LECTURE 10 - LIMITS & CONTINUITY OF MULTIVARIABLE FUNCTIONS

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Abstract. In the last lecture we introduced multivariable functions. In this lecture we pave the way for doing calculus with multivariable functions by introducing limits and continuity of such functions.

1. Limits

Informally, the notation $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ means that as the inputs (x, y) gets "really close" to (x_0, y_0), the outputs $f(x, y)$ get "really close" to L. We won't spend the time to make this notion precise, but it comes down to an ϵ-δ definition of the limit, like we saw when we defined limits of vector-valued functions.

Recall that for functions of a single variable, we could talk about right-hand and left-hand limits. That is, if our inputs were from the real line, then we could approach a value from one of two directions. When our inputs live in the plane, there infinitely many different ways for the inputs (x, y) to approach (x_0, y_0). In order for the limit $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ to exist, we must get the same value for all possible ways of approaching (x_0, y_0). Put another way, if any two paths give different values, the limit does not exist.

Example 1.1. Determine whether or not the limit $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ exists.

If we approach $(0, 0)$ from the x axis (so $y = 0$), we have

$$\lim_{x\to 0} \frac{x^2}{x^2} = 1.$$

If we instead approach $(0, 0)$ from the y axis (so $x = 0$), we have

$$\lim_{y\to 0} \frac{-y^2}{y^2} = -1.$$

Since these two values disagree, the limit can not exist.

Example 1.2. Determine whether or not the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}$ exists.

Let's first approach from the x-axis to get

$$\lim_{x\to 0} \frac{0}{x^4+3y^4} = 0.$$

Approaching from the y-axis,

$$\lim_{y\to 0} \frac{0}{x^4+3y^4} = 0.$$

Finally, let's approach from the line $y = x$:

$$\lim_{x\to 0} \frac{x^4}{x^4+3x^4} = \lim_{x\to 0} \frac{x^4}{4x^4} = \frac{1}{4}.$$

These values don't all agree, so the limit can not exist.

Given that there are infinitely-many different paths we'd need to check to see if a limit exists, you may wonder how on earth we're supposed to check if limits exist. The answer is that we need to some tools to help us. The main tool we need is continuity of multivariable functions, since this will let us easily calculate limits.

2. Continuity

We say that a function $f(x,y)$ is *continuous at the point* (x_0,y_0) if the following three conditions are met:

(i) $f(x_0,y_0)$ is defined (i.e., (x_0,y_0) is in the domain of f).
(ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
(iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

If a function is continuous at every point in its domain, then we simply say that the function is *continuous.*

This means that if we know a function is continuous, then its easy to take limits: we just evaluate the function. Now what we need is a repertoire of continuous functions.

Theorem 2.1. *The following types of multivariable functions are continuous:*

(i) Polynomials are always continuous on all of $\mathbb{R}^2$*.*
(ii) Rational functions (ratios of polynomials) are continuous where they're defined (i.e., where the denominator is not zero)
(iii) If $g : \mathbb{R} \to \mathbb{R}$ *is continuous and* $f(x,y) : \mathbb{R} to \mathbb{R}$ *is continuous, then* $g(f(x,y))$ *is continuous.*
(iv) Products and sums of continuous functions are always continuous.

(v) *Quotients of continuous functions are continuous where they're defined.*
(vi) *A composition of continuous functions, in any number of variables, is continuous.*

Let's spend a little bit of time describing each of the types of functions described in the theorem above.

A *polynomial* in two variables is a sum where each term has the form cx^iy^j where c is a real number, and i and j are positive integers. So the following functions are all polynomials:

$$5x^3y^2 + 3x^2 - 2y^3 + 4xy + 5$$
$$-3x^{17} + y^3$$
$$32x^5y^4$$
$$13$$

Polynomials are very nice functions because they're built from the basic operations of arithmetic: addition and multiplication. Since the above theorem tells us that polynomials are continuous, it's very easy to take limits of polynomials.

Example 2.1. Calculate the limit $\lim_{(x,y)\to(3,-1)} (3x^2y - 2y^2 + x)$.

$$\lim_{(x,y)\to(3,-1)} \left(3x^2y - 2y^2 + x\right) = 3{\cdot}3^2{\cdot}(-1) - 2{\cdot}(-1)^2 + 3 = -27 - 2 + 3 = 26$$

A *rational function* in two variables is just a ratio of two polynomials. So the functions below are examples of rational functions:

$$\frac{3x^2 + xy}{4y^2 - x}$$

$$\frac{3}{x + y}$$

$$\frac{x + y^2 - 2x}{2xy}$$

The theorem above tells us that rational functions are continuous everywhere they're defined. So taking limits of rational functions is also very easy, provided that we're taking the limit at a point that's in the domain of the function.

Example 2.2. Calculate the following limit:

$$\lim_{(x,y)\to(1,2)} \frac{3xy - y^2}{4x + 3y}.$$

Notice that the denominator, $4x+3y$ is not zero at the point $(x, y) = (1, 2)$, so this point is in the domain of the rational function, so to take the limit we just evaluate the function:

$$\lim_{(x,y)\to(1,2)} \frac{3xy - y^2}{4x + 3y} = \frac{3 \cdot 1 \cdot 2 - 2^2}{4 \cdot 1 + 3 \cdot 2} = \frac{2}{10} = \frac{1}{5}.$$

The third condition of the theorem above, that a composition of the form $g(f(x, y))$ is continuous when $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous tells us that functions such as the following are continuous:

$$\cos(x + y)$$
$$\tan^{-1}(2x^3 y)$$
$$e^{x-y}$$

Once nice property about such functions is the following:

Theorem 2.2. *If $g : \mathbb{R} \to \mathbb{R}$ is continuous and $f : \mathbb{R}^2 \to \mathbb{R}$ is any function, then*

$$\lim_{(x,y)\to(a,b)} g(f(x, y)) = g\left(\lim_{(x,y)\to(a,b)} f(x, y)\right).$$

provided $\lim_{(x,y)\to(a,b)} f(x, y)$ *is in the domain of g.*

That is, we can move limits inside of continuous functions.

Example 2.3. Calculate the following limit:

$$\lim_{(x,y)\to(-1,4)} e^{x+\sqrt{y}}.$$

$$\lim_{(x,y)\to(-1,4)} e^{x+\sqrt{y}} = e^{\lim_{(x,y)\to(-1,4)}\left(x+\sqrt{y}\right)} = e^{-1+\sqrt{4}} = e$$

Knowing that all of these functions are continuous is very helpful, but there are still times when they aren't able to help us take limits. For example, if we wanted to calculate the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^2 + y^2},$$

continuity doesn't help us since $(0, 0)$ isn't in the domain of $\frac{x^2y^2}{x^2+y^2}$. To evaluate limits like this we need one more tool: the sandwich theorem.

One of the interesting thing about limits of multivariable functions is that, if the limit exists, we can write it as a *double limit*:

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y).$$

Again, this is contingent on the fact that we already know $\lim_{(x,y)\to(a,b)} f(x,y)$! This may seem like a minor observation, but it turns out to be a very useful fact, particularly for proving certain theorems.

3. The Sandwich Theorem

The sandwich theorem tells us that if we have a function that's "sandwiched" between two other functions, then the limit has to be sandwiched as well.

Theorem 3.1 (Sandwich theorem, (aka the squeeze theorem))**.** *Suppose that $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ are three multivariable functions defined near the point $(a, b) \in \mathbb{R}^2$ and such that $f(x, y) \leq g(x, y) \leq h(x, y)$ for all (x, y) near (a, b). If*

$$\lim_{(x,y)\to(a,b)} f(x,y) = L = \lim_{(x,y)\to(a,b)} h(x,y),$$

then we must also have that

$$\lim_{(x,y)\to(a,b)} g(x,y) = L$$

as well.

Example 3.1. Evaluate the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}.$$

Let's notice first that since $(0, 0)$ isn't in the domain of this function, we can't use continuity to help us evaluate this limit. To use the sandwich theorem we need to find two functions which sandwich our $\frac{x^2y^2}{x^2+y^2}$ from above and below.

Let's first notice that $\frac{x^2y^2}{x^2+y^2}$ is never negative, and so we have

$$0 \leq \frac{x^2y^2}{x^2+y^2}.$$

Let's also notice that

$$x^2 \leq x^2 + y^2$$

since adding y^2 will always make x^2 larger (as $y^2 > 0$). This means

$$\frac{x^2}{x^2+y^2} \leq 1$$

If we multiply both sides by y^2 we have $\frac{x^2y^2}{x^2+y^2} \leq y^2$. Now we have our sandwich functions:

$$0 \leq \frac{x^2y^2}{x^2+y^2} \leq y^2.$$

Taking the limit as $(x, y) \to (0, 0)$ we have

$$\begin{aligned} 0 \leq & \quad \frac{x^2y^2}{x^2+y^2} & \leq y^2 \\ \implies \lim_{(x,y)\to(0,0)} 0 \leq & \quad \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} & \leq \lim_{(x,y)\to(0,0)} y^2 \\ \implies 0 \leq & \quad \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} & \leq 0 \\ \implies & \quad \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0 & \end{aligned}$$