Quiz 2 Key and Grading Guidelines

There are specific notes for the graders about how to grade the problems appearing after the solution in the key, but here are some general guidelines:

1. If you can not easily read a student's handwriting or follow their logic, mark the problem as incorrect. If their work makes no sense whatsoever, write a large red question mark over their work.

2. If a student leaves a problem blank, make a large red slash through the space where the solution would be written (to prevent students from going back and writing in the correct solution later).

3. Students do not need to use the implication symbol, $\implies$, but if they use it incorrectly (e.g., in place of $=$; or using $=$ to mean implication), take off 0.5 points.

4. Take off 0.5 point on any problem where the student makes minor arithmetic mistakes (e.g., adds or multiplies incorrectly, drops a sign, etc.), but otherwise has correct work.

5. If the student magically has the right answer but no supporting work, mark the problem as incorrect. Also, write the names of these students down on a Post-It note and attach that to the top of the piles of tests when you give them back to me so that I can look into whether these students may be cheating.

6. If an indeterminate form like $^0/_0$ appears anywhere in the student's answer, mark the the problem as incorrect and write a big red "NO!" next to $^0/_0$.

7. Students are not to use L'Hôpital's rule. If they try to use L'Hôpital's rule in any problem, mark the problem as incorrect.

Name:
Table:

No electronic devices (phones, calculators, computers, etc.) are allowed during the exam.

You may only use techniques on this quiz which we have discussed in class. In particular, you may not use L'Hôpital's rule in evaluating any of the limits below. You must have provide clear, logical solutions to each of the problems below to receive full credit.

Problem 1 (1 point)

Use the precise definition of the limit at infinity to show $\lim\limits_{x\to\infty} \frac{1}{x^2} = 0$.

Solution

Let $\epsilon > 0$ and set $N = 1/\sqrt{\epsilon}$. We claim that $\left|\frac{1}{x^2} - 0\right| < \epsilon$ if $x > N$. So suppose $x > N$ and note the following:

$$\begin{aligned} & x > N \\ \implies & x > 1/\sqrt{\epsilon} \\ \implies & \frac{1}{1/\sqrt{\epsilon}} > \frac{1}{x} \\ \implies & \frac{1}{x} < \sqrt{\epsilon} \\ \implies & \frac{1}{x^2} < \epsilon \\ \implies & \left|\frac{1}{x^2} - 0\right| < \epsilon. \end{aligned}$$

Thus $\lim_{x\to\infty} 1/x^2 = 0$.

Note to graders: Several students are not using the precise definition of the limit at all. If they do not use the precise definition, then mark the problem as incorrect. In particular, if the student writes something like "when x gets big, $1/x^2$ gets small, so the limit is zero," mark the problem as incorrect since they did not use the precise definition.

Problem 2 (1 points)

Calculate the following limit:

$$\lim_{x\to 0} \sqrt{\frac{2\sin(x)}{x}}.$$

Solution

Note that $\sqrt{x}$ is a continuous function, and so we may write

$$\lim_{x\to 0} \sqrt{\frac{2\sin(x)}{x}} = \sqrt{\lim_{x\to 0} \frac{2\sin(x)}{x}}.$$

Using the limit law stating that the limit of a product is the product of limits, and recalling that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ (one of the first examples of limits we did in class), we

have
$$\lim_{x\to 0}\sqrt{\frac{2\sin(x)}{x}} = \sqrt{2}.$$

Problem 3 (2 points)

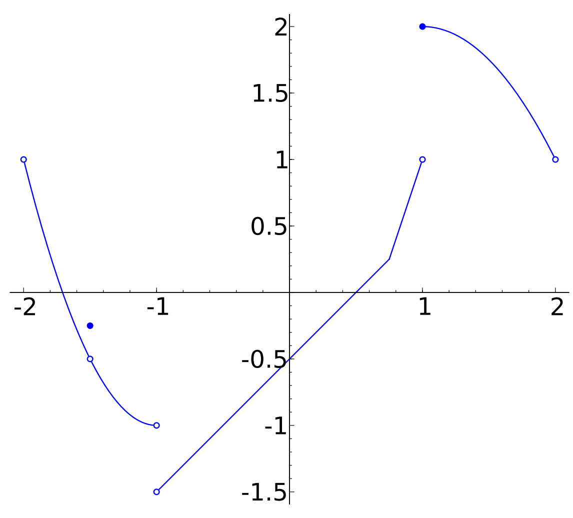
Calculate the following limit:
$$\lim_{x\to 3}\log_2\left(\frac{2x^3-18x}{9x-27}\right).$$

Solution

Recalling that logarithms are continuous, we have the following:
$$\begin{aligned}
\lim_{x\to 3}\log_2\left(\frac{2x^3-18x}{9x-27}\right) &= \log_2\left(\lim_{x\to 3}\frac{2x^3-18x}{9x-27}\right)\\
&= \log_2\left(\lim_{x\to 3}\frac{2x\left(x^2-9\right)}{9(x-3)}\right)\\
&= \log_2\left(\lim_{x\to 3}\frac{2x(x+3)(x-3)}{9(x-3)}\right)\\
&= \log_2\left(\lim_{x\to 3}\frac{2x(x+3)}{9}\right)\\
&= \log_2\left(\frac{2\cdot 3\cdot(3+3)}{9}\right)\\
&= \log_2\left(\frac{36}{9}\right)\\
&= \log_2(4)\\
&= 2.
\end{aligned}$$

Problem 4 (1 point)

Where is the following function not continuous?



Solution
The discontinuities of this function include the intervals those points where the function is not defined, those points where the limit does not exist, and those points where the limit value does not equal the function value.

The function is undefined in the intervals $(-\infty, -2)$, $(2, \infty)$, and the point -1. The limit does not exist at -1 or at 1. The limit value does not equal the function value at -1.5. Thus the set of discontinuities is $(-\infty, -2) \cup \{-1.5, -1, 1\} \cup (2, \infty)$.

Note to graders: Give full credit if the student only lists the discontinuities -1.5, -1, and 1. Take off 0.5 points for each discontinuity they forget; take off the full point if the student does not list 2 or more of these discontinuities.

Problem 5 (1 point)
What are the horizontal asymptotes of the following function?

$$f(x) = \frac{6x^3 - 5x^2 + 7}{2x^3 + x - 5}$$

(To receive credit you must justify your answer mathematically.)

Solution
We need to take the limits of the function at ∞ and $-\infty$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{6x^3 - 5x^2 + 7}{2x^3 + x - 5} &= \lim_{x\to\infty} \frac{6x^3 - 5x^2 + 7}{2x^3 + x - 5} \cdot \frac{1/x^3}{1/x^3} \\
&= \lim_{x\to\infty} \frac{6 - 5/x + 7}{2 + 1/x^2 - 5/x^3} \\
&= \frac{\lim_{x\to\infty} (6 - 5/x + 7)}{\lim_{x\to\infty} (2 + 1/x^2 - 5/x^3)} \\
&= \frac{6 - 0 + 0}{2 + 0 - 0} = 3.
\end{aligned}$$

The limit as x goes to $-\infty$ is exactly the same. Thus there is only one horizontal asymptote, $y = 3$.

Note to graders: If students do not justify their answer by multiplying and dividing by $1/x^3$, then mark the problem as incorrect. If they calculate the limit correctly, but then never state what the horizontal asymptotes are, take off half a point and make a note on their paper that they never said what the horizontal asymptotes were.

Problem 6 (1 point)
What are the horizontal asymptotes of the following function?

$$f(x) = \frac{3x + 1}{\sqrt{5x^2 + 2x - 3}}.$$

(To receive credit you must justify your answer mathematically.)

Solution

We need to take the limits of the function at ∞ and $-\infty$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x+1}{\sqrt{5x^2+2x-3}} &= \lim_{x\to\infty} \frac{3x+1}{\sqrt{5x^2+2x-3}} \cdot \frac{1/x}{1/x} \\
&= \lim_{x\to\infty} \frac{3x+1}{\sqrt{5x^2+2x-3}} \cdot \frac{1/x}{\sqrt{1/x^2}} \\
&= \lim_{x\to\infty} \frac{(3x+1)\cdot 1/x}{\sqrt{(5x^2+2x-3)\cdot 1/x^2}} \\
&= \lim_{x\to\infty} \frac{3+1/x}{\sqrt{5+2/x-3/x^2}} \\
&= \frac{\lim_{x\to\infty}(3+1/x)}{\sqrt{\lim_{x\to\infty}(5+2/x-3/x^2)}} \\
&= 3/\sqrt{5}
\end{aligned}$$

The limit as x goes to $-\infty$ is similar, except for one caveat: when x is negative, $\sqrt{x^2} = -x$, and this introduces a negative in the work above:

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x+1}{\sqrt{5x^2+2x-3}} &= \lim_{x\to\infty} \frac{3x+1}{\sqrt{5x^2+2x-3}} \cdot \frac{1/x}{1/x} \\
&= \lim_{x\to\infty} \frac{3x+1}{\sqrt{5x^2+2x-3}} \cdot \frac{1/x}{-\sqrt{1/x^2}} \\
&= -\lim_{x\to\infty} \frac{(3x+1)\cdot 1/x}{\sqrt{(5x^2+2x-3)\cdot 1/x^2}} \\
&= -\lim_{x\to\infty} \frac{3+1/x}{\sqrt{5+2/x-3/x^2}} \\
&= -\frac{\lim_{x\to\infty}(3+1/x)}{\sqrt{\lim_{x\to\infty}(5+2/x-3/x^2)}} \\
&= -3/\sqrt{5}
\end{aligned}$$

And thus the horizontal asymptotes are $y = 3/\sqrt{5}$ and $y = -3/\sqrt{5}$.

Note to graders: It is okay if the students write $y = \pm 3/\sqrt{5}$ as their answer. If they calculate the limit correctly, but then never state what the horizontal asymptotes are, take off half a point and make a note on their paper that they never said what the horizontal asymptotes were.

Problem 7 (2 points)

Suppose $f(x)$ is a continuous function defined for all real numbers, and that $f(x)$ satisfies the following equation:

$$\lim_{x\to 2} \frac{f(x)^2-16}{2x^2-8} = 5.$$

What is $f(2)$?

Solution
Note that by continuity, $f(2) = \lim_{x\to 2} f(x)$. Note too that because composition of continuous functions yields a continuous function, $f(2)^2 = \lim_{x\to 2} f(x)^2$. Now we simply try to manipulate $\lim_{x\to 2} f(x)$ until we have a value we can calculate.

$$\begin{aligned} f(2)^2 &= \lim_{x\to 2} f(x)^2 \\ &= \lim_{x\to 2} f(x)^2 - 16 + 16 \\ &= \lim_{x\to 2} \frac{f(x)^2 - 16}{2x^2 - 8} \cdot \left(2x^2 - 8\right) + 16 \\ &= \left(\lim_{x\to 2} \frac{f(x)^2 - 16}{2x^2 - 8}\right) \cdot \left(\lim_{x\to 2} \left(2x^2 - 8\right)\right) + 16 \\ &= 5 \cdot 0 + 16 \\ &= 16 \end{aligned}$$

As $f(2)^2 = 16$, we have that $f(2) = 4$ or $f(2) = -4$.

Note to graders: Give full credit if the student says $f(2) = 4$ or $f(2) = -4$.

Problem 8 (1 point)
Use the ϵ-δ definition of limit to show that $\lim\limits_{x\to -1} (4x - 7) = -11$.

Solution: We first need to find a guess for δ. To do this we work backwards from $|(4x - 7) - (-11)| < \epsilon$ until we have an inequality of the form $|x - (-1)| < C$. We will then claim that C is the appropriate choice of δ.

$$\begin{aligned} & |(4x - 7) - (-11)| < \epsilon \\ \implies & |4x - 7 + 11| < \epsilon \\ \implies & |4x + 4| < \epsilon \\ \implies & 4\,|x + 1| < \epsilon \\ \implies & |x - (-1)| < \epsilon/4. \end{aligned}$$

At this point we have a "guess" for δ: $\epsilon/4$. We now need to verify that this choice of δ implies $|4x - 7 - (-11)| < \epsilon$ whenever $0 < |x - (-1)| < \delta$.

Let $\epsilon > 0$ and set $\delta = \epsilon/4$.

$$\begin{aligned} & 0 < |x - (-1)| < \delta \\ \implies & |x + 1| < \epsilon/4 \\ \implies & 4|x + 1| < \epsilon \\ \implies & |4x + 4| < \epsilon \\ \implies & |4x - 7 + 11| < \epsilon \\ \implies & |4x - 7 - (-11)| < \epsilon. \end{aligned}$$

Thus $\lim_{x\to -1} (4x - 7) = -11$.