 - 2)\cdot {}^{1}\!/\!_{x^8}}}$$

$$= \lim_{x\to\pm\infty} \frac{6 + {}^{4}\!/\!_{x^2}}{\sqrt[4]{5 - {}^{2}\!/\!_{x^8}}}$$

$$= \frac{\lim_{x\to\pm\infty} (6 + {}^{4}\!/\!_{x^2})}{\lim_{x\to\pm\infty} \sqrt[4]{5 - {}^{2}\!/\!_{x^8}}}$$

$$= \frac{\lim_{x\to\pm\infty} (6 + {}^{4}\!/\!_{x^2})}{\sqrt[4]{\lim_{x\to\pm\infty} (5 - {}^{2}\!/\!_{x^8})}}$$

$$= \frac{6+0}{\sqrt[4]{5-0}} = \frac{6}{\sqrt[4]{5}}$$

The horizontal asymptote of $\frac{6x^2+4}{\sqrt[4]{5x^8-2}}$ is thus $y = {}^{6}\!/\!_{\sqrt[4]{5}}$.

Warning!

It is easy to notice certain "patterns" and come up with tricks and shortcuts for solving these types of problems (limits of rational functions at infinity). *You can not use those shortcuts for solving problems on quizzes or tests!*

For example: if you say $\lim_{x \to \infty} \frac{3x^4 - 6x^2}{2x^4 + 5} = \frac{3}{2}$ without any justification, or simply saying this is the limit "because the numerator and denominator have the same degree," *then you will not receive any credit!*

Instead, you must justify your answer by doing the algebra to rewrite the function as something simpler that you can actually take the limit of.

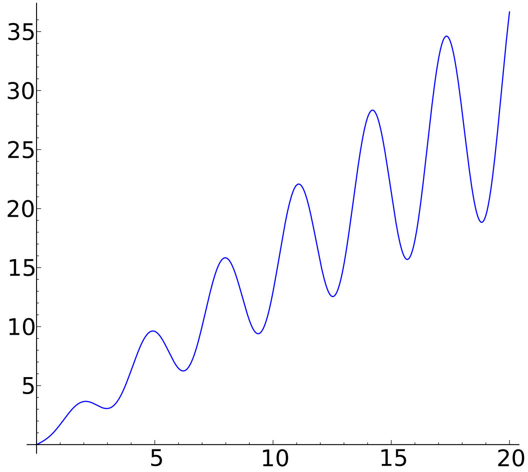
Warning!

If you know more advanced tricks that we have not yet discussed in class (e.g., L'Hôpital's rule), *you can not use them on quizzes or tests (yet)*.

If you want to use these tricks to double-check your answers, then that's fine, but you can't use them as justification for your answer on a quiz or test.

Infinite limits at infinity

If $f(x)$ grows without bound as x goes to ∞, then we write $\lim_{x\to\infty} f(x) = \infty$.



Infinite limits at infinity

The precise definition of having an infinite limit at infinity is the following: we say that $\lim_{x \to \infty} f(x) = \infty$ if for every $M > 0$ there exists an N such that $f(x) > M$ whenever $x > N$.

The definitions of $\lim_{x \to \infty} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = \infty$, and $\lim_{x \to -\infty} f(x) = -\infty$ are comparable.

Infinite limits at infinity

Let's show that $\lim_{x\to\infty} \ln(x) = \infty$.

Let $M > 0$ and set $N = e^M$. If $x > e^M$, then

$$x > e^M$$

$$\implies \ln(x) > \ln(e^M) = M.$$

And so we can make $\ln(x)$ arbitrarily large.

Infinit limits at infinity

Example: Determine $\lim_{x\to\infty} \frac{3x^5+6x^2+2}{2x^3-7}$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^5+6x^2+2}{2x^3-7} &= \lim_{x\to\infty} \frac{3x^5+6x^2+2}{2x^3-7} \\
&= \lim_{x\to\infty} \frac{3x^5+6x^2+2}{2x^3-7} \cdot \frac{1/x^3}{1/x^3} \\
&= \lim_{x\to\infty} \frac{3x^2+\frac{6}{x}+\frac{2}{x^3}}{2-7\frac{7}{x^3}} \\
&= \frac{\lim_{x\to\infty}\left(3x^2+\frac{6}{x}+\frac{2}{x^3}\right)}{\lim_{x\to\infty}\left(2-7\frac{7}{x^3}\right)} \\
&= \frac{\lim_{x\to\infty} 3x^2+0+0}{2-0} = \lim_{x\to\infty} \frac{3}{2}x^2 = \infty.
\end{aligned}$$

Arithmetic with infinity

Finally, let's consider an example to show that you have to be very careful when dealing with infinite limits: ∞ does not obey the normal laws of arithmetic.

$$\lim_{x\to\infty}\left(\sqrt{x^2-2}-\sqrt{x^2+x}\right)$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2-2}-\sqrt{x^2+x}\right)\cdot\left(\sqrt{x^2-2}+\sqrt{x^2+x}\right)}{\sqrt{x^2-2}+\sqrt{x^2+x}}$$

$$=\lim_{x\to\infty}\frac{x^2-2-x^2-x}{\sqrt{x^2-2}+\sqrt{x^2+x}}$$

$$=\lim_{x\to\infty}\frac{-2-x}{\sqrt{x^2-2}+\sqrt{x^2+x}}$$

Arithmetic with infinity

$$\ldots = \lim_{x\to\infty} \frac{-2-x}{\sqrt{x^2-2}+\sqrt{x^2+x}}$$

$$= \lim_{x\to\infty} \frac{-2-x}{\sqrt{x^2-2}+\sqrt{x^2+x}} \cdot \frac{1/x}{1/x}$$

$$= \lim_{x\to\infty} \frac{-2/x-1}{\sqrt{1-2/x^2}+\sqrt{1+1/x}}$$

$$= -\frac{1}{2}$$

Thus you can not write $\infty - \infty = 0$!

Homework

Due Monday, September 8 :

- Read §2.5 in Stewart.
- Homework set listed on the web site.

Due Wednesday, September 10 :

- Read §2.6 in Stewart.
- Complete 33% of ALEKS, or risk receiving nagging emails reminding you to work on ALEKS!

There will be a quiz on Wednesday, September 10. The quiz will focus on continuity, limits at infinity, and horizontal asymptotes; however, earlier material (e.g., the ε-δ definitions) may also make an appearance.