



MATH 1060

LECTURE 9

THE POWER RULE, POLYNOMIALS, AND EXPONENTIALS

# Outline

## Summary of last lecture

### Derivative laws

- The derivative of a constant function

- The derivative of the identity function

- The power law

- The constant multiple law

- The sum and difference laws

- The general power law

- The derivative of  $e^x$

### Homework

## Summary of last lecture

- ▶ Defined derivative of a function in terms of limits.
- ▶ Discussed three particular interpretations of the derivative: the slope of a tangent line, the velocity of a particle, and the instantaneous rate of change of a function.
- ▶ Proved that a function must be continuous in order to be differentiable.
- ▶ Discussed two common obstacles to differentiability: corners and cusps.
- ▶ Defined higher-order derivatives.
- ▶ Saw several different notations for derivatives.
- ▶ Briefly mentioned applications of derivatives within mathematics, physics, computer science, and engineering.

## In this lecture...

Over the next two weeks we will learn several computational shortcuts that can be used to make calculating derivatives easier.

Since derivatives are defined in terms of limits, we will justify these shortcuts using the limit definition of the derivative, and consider several examples of how to use the shortcuts.

In particular, by the end of this lecture we will understand how to differentiate polynomials and exponential functions.

We will, however, start off with very simple functions and work our way up to more interesting situations.

# The derivative of a constant function

## Theorem

*For any constant  $c$ , the function  $f(x) = c$  is differentiable and  $\frac{d}{dx}c = 0$ .*

## Proof.

Applying the limit definition of the derivative we have

$$\begin{aligned}\frac{d}{dx}c &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c - c}{h} \\ &= 0.\end{aligned}$$



# The derivative of a constant function

## Examples:

- ▶  $\frac{d}{dx}5 = 0$ .
- ▶  $\frac{d}{dx}\pi = 0$ .
- ▶ If  $f(x) = -22/7$ , then  $f'(x) = 0$ .
- ▶ If  $f(x) = 0$ , then  $f'(x) = 0$ .

In terms of slopes of tangent lines, all this says is that the tangent lines to graphs of the form  $y = c$  always have slope zero.

# The derivative of the identity function

## Theorem

*The derivative of  $f(x) = x$  is  $f'(x) = 1$  for every  $x$ .*

## Proof.

If  $f(x) = x$ , then

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{x+h-x}{h} \\&= \lim_{h \rightarrow 0} \frac{h}{h} \\&= 1.\end{aligned}$$



# The derivative of the identity function

In terms of tangent lines, the line tangent to  $y = x$  has slope 1.

In terms of velocity, if the position of a particle at time  $t$  is  $f(t) = t$ , then its velocity is always  $f'(t) = 1$ .



# The power law

## Theorem (The power law)

*For every positive integer  $n$ , the derivative of  $x^n$  is  $\frac{d}{dx}x^n = nx^{n-1}$ .*

## Proof.

Let  $f(x) = x^n$ . We will calculate  $f'(a)$  using one of our alternative limit definitions:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a}.$$

Now we factor the numerator,

$$x^n - a^n = (x - a) \cdot (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

This can be verified by multiplying out the right-hand side and simplifying.

# The power law

Proof (continued).

Now,

$$\begin{aligned}f'(a) &= \lim_{x \rightarrow a} \frac{(x - a) \cdot (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\&= \lim_{x \rightarrow a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\&= \underbrace{a^{n-1} + \cdots + a^{n-1}}_{n \text{ times}} \\&= na^{n-1}.\end{aligned}$$

So for every  $a$ ,  $f'(a) = na^{n-1}$  and thus  $\frac{d}{dx}x^n = nx^{n-1}$ . □

# The power law

## Examples

- ▶ If  $f(x) = x^9$ , then  $f'(x) = 9x^8$ .
- ▶ If  $f(x) = x^3$ , then  $\frac{d}{dx}f(x) = 3x^2$ .
- ▶  $\frac{d}{dx}x^{13} = 13x^{12}$ .
- ▶  $\left.\frac{d}{dx}x^4\right|_{x=2} = 4 \cdot 2^3 = 32$ .

## The power law

**Example:** Find the equation of the line tangent to the curve  $y = x^3$  at the point  $(4, 64)$ .

**Solution**

We know the line will go through  $(x_0, y_0) = (4, 64)$ , so if we can find the slope  $m$ , the equation of the line will be

$$y - y_0 = m(x - x_0).$$

To find the slope we differentiate  $x^3$  and then evaluate at  $x = 4$ . Since  $y = x^3$ , we have  $y' = 3x^2$ . Evaluating this at  $x = 4$  tells us the slope is 48. Thus the line is

$$y - 64 = 48(x - 4).$$

Or, in slope-intercept form,

$$y = 48x - 128.$$

# The constant multiple law

## Theorem

If  $f(x)$  is a differentiable function and if  $c \in \mathbb{R}$  is any constant, then

$$\frac{d}{dx} cf(x) = cf'(x).$$

Proof.

$$\begin{aligned}\frac{d}{dx} cf(x) &= \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} \\ &= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= cf'(x).\end{aligned}$$



# The constant multiple law

## Examples:

- ▶ If  $f(x) = 7x^2$ , then  $f'(x) = 14x$ .
- ▶  $\frac{d}{dx}\pi x^9 = 9\pi x^8$ .
- ▶ If  $y = \frac{-4x^5}{7}$ , then  $y' = \frac{-20x^4}{7}$ .

# The sum and difference laws

## Theorem

*If  $f$  and  $g$  are differentiable functions, then  $f + g$  and  $f - g$  are also differentiable with derivatives*

$$\frac{d}{dx} (f(x) + g(x)) = f'(x) + g'(x), \text{ and}$$

$$\frac{d}{dx} (f(x) - g(x)) = f'(x) - g'(x).$$

Proof.

Homework.



# The sum and difference laws

## Examples:

- ▶ If  $f(x) = 7x^3$  and  $g(x) = 4x^5$ , then  $\frac{d}{dx}(f(x) + g(x)) = 21x^2 + 20x^4$ .
- ▶  $\frac{d}{dx}(x^2 + 3x) = 2x + 3$
- ▶ If  $y = 15x^4 - 6x^3 + \pi x + \frac{13}{2}$ , then  $y' = 60x^3 - 18x^2 + \pi$ .



## The sum and difference laws

The position of a ball thrown straight up from a height of four feet with an initial 30 feet per second is given, in feet above the ground,  $t$  seconds after being thrown by

$$r(t) = -16t^2 + 30t + 4.$$

1. What is the ball's instantaneous velocity at time  $t = 3$  seconds?
2. What is the ball's speed at time  $t = 3$  seconds?

## The sum and difference laws

We've seen before that instantaneous velocity of an object is the derivative of the object's position. Thus the velocity of the ball at time  $t$  is

$$v(t) = -32t + 30$$

When  $t = 3$  seconds, the velocity is  $v(3) = -66$  feet per second. Speed is the magnitude (absolute value) of velocity, thus the speed of the ball is 66 feet per second.

## The sum and difference laws

What is the equation of the line tangent to  $y = 5x^3 - 6x + 3$  at the point  $(2, 31)$ ?

The slope of the line is the derivative of  $5x^3 - 6x + 3$ , evaluated at  $x = 2$ . As

$$\frac{d}{dx} (5x^3 - 6x + 3) = 15x^2 - 6,$$

the slope is 54, and so the equation of the tangent line is

$$y - 31 = 54(x - 2)$$

or, in slope-intercept form,

$$y = 54x - 77.$$

# The general power law

## Theorem

*For any real number  $r \in \mathbb{R}$ ,  $\frac{d}{dx}x^r = rx^{r-1}$ .*

(We will have to wait until we learn about the chain rule next week to prove this, but you can go ahead and use this theorem.)

## Examples:

- ▶  $\frac{d}{dx}x^\pi = \pi x^{\pi-1}$ .
- ▶  $\frac{d}{dx}x^e = ex^{e-1}$ .

## The general power law

This allows us to differentiate roots:

$$\frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{1/2}$$

$$= \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}}.$$

$$\frac{d}{dx} \sqrt[5]{x} = \frac{d}{dx} x^{1/5}$$

$$= \frac{1}{5} x^{-4/5}$$

$$= \frac{1}{5\sqrt[5]{x^4}}$$

## The general power law

We can also differentiate some simple fractions:

$$\frac{d}{dx} \frac{1}{x} = \frac{d}{dx} x^{-1}$$

$$= -1x^{-2}$$

$$= -\frac{1}{x^2}$$

$$\frac{d}{dx} \frac{1}{x^3} = \frac{d}{dx} x^{-3}$$

$$= -3x^{-4}$$

$$= -\frac{3}{x^4}$$

# The derivative of $e^x$

## Theorem

The derivative of  $e^x$  is  $\frac{d}{dx}e^x = e^x$ .

## Proof.

Recall that  $e$  is defined as the value of  $a$  such that the line tangent to  $y = a^x$  has slope 1 at  $(0, 1)$ . That is, if  $f(x) = e^x$ , then  $f'(0) = 1$ : the number  $e$  is defined by this property. This means

$$\begin{aligned}f'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\&= \lim_{h \rightarrow 0} \frac{e^h - 1}{h} \\&= 1.\end{aligned}$$

# The derivative of $e^x$

Proof (continued).

In general,

$$\frac{d}{dx}e^x = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^x(e^h - 1)}{h}$$

$$= e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h}$$

$$= e^x.$$





# The derivative of $e^x$

## Examples:

- ▶  $\frac{d}{dx} e^x \Big|_{x=2} = e^2$
- ▶ If  $f(x) = e^x$ , then  $f'(\ln(3)) = 3$ .
- ▶ Differentiate the function  $-3e^x + 4x^4 - 19\pi x^2 + ex + e$ :

$$-3e^x + 16x^3 - 38\pi x + e.$$

# Homework

Due Thursday, September 18 :

1. Read §3.1 in Stewart.
2. Do the problem set listed online at  
<http://ccjohnson.org/math1060/homework>
3. Seriously, complete ALEKS if you haven't already.