CLEMSON UNIVERSITY

Math 1060

Lecture 8
Derivatives

Outline

Summary of last lecture

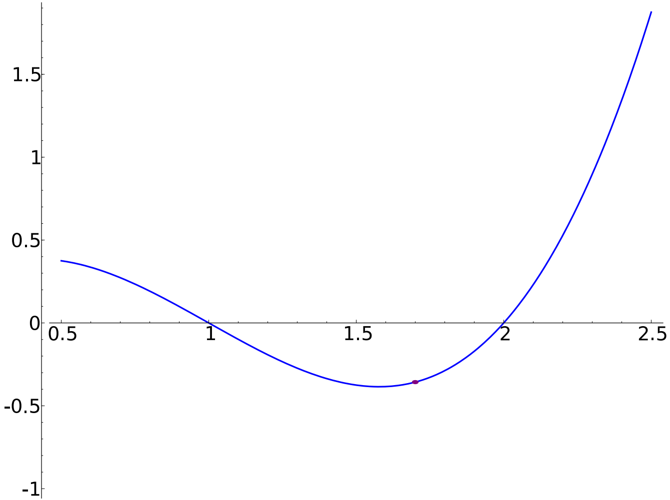
- Described limits at $\pm\infty$.
- Gave the precise definition of $\lim_{x\to\infty} f(x) = L$:

  For every $\varepsilon > 0$ there exists an $N > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > N$.

- Mentioned the relationship between horizontal asymptotes and infinite limits.
- Saw the main technique for calculating limits at infinity for rational functions: multiply and divide by $\frac{1/x^n}{1/x^n}$ where n is the degree of the denominator.
- Saw some examples to show that you have to be careful when doing arithmetic with ∞.

Tangent lines

Motivating Problem: Calculate the equation of the line tangent to the graph $y = f(x)$ at the point $(x_0, f(x_0))$.



Tangent lines

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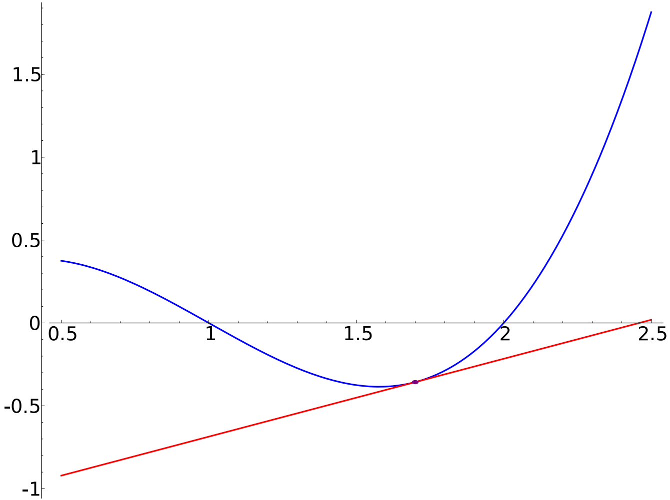
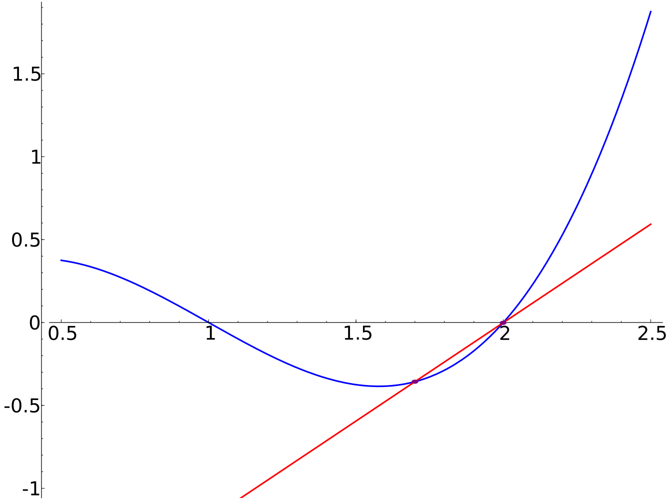


Tangent lines

Solution: Calculate the *secant line* through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ as an approximation. Then move x_1 closer to x_0.



Secant lines

Let's now make this idea of approximating the tangent line with secant lines more precise.

Recall that a *secant line* for a curve $y = f(x)$ is simply a line that passes through two points on the curve, let's call these two points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$

Since $y = f(x)$, the y-coordinate of P_0 is $y_0 = f(x_0)$; thus $P_0 = (x_0, f(x_0))$. Similarly, $P_1 = (x_1, f(x_1))$.

Secant lines

To write down the equation of a line we need two pieces of information: the slope of the line, and a point the line passes through.

For our secant line through $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ the slope is

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Secant lines

Recall that the equation of the line with slope m through the point $P_0 = (x_0, y_0)$ is

$$y - y_0 = m(x - x_0).$$

The equation of our secant line is thus

$$y - y_0 = m(x - x_0)$$

$$\Longrightarrow y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$\Longrightarrow y = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + f(x_0)$$

Secant lines

Example: Find the equation of the secant line for the curve $y = x^2 + 1$ which passes through the points with x-coordinates $x_0 = 1$ and $x_1 = 2$.

First notice that our points are $P_0 = (1, 2)$, and $P_1 = (2, 5)$.

The slope of the line is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^2 - x_0^2}{x_1 - x_0} = \frac{5-2}{2-1} = 3.$$

Thus the secant line is

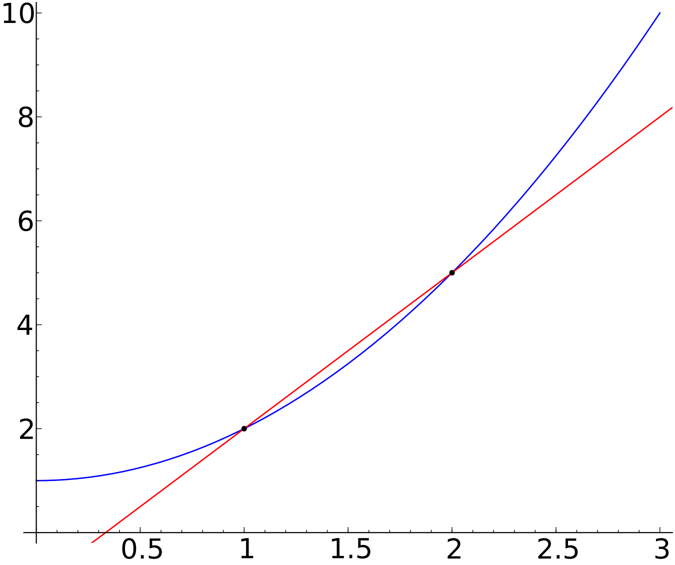
$$y - f(x_0) = m(x - x_0)$$

$$\implies y - 2 = 3(x - 1)$$

$$\implies y = 3(x - 1) + 2$$

$$\implies y = 3x - 1.$$

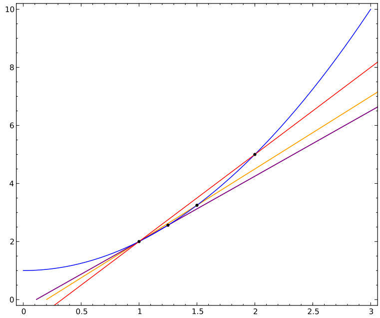
Secant lines



Secant lines

Each of our secant lines gives us an approximation of the tangent line. Our goal now is to make the approximation better, and we do this by moving our two points closer together.

Notice that if we are approximating the tangent line at $(x_0, f(x_0))$, then we are simply going to move x_1 closer to x_0.



Thus in our formula for the secant line, $y - f(x_0) = m(x - x_0)$, the only quantity that will change is the slope.

Secant lines

So all we really need to do is understand how the slope is changing.

To make our calculations a little bit easier, let's suppose we obtain x_1 by moving a distance of h from x_0:

$$x_1 = x_0 + h.$$

Note: h may be positive or negative. If h is positive, we move h-units to the right from x_0; if h is negative, we move $|h|$-units to the left from x_0.

The slope of our secant line is thus

$$\begin{aligned}\frac{f(x_1) - f(x_0)}{x_1 - x_0} &= \frac{f(x_0 + h) - f(x_0)}{x_0 + h - x_0} \\ &= \frac{f(x_0 + h) - f(x_0)}{h}.\end{aligned}$$

Tangent lines

To obtain the slope of the line tangent to $y = f(x)$ at $(x_0, f(x_0))$, we want to move $x_1 = x_0 + h$ closer and closer to x_0. I.e., we want to take the limit as h goes to zero:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

This limit, if it exists, tells us the slope of the tangent line.

Tangent lines

Example: Determine the equations of the line tangent to $y = x^2 + 1$ at the point $(1, 2)$.

Solution: We first need to find the slope of this line. Here, $x_0 = 1$, and so we consider the slopes of secant lines through x_0 and $x_1 = 1 + h$.

In this problem, $f(x) = x^2 + 1$. Taking the limit of these slopes as h goes to zero, we have

$$\begin{aligned}
\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0} \frac{\left[(1+h)^2 + 1\right] - \left[1^2 + 1\right]}{h} \\
&= \lim_{h \to 0} \frac{1 + 2h + h^2 + 1 - 2}{h} \\
&= \lim_{h \to 0} \frac{2h + h^2}{h}
\end{aligned}$$

Tangent lines

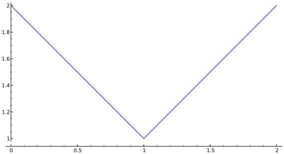
$$
\begin{aligned}
\ldots &= \lim_{h\to 0} \frac{2h + h^2}{h} \\
&= \lim_{h\to 0} (2 + h) \\
&= 2
\end{aligned}
$$

Thus the equation of the line tangent to $y = x^2 + 1$ at the point $(1, 2)$ is $y - 2 = 2(x - 1)$ or simply $y = 2x$.

Tangent lines

Notice that any time we talk about limits, we always have to worry about whether the limit exists or not; it could be that $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ does not exist.

Here's one example of a function where this may happen: consider $f(x) = |x-1|+1$. If we try to determine the tangent line at the point $(1,1)$, we will run into trouble:



$$\lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} = 1 \qquad \lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h} = -1.$$

Instantaneous velocity

As another example of using limits of approximations, let's determine the instantaneous velocity of a moving object.

Suppose you drop a ball from a height of 100 metres. The height of the ball t seconds after being dropped is given, in metres, by $f(t) = 100 - 4.9t^2$.

What is the instantaneous velocity of the ball two seconds after being dropped?

Average velocity

To answer this question, we will approximate the instantaneous velocity with average velocities over smaller and smaller intervals of time.

Let's calculate the average velocity of the ball over the time interval $[2, 2+h]$:

$$\begin{aligned}\frac{f(2+h)-f(2)}{2+h-2} &= \frac{\left[100-4.9(2+h)^2\right]-\left[100-4.9\cdot 2^2\right]}{h} \\ &= \frac{100-4.9\cdot(4+4h+h^2)-100+19.6}{h} \\ &= \frac{-19.6h-4.9h^2}{h} \\ &= -19.6-4.9h\end{aligned}$$

Instantaneous velocity

That is, the average velocity of the ball over the time interval $[2, 2+h]$ is $(-19.6 - 4.9h)\ \frac{\text{m}}{\text{s}}$.

To get the instantaneous velocity, we consider the average velocities over smaller and smaller intervals of time: I.e., we take the limit as h goes to zero.

The instantaneous velocity of the ball at time $t = 2$ seconds is

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} (-19.6 - 4.9h)$$

$$= -19.6.$$

And so, the instantaneous velocity of the ball one second after being dropped is $-19.6\frac{\text{m}}{\text{s}}$.

Instantaneous rates of change

As a slightly different example, suppose that water is being pumped into an industrial sized, cone shaped kettle. Suppose that due to the height and radius of the kettle, when g gallons of water have been pumped into the kettle, the depth of the water measured in feet is

$$d(g) = \sqrt[3]{\frac{12g}{\pi}}.$$

What is the instantaneous rate of change in the depth of the water when there are already 200 gallons of water?

We will find this instantaneous rate of change by taking the limit of the average rates of change.

Average rate of change

In general, the *average rate of change* of a function $f(x)$ over the interval $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

In the problem described on the previous slide we are considering the rate of change in the depth of water as we pour more water into the kettle, supposing there are already 200 gallons of water in the kettle.

That is, we will consider the average rate of change of $d(g)$ over intervals of the form $[200, 200 + h]$:

$$\frac{d(200 + h) - d(200)}{200 + h - 200} = \frac{d(200 + h) - d(200)}{h}.$$

Notice this average rate of change has the units feet per gallon.

Instantaneous rate of change

The *instantaneous rate of change* of a function $f(x)$ at $x = a$ is the limit as h goes to zero of the average rates of change of f over the interval $[a, a + h]$.

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

For the problem at hand, we thus need to calculate

$$\lim_{h\to 0} \frac{d(200+h) - d(200)}{h}.$$

Instantaneous rate of change

Our instantaneous rate of change in the depth of the water in the kettle, when there is already 200 gallons in the kettle is

$$
\begin{aligned}
\lim_{h\to 0} \frac{d(200+h)-d(200)}{h} &= \lim_{h\to 0} \frac{\sqrt[3]{\frac{12(200+h)}{\pi}} - \sqrt[3]{\frac{2400}{\pi}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt[3]{\frac{2400}{\pi} - \frac{12h}{\pi}} - \sqrt[3]{\frac{2400}{\pi}}}{h} \\
&= \text{(algebra to rewrite as a difference of cubes)} \\
&= \frac{1}{3}\left(\sqrt[3]{\frac{2400}{\pi}}\right)^{-2/3} \cdot \frac{12}{\pi}\,\frac{\text{feet}}{\text{gallon}}
\end{aligned}
$$

Derivatives

All of the examples above follow a familiar pattern: whether we are calculating tangent lines, instantaneous velocities, or instantaneous rates of change, we evaluate a limit of the form

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

These types of limits come up very frequently in mathematics and physics and have a special name. We call the quantity $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, if this limit exists, the *derivative* of $f(x)$ at $x = a$.

We denote this limit (if it exists) by $f'(a)$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Derivatives

Example: Let $f(x) = x^2 + 2x$. Calculate the derivative $f'(-1)$.
Solution: In this problem the value of a is -1, and we simply apply the definition of the derivative and calculate the limit.

$$
\begin{aligned}
f'(-1) &= \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} \\
&= \lim_{h \to 0} \frac{\left[(-1+h)^2 + 2(-1+h)\right] - \left[(-1)^2 + 2(-1)\right]}{h} \\
&= \lim_{h \to 0} \frac{\left[(-1)^2 - 2h + h^2\right] + [-2 + 2h] - [1 - 2]}{h} \\
&= \lim_{h \to 0} \frac{h^2}{h} \\
&= \lim_{h \to 0} h = 0
\end{aligned}
$$

Derivatives

Because the derivative of $f(x) = x^2 + 2x$ at $x = -1$ is $f'(-1) = 0$, we know the equation of the line tangent to $y = x^2 + 2x$ at $(-1, -1)$ is $y = -1$.

Warning: There are some shortcuts for calculating the derivatives which we will learn about later. If you already know the shortcuts you may use them to double-check your work, but for the time being you must use this "limit definition of the derivative" when doing calculations on tests and quizzes!

Derivatives

Example: Determine the derivative of $f(x) = 3x + 4$ at the point $x = 5$.

Solution: We simply calculate the limit:

$$
\begin{aligned}
f'(5) &= \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} \\
&= \lim_{h \to 0} \frac{[3(5+h)+4] - [3 \cdot 5 + 4]}{h} \\
&= \lim_{h \to 0} \frac{15 + 3h + 4 - 15 - 4}{h} \\
&= \lim_{h \to 0} \frac{3h}{h} \\
&= \lim_{h \to 0} 3 = 3.
\end{aligned}
$$

Derivatives

Before we go any further, let's notice that the derivative $f'(a)$ may actually be written in several different ways.

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{\Delta y}{\Delta x}.
\end{aligned}
$$

Even though these limits are all equal, it is sometimes convenient to express the derivative in one of these alternative forms; it makes some calculations slightly easier.

Derivatives

Let's take a moment to understand why

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

Notice that $a+h$ is some x-value that we plug into f; let's just write $x=a+h$.

Since $h=h+a-a=a+h-a$, we may also write so $h=x-a$.

Combining these two facts, the *difference quotient* in the derivative becomes

$$\frac{f(a+h)-f(a)}{h}=\frac{f(x)-f(a)}{x-a}.$$

Finally, note that when h goes to 0, $x=a+h$ goes to a. Thus

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

Differentiability at a point

If the derivative $f'(a)$ of a function is defined at $x = a$, then we say that f is *differentiable at* $x = a$.

If the derivative $f'(a)$ is defined for every point $x = a$ in the domain of the function, we just say the function is *differentiable*.

Most of (but not all) functions we care about are differentiable: polynomials, rational functions, trig functions, inverse trig functions, exponentials, and logarithms are differentiable.

Differentiability implies continuity

One basic property of differentiability is that a function *must* be continuous in order to be differentiable.

If f is differentiable at $x = a$, then the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. We want to show that this implies $\lim\limits_{x \to a} f(x) = f(a)$.

This is the same as showing

$$\lim_{x \to a} (f(x) - f(a)) = 0.$$

Differentiability implies continuity

$$
\begin{aligned}
\lim_{x\to a}\left(f(x)-f(a)\right) &= \lim_{x\to a}\left(f(x)-f(a)\right)\cdot\frac{x-a}{x-a} \\
&= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a) \\
&= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\cdot\left(\lim_{x\to a}(x-a)\right) \\
&= f'(a)\cdot 0 \\
&= 0
\end{aligned}
$$

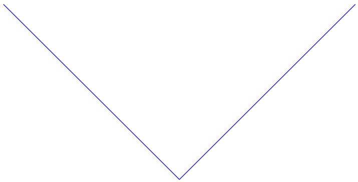
What we've shown is that if f is differentiable at a, then f must be continuous at a.
Said another way, if f is not continuous at a, then f can not be differentiable at a.

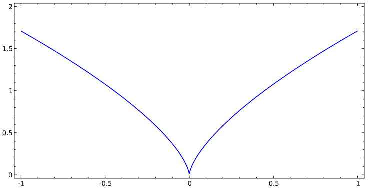
When is a function not differentiable

As differentiability implies continuity, a function can not be differentiable if it is not continuous.

Are there places where the function is continuous, but not differentiable?

Yes! These correspond to places where the function has a *corner* or a *cusp*.





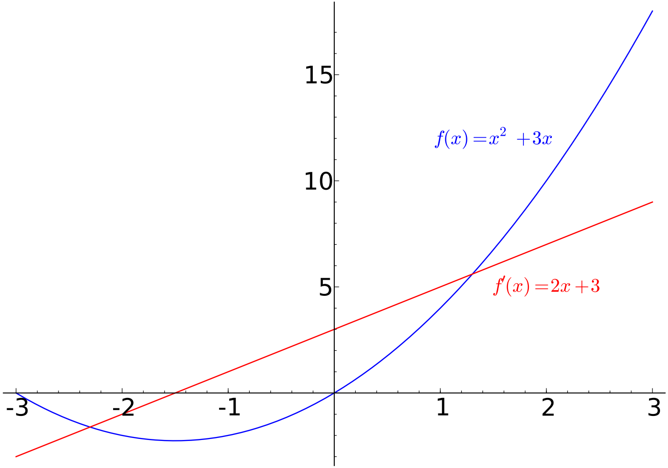
The derivative is a function

Suppose that f is a differentiable function, so that $f'(a)$ is defined for each a in the domain of f. Then we can think of the derivative as being a function.

For example, let $f(x) = x^2 + 3x$. For every value of x we have

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\left[(x+h)^2 + 3(x+h)\right] - \left[x^2 + 3x\right]}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h} \\
&= \lim_{h \to 0} (2x + 3 + h)
\end{aligned}$$

The derivative is a function



The derivative is a function

Since f' is a function, we can try to take its derivative.

The derivative of f' is denoted f'' and is also called the *second derivative* of f.

$$\begin{aligned}
f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\
&= \lim_{h\to 0} \frac{[2(x+h)+3] - [2x+3]}{h} \\
&= \lim_{h\to 0} \frac{2x+2h+3-2x-3}{h} \\
&= \lim_{h\to 0} \frac{2h}{h} \\
&= \lim_{h\to 0} 2 = 2.
\end{aligned}$$

Higher-order derivatives

We could then differentiate f'' again to obtain the *triple derivative* of f, denoted f'''.

Notice we can continue doing this pattern forever, obtaining *quadruple derivatives* and so on.

Any of these functions obtained by differentiating f multiple times is called a *higher-order derivative*.

Writing a single $'$ (called a *prime*) for each derivative we take becomes unrealistic if we want to differentiate a function several times. Luckily there is some terser notation.

Notation

The n-th derivative of f (the function obtained by differentiating n times) is sometimes denoted $f^{(n)}$. For example,

$$f' = f^{(1)}$$

$$f'' = f^{(2)}$$

$$f''' = f^{(3)}$$

$$f'''' = f^{(4)}$$

$$\vdots$$

Notation

An alternative notation is to write

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n}$$

for the n-th derivative. When $n = 1$, we just write $\frac{\mathrm{d}f}{\mathrm{d}x}$.

Note $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}$ and $f^{(n)}$ mean the same thing, and we will use both notations throughout the course.

By convention the "zero-th derivative" of a function just means the function itself:

$$f = f^{(0)} = \frac{\mathrm{d}^0 f}{\mathrm{d}x^0}.$$

Notation

When $y = f(x)$, we will sometimes write y' or $\frac{\mathrm{d}y}{\mathrm{d}x}$ in place of f' or $\frac{\mathrm{d}f}{\mathrm{d}x}$.

When evaluating the derivative at a point $x = a$, we write

$$f'(a) \qquad \text{or} \qquad \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{a}$$

All of these notations are very common, so you should be comfortable with each.

Notation

Another notation that is common in physics, especially if f is a function of time, is to write $\dot{f}$ in place of f'; $\ddot{f}$ in place of f''; and so on.

We will not use this "dot notation" in this course, but you should be aware of what it means in case you come across it in another class.

Applications

Finally, we end by mentioning a few applications of derivatives to mathematics, physics, computer science, and engineering.

We are just beginning our study of derivatives in this course, and learning all of the tricks of the trade for derivatives will occupy the majority of the remainder of this course.

Right now we don't have the technical abilities to go through the nitty-gritty details of all of these examples, but it's good to know that the topics we are discussing have practical, real-world applications.

Applications within mathematics

Several other areas of mathematics use derivatives in one way or another.

One example you will see later, in Math 2060, is curvature. The curvature of a curve is the (magnitude of the) derivative of the curve's tangent vectors. The curvature of a surface is the product of the largest and smallest curvatures of curves on the surface.

This curvature has a lot of consequences in geometry: the three types of geometries for surfaces (Euclidean, spherical, or hyperbolic) correspond to three possibilities for curvature (zero, positive, or negative).

Applications within physics

Very, very many quantities in physics are defined as derivatives. Whether you're talking about classical mechanics; relativistic mechanics; electrodynamics; or quantum theory, derivatives are ubiquitous in physics.

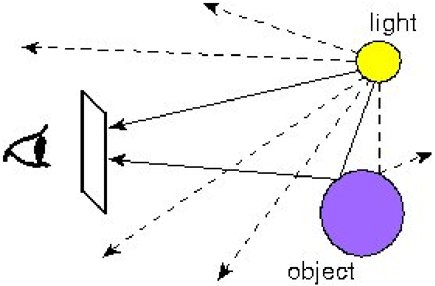
One important set of examples: Maxwell's equations describe how electromagnetic fields propagate and are at the heart of all modern electronics (computers, phones, etc.). Maxwell's equations are four *differential equations*, simply meaning they are equations involving derivatives.

Applications within computer science

One of the simplest ways to create three-dimensional graphics on a computer is to use a technique called *ray tracing*. Ray tracing produces extremely realistic three-dimensional images and is used, for example, to create special effects in movies.

Applications within computer science

The idea behind ray tracing is surprisingly simple: model the path a ray of light would take between a light source and your eye, bouncing off any objects in the scene.



Calculating how the light bounces off an object involves determining the *tangent plane* of the object, and this requires differentiating the equation whose graph gives the surface.

Applications within engineering

One important aspect in many types of engineering problems involves not simply finding a solution to a problem, but finding the most efficient solution (e.g., the solution which uses the least power, or costs the least amount of money to implement).

The most commonly used method for solving special types of these problems (called *linear programs*) is to use the *simplex algorithm* which iteratively works its way to the most efficient solution to the problem.

As we will see later, derivatives tell when a function is increasing or decreasing. The basic idea behind the simplex algorithm is to start with a possible solution, then consider derivatives of the function to determine where the next best point is. Repeat this process until you are at the point maximizing (or minimizing) the *objective function*.