Math 1060

Lecture 8
Derivatives

Outline

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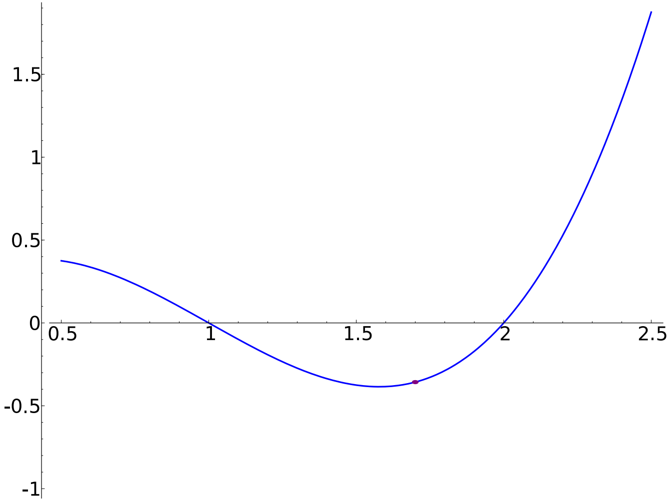
- Saw some examples to show that you have to be careful when doing arithmetic with ∞.

Tangent lines

Motivating Problem: Calculate the equation of the line tangent to the graph $y = f(x)$ at the point $(x_0, f(x_0))$.



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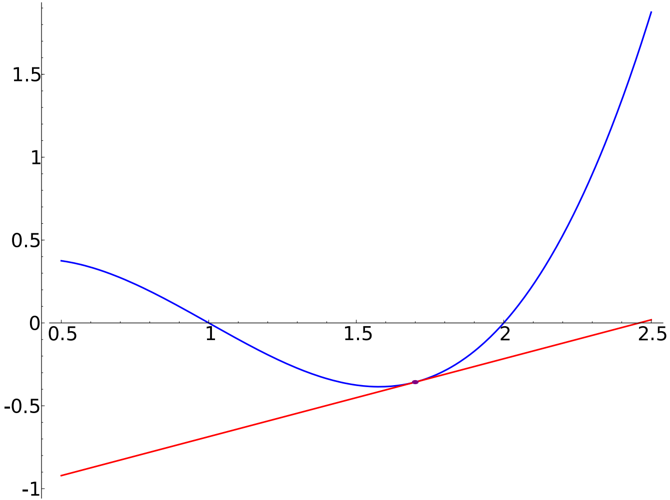
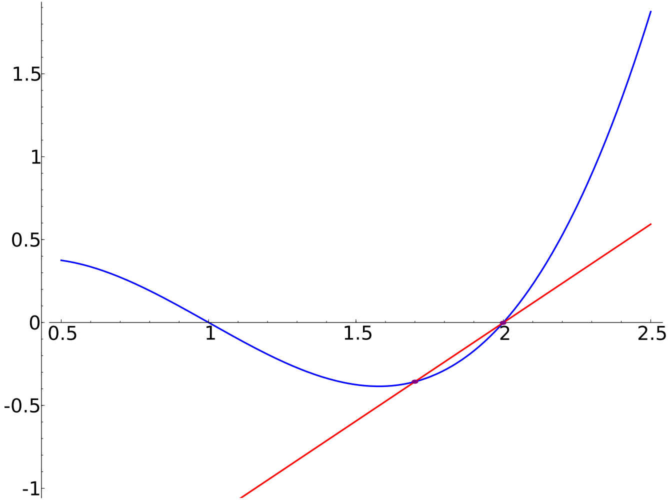


Tangent lines

Solution: Calculate the *secant line* through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ as an approximation. Then move x_1 closer to x_0.



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Recall that a *secant line* for a curve $y = f(x)$ is simply a line that passes through two points on the curve, let's call these two points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$

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Since $y = f(x)$, the y-coordinate of P_0 is $y_0 = f(x_0)$; thus $P_0 = (x_0, f(x_0))$. Similarly, $P_1 = (x_1, f(x_1))$.

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$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

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$$\implies y = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + f(x_0)$$

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Example: Find the equation of the secant line for the curve $y = x^2 + 1$ which passes through the points with x-coordinates $x_0 = 1$ and $x_1 = 2$.

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The slope of the line is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^2 - x_0^2}{x_1 - x_0} = \frac{5 - 2}{2 - 1} = 3.$$

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Thus the secant line is

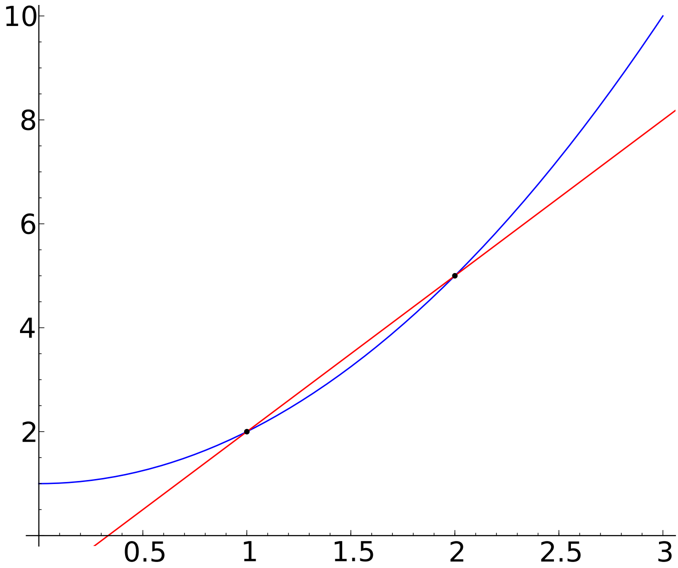
$$y - f(x_0) = m(x - x_0)$$

$$\implies y - 2 = 3(x - 1)$$

$$\implies y = 3(x - 1) + 2$$

$$\implies y = 3x - 1.$$

Secant lines



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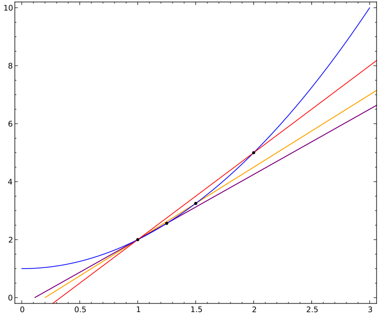
Notice that if we are approximating the tangent line at $(x_0, f(x_0))$, then we are simply going to move x_1 closer to x_0.



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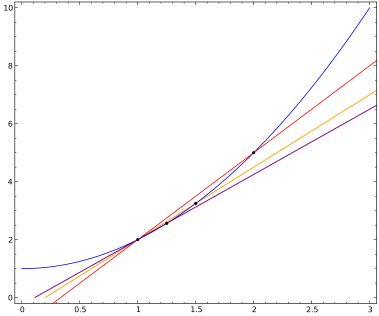

Thus in our formula for the secant line, $y - f(x_0) = m(x - x_0)$, the only quantity that will change is the slope.

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The slope of our secant line is thus

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + h) - f(x_0)}{x_0 + h - x_0}$$

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$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

This limit, if it exists, tells us the slope of the tangent line.

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In this problem, $f(x) = x^2 + 1$. Taking the limit of these slopes as h goes to zero, we have

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

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Thus the equation of the line tangent to $y = x^2 + 1$ at the point $(1, 2)$ is $y - 2 = 2(x - 1)$ or simply $y = 2x$.

Tangent lines

Notice that any time we talk about limits, we always have to worry about whether the limit exists or not; it could be that $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ does not exist.

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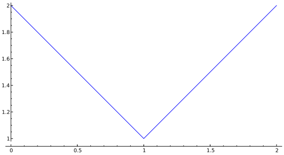
Here's one example of a function where this may happen: consider $f(x) = |x-1|+1$. If we try to determine the tangent line at the point $(1,1)$, we will run into trouble:



$$\lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} = 1 \qquad \lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h} = -1.$$

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What is the instantaneous velocity of the ball two seconds after being dropped?

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$$\frac{f(2+h) - f(2)}{2+h-2}$$

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Let's calculate the average velocity of the ball over the time interval $[2, 2+h]$:

$$\frac{f(2+h)-f(2)}{2+h-2} = \frac{\left[100-4.9(2+h)^2\right]-\left[100-4.9\cdot 2^2\right]}{h}$$

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&= \frac{100 - 4.9 \cdot (4 + 4h + h^2) - 100 + 19.6}{h} \\
&= \frac{-19.6h - 4.9h^2}{h} \\
&= -19.6 - 4.9h
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The instantaneous velocity of the ball at time $t = 2$ seconds is

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$$\begin{aligned}\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h\to 0} (-19.6 - 4.9h) \\ &= -19.6.\end{aligned}$$

And so, the instantaneous velocity of the ball one second after being dropped is $-19.6\frac{\text{m}}{\text{s}}$.

Instantaneous rates of change

As a slightly different example, suppose that water is being pumped into an industrial sized, cone shaped kettle. Suppose that due to the height and radius of the kettle, when g gallons of water have been pumped into the kettle, the depth of the water measured in feet is

$$d(g) = \sqrt[3]{\frac{12g}{\pi}}.$$

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What is the instantaneous rate of change in the depth of the water when there are already 200 gallons of water?

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What is the instantaneous rate of change in the depth of the water when there are already 200 gallons of water?

We will find this instantaneous rate of change by taking the limit of the average rates of change.

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That is, we will consider the average rate of change of $d(g)$ over intervals of the form $[200, 200 + h]$:

$$\frac{d(200 + h) - d(200)}{200 + h - 200} = \frac{d(200 + h) - d(200)}{h}.$$

Notice this average rate of change has the units feet per gallon.

Instantaneous rate of change

The *instantaneous rate of change* of a function $f(x)$ at $x = a$ is the limit as h goes to zero of the average rates of change of f over the interval $[a, a+h]$.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

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The *instantaneous rate of change* of a function $f(x)$ at $x = a$ is the limit as h goes to zero of the average rates of change of f over the interval $[a, a + h]$.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

For the problem at hand, we thus need to calculate

$$\lim_{h \to 0} \frac{d(200+h) - d(200)}{h}.$$

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Our instantaneous rate of change in the depth of the water in the kettle, when there is already 200 gallons in the kettle is

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These types of limits come up very frequently in mathematics and physics and have a special name. We call the quantity $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$, if this limit exists, the *derivative* of $f(x)$ at $x = a$.

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We denote this limit (if it exists) by $f'(a)$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

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f'(-1) &= \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} \\
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Because the derivative of $f(x) = x^2 + 2x$ at $x = -1$ is $f'(-1) = 0$, we know the equation of the line tangent to $y = x^2 + 2x$ at $(-1, -1)$ is $y = -1$.

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Warning: There are some shortcuts for calculating the derivatives which we will learn about later. If you already know the shortcuts you may use them to double-check your work, but for the time being you must use this “limit definition of the derivative” when doing calculations on tests and quizzes!

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Even though these limits are all equal, it is sometimes convenient to express the derivative in one of these alternative forms; it makes some calculations slightly easier.

Derivatives

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Finally, note that when h goes to 0, $x = a + h$ goes to a. Thus

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Differentiability at a point

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Most of (but not all) functions we care about are differentiable: polynomials, rational functions, trig functions, inverse trig functions, exponentials, and logarithms are differentiable.

Differentiability implies continuity

One basic property of differentiability is that a function *must* be continuous in order to be differentiable.

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If f is differentiable at $x = a$, then the limit

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exists. We want to show that this implies $\lim_{x \to a} f(x) = f(a)$.

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This is the same as showing

$$\lim_{x \to a} (f(x) - f(a)) = 0.$$

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What we've shown is that if f is differentiable at a, then f must be continuous at a.
Said another way, if f is not continuous at a, then f can not be differentiable at a.

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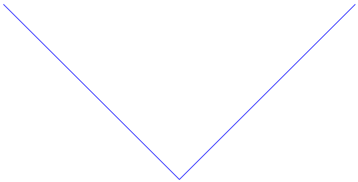
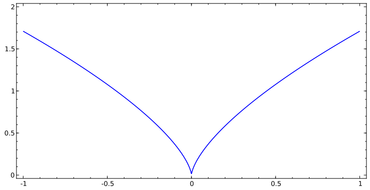
Yes! These correspond to places where the function has a *corner* or a *cusp*.





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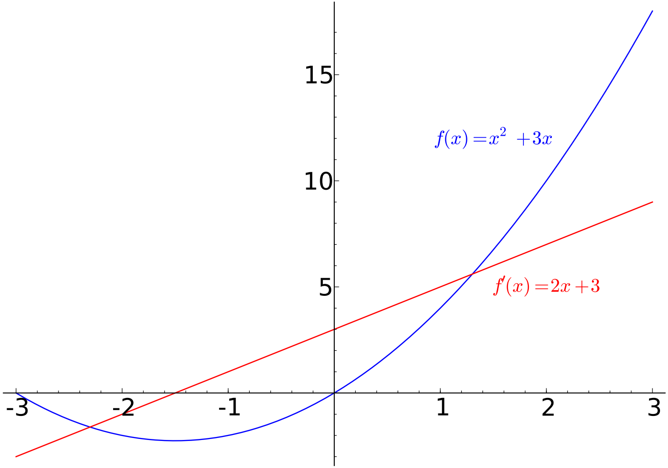

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$$\begin{aligned} f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h\to 0} \frac{[2(x+h)+3] - [2x+3]}{h} \\ &= \lim_{h\to 0} \frac{2x+2h+3-2x-3}{h} \\ &= \lim_{h\to 0} \frac{2h}{h} \end{aligned}$$

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Higher-order derivatives

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Any of these functions obtained by differentiating f multiple times is called a *higher-order derivative*.

Writing a single $'$ (called a *prime*) for each derivative we take becomes unrealistic if we want to differentiate a function several times. Luckily there is some terser notation.

Notation

The n-th derivative of f (the function obtained by differentiating n times) is sometimes denoted $f^{(n)}$. For example,

$$f' = f^{(1)}$$

$$f'' = f^{(2)}$$

$$f''' = f^{(3)}$$

$$f'''' = f^{(4)}$$

$$\vdots$$

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By convention the "zero-th derivative" of a function just means the function itself:

$$f = f^{(0)} = \frac{\mathrm{d}^0 f}{\mathrm{d}x^0}.$$

Notation

When $y = f(x)$, we will sometimes write y' or $\frac{\mathrm{d}y}{\mathrm{d}x}$ in place of f' or $\frac{\mathrm{d}f}{\mathrm{d}x}$.

When evaluating the derivative at a point $x = a$, we write

$$f'(a) \qquad \text{or} \qquad \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{a}$$

All of these notations are very common, so you should be comfortable with each.

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We will not use this "dot notation" in this course, but you should be aware of what it means in case you come across it in another class.

Applications

Finally, we end by mentioning a few applications of derivatives to mathematics, physics, computer science, and engineering.

We are just beginning our study of derivatives in this course, and learning all of the tricks of the trade for derivatives will occupy the majority of the remainder of this course.

Right now we don't have the technical abilities to go through the nitty-gritty details of all of these examples, but it's good to know that the topics we are discussing have practical, real-world applications.

Applications within mathematics

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One example you will see later, in Math 2060, is curvature. The curvature of a curve is the (magnitude of the) derivative of the curve's tangent vectors. The curvature of a surface is the product of the largest and smallest curvatures of curves on the surface.

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This curvature has a lot of consequences in geometry: the three types of geometries for surfaces (Euclidean, spherical, or hyperbolic) correspond to three possibilities for curvature (zero, positive, or negative).

Applications within physics

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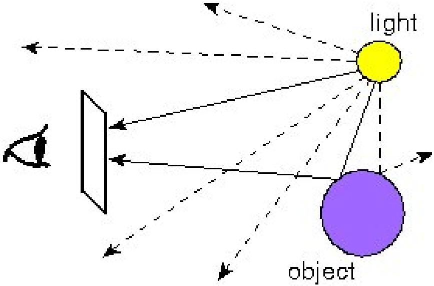
One important set of examples: Maxwell's equations describe how electromagnetic fields propagate and are at the heart of all modern electronics (computers, phones, etc.). Maxwell's equations are four *differential equations*, simply meaning they are equations involving derivatives.

Applications within computer science

One of the simplest ways to create three-dimensional graphics on a computer is to use a technique called *ray tracing*. Ray tracing produces extremely realistic three-dimensional images and is used, for example, to create special effects in movies.

Applications within computer science

The idea behind ray tracing is surprisingly simple: model the path a ray of light would take between a light source and your eye, bouncing off any objects in the scene.



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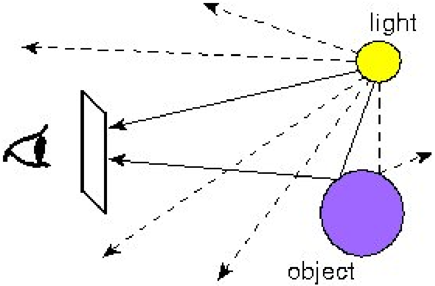

Calculating how the light bounces off an object involves determining the *tangent plane* of the object, and this requires differentiating the equation whose graph gives the surface.

Applications within engineering

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The most commonly used method for solving special types of these problems (called *linear programs*) is to use the *simplex algorithm* which iteratively works its way to the most efficient solution to the problem.

As we will see later, derivatives tell when a function is increasing or decreasing. The basic idea behind the simplex algorithm is to start with a possible solution, then consider derivatives of the function to determine where the next best point is. Repeat this process until you are at the point maximizing (or minimizing) the *objective function*.