HOMEWORK 01

MATH 4100

Problem 1.6:

For each of the following statements, fill in the blank with an easy-to-check statement.

(a) M is a triangular number if and only if $\underline{8M+1}$ is an odd square.

(b) N is an odd square if and only if $\underline{(N-1)/8}$ is triangular.

(c) Prove your answers to (a) and (b).

(a) First suppose that M is a triangular number. Then for some $N \in \mathbb{N}$, $M = \frac{N^2+N}{2}$. Note that

$$\begin{aligned} 8M+1 =& 4(N^2+N)+1 \\ =& 4N^2+4N+1 \\ =& (2N+1)^2, \end{aligned}$$

which is an odd square. Now suppose that $8M+1$ is an odd square, so for some $N \in \mathbb{N}$ we may write $8M+1 = (2N+1)^2$. Thus $8M+1 = 4(N^2+N)+1$, and so $M = \frac{N^2+N}{2}$ and M is triangular.

(b) Suppose N is an odd square, and so $N = (2n+1)^2$ for some $n \in \mathbb{N}$. Thus

$$\begin{aligned} \frac{N-1}{8} =& \frac{(2n+1)^2-1}{8} \\ =& \frac{4n^2+4n+1-1}{8} \\ =& \frac{n^2+n}{2} \end{aligned}$$

and so $\frac{N-1}{8}$ is triangular.

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Now suppose that $\frac{N-1}{8}$ is triangular, and so $\frac{N-1}{8} = \frac{n^2+n}{2}$ for some $n \in \mathbb{N}$. Then

$$\frac{N-1}{8} = \frac{n^2+n}{2}$$

$$\implies N - 1 = 4(n^2 + n)$$

$$\implies N = 4n^2 + 4n + 1 = (2n+1)^2,$$

and so N is an odd square.

Problem 2.1:

(a) Suppose (a, b, c) is a primitive Pythagorean triple. Show that one of a or b must be a multiple of 3.

Solution

Note that for any natural number n, n falls into one of three categories: n is a multiple of 3 (i.e., $n = 3x$ for some x), n is one greater than a multiple of 3 (i.e., $n = 3x + 1$ for some x), or n is two greater than a multiple of 3 (i.e., $n = 3x + 2$ for some x).

Note also that if n is a multiple of 3, then n^2 is also clearly a multiple of 3: if $n = 3x$, then $n^2 = 9x^2 = 3(3x^2)$. If n is one greater than a multiple of 3, then so is n^2: $(3x+1)^2 = 9x^2 + 6x + 1 = 3(3x^2 + 2x) + 1$. If n is two greater than a multiple of 3, then n^2 is one greater than a multiple of 3:

$$\begin{aligned}(3x + 2)^2 =& 9x^2 + 12x + 4\\ =& 9x^2 + 12x + 3 + 1\\ =& 3(3x^2 + 4x + 1) + 1.\end{aligned}$$

In a primitive Pythagorean triple there are three possibilities for a and b: they are both multiples of 3, neither one is a multiple of 3, or one is a multiple of 3 and the other is not. We will show the first two possibilities are impossible, meaning every primitive Pythagorean triple belongs to the third category.

If a and b were both multiples of 3, then a^2 and b^2 would also be multiples of 3, but this would imply that c^2 is a multiply of 3, hence c would be a multiple of 3. This is impossible if (a, b, c) is a primitive Pythagorean triple, however, as the numbers would then all have a common divisor of 3.

Suppose that neither a nor b were multiples of 3. Then a^2 and b^2 would both be one greater than a multiple of 3: say $a^2 = 3x+1$ and $b^2 = 3y+1$. This would mean $c^2 = 3z+2$ where $z = x + y$. However this impossible because c is a square number, but squaring any number only results in a number which is either a multiple of 3 or is one greater than a multiple of 3.

Only the third possibility remains, and so one of a or b is a multiple of 3, but the other is not.

(b) By examining a list of primitive Pythagorean triples, make a guess about when a, b, or c is a multiple of 5. Try to show your guess is correct.

Solution

Exactly one of a, b, or c will be a multiple of 5. To see this, first note that for any $n \in \mathbb{N}$, n falls into one of five categories: $n = 5x + m$ where $m \in \{0, 1, 2, 3, 4\}$. We consider the squares of numbers in each category below:
If $n = 5x$, then $n^2 = 5y$ with $y = 5x^2$.
If $n = 5x + 1$, then $n^2 = 5y + 1$ with $y = 5x^2 + 2x$.
If $n = 5x + 2$, then $n^2 = 5y + 4$ with $y = 5x^2 + 4x$.
If $n = 5x + 3$, then $n^2 = 5y + 4$ with $y = 5x^2 + 6x + 1$.
If $n = 5x + 4$, then $n^2 = 5y + 1$ with $y = 5x^2 + 8x + 3$.
Note that a and b can not both be a multiple of 5, for then c would also be a multiple of 5 and so (a, b, c) would not be primitive: If $a^2 = 5\alpha$ and $b^2 = 5\beta$, then $c^2 = 5(\alpha + \beta)$, but the only numbers whose squares are a multiple of 5 are the multiples of 5 themselves, so c would be a multiple of 5.
Now suppose that neither a nor b is a multiple of 5. If $a^2 = 5\alpha + 1$ and $b^2 = 5\beta + 1$, then $c^2 = 5(\alpha + \beta) + 2$, but this is not possible since all squares are either multiples of 5, or are one or four greater than a multiple of 5. Similarly, we can not have that $a^2 = 5\alpha + 4$ and $b^2 = 5\beta + 4$ as then $c^2 = 5(\alpha+\beta+1)+3$, and there are no numbers whose square is three greater than a multiple of 5. Thus if neither a nor b is a multiple of 5, then one of a^2 and b^2 must be one greater than a multiple of 5, and one must be four greater than a multiple of 5. Thus $c^2 = 5x^2 + 1 + 5y^2 + 4 = 5(x^2 + y^2 + 1)$ and so c^2, and hence c, is a multiple of 5.
Now suppose that neither b nor c is a multiple of 5. We will show that a must be a multiple of 5. Write $a^2 = c^2 - b^2$. If $c^2 = 5\gamma + 1$ and $b^2 = 5\beta + 1$, then $a^2 = 5(\gamma - \beta)$, and so a is a multiple of 5. If $c^2 = 5\gamma + 4$ and $b^2 = 5\beta + 4$, then $a^2 = 5(\gamma - \beta)$ and a is a multiple of 5. If $c^2 = 5\gamma + 4$ and $b^2 = 5\beta + 1$, then $a^2 = 5(\gamma - \beta) + 3$ which is impossible. If $c^2 = 5\gamma + 1$ and $b^2 = 5\beta + 4$, then $a^2 = 5(\gamma - \beta) - 3 = 5(\gamma - \beta - 1) + 2$ which is impossible. Thus if neither b nor c is a multiple of 5, then a must be.
Switching the roles of a and b in the above shows that if neither a nor c is a multiple of 5, then b must be.

Problem 3.3:
Find a formula for all points on the hyperbola $x^2 - y^2 = 1$ whose coordinates are rational numbers.

Solution
To find the formula we consider the point of intersection between the hyperbola $x^2 - y^2 = 1$ and the line with slope $m \in \mathbb{Q}$ through $(-1, 0)$. This line is given by $y = mx + m$. Points on this line have the form $(x, mx + m)$. If such a point is to live on the hyperbola $x^2 - y^2 = 1$, then we must have

$$x^2 - (mx + m)^2 = 1.$$

However, this is simply a quadratic in x:

$$(1 - m^2)x^2 + (-2m^2)x + (-m^2 - 1) = 0.$$

Applying the quadratic formula to solve for x we have

$$\begin{aligned} x =& \frac{2m^2 \pm \sqrt{4m^4 - 4(1 - m^2)(-m^2 - 1)}}{2(1 - m^2)} \\ =& \frac{m^2 \pm \sqrt{m^4 + (1 - m^2)(1 + m^2)}}{1 - m^2} \\ =& \frac{m^2 \pm 1}{1 - m^2}. \end{aligned}$$

Notice that m can not equal 1 or -1. This is because the hyperbola $x^2 - y^2 = 1$ has oblique asymptotes of $y = \pm x$ which it never intersects. But supposing $m \in \mathbb{Q}$ and m is neither 1 nor -1, the above gives us two points of intersection between the hyperbola and the line.

One of these solutions, $\frac{m^2-1}{1-m^2}$, is simply the point $(-1, 0)$ which we already know is on the hyperbola and the line. The other point has x-coordinate $\frac{1+m^2}{1-m^2}$, and so the y-coordinate is $mx + m = \frac{2m}{1-m^2}$.

Thus given a rational slope m, $m \neq \pm 1$, we have a point on the hyperbola with rational coordinates:

$$\left(\frac{1 + m^2}{1 - m^2}, \frac{2m}{1 - m^2} \right).$$

Furthermore, since arithmetic with rational numbers always produces a rational number, it's clear that every point on the hyperbola with rational coordinates is given by the above, except for $(-1, 0)$.