Recall that the Euler totient function φ tells us the number of values between 1 and n which are relatively prime to n:

$$\varphi(9) = 1, 2, 4, 5, 6, 8$$

In general, if p is prime, then $\varphi(p) = p-1$, $\varphi(p^k) = p^k - p^{k-1}$, and $\varphi(mn) = \varphi(m)\varphi(n)$, if $\gcd(m,n) = 1$

Given a number n, let's record all of the divisors of n — including 1 and n:

n	divisors
1	1
2	1, 2
3	1, 3
4	1, 2, 4
5	1, 5
6	1, 2, 3, 6
7	1, 7
8	1, 2, 4, 8
9	1, 3, 9
10	1, 2, 5, 10
11	1, 11
12	1, 2, 3, 4, 6, 12
13	1, 13
14	1, 2, 7, 14

Now let's evaluate φ of each of the divisors of n and sum them up

n	divisors	φ's	sums
1	1	$\varphi(1) = 1$	$= 1$
2	1, 2	$\varphi(1) + \varphi(2) = 1 + 1$	$= 2$
3	1, 3	$\varphi(1) + \varphi(3) = 1 + 2$	$= 3$
4	1, 2, 4	$\varphi(1) + \varphi(2) + \varphi(4) = 1 + 1 + 2$	$= 4$
5	1, 5	$\varphi(1) + \varphi(5) = 1 + 4$	$= 5$
6	1, 2, 3, 6	$\varphi(1) + \varphi(2) + \varphi(3) + \varphi(6) = 1 + 1 + 2 + 2\cdot 1$	$= 6$
7	1, 7	$\varphi(1) + \varphi(7) = 1 + 6$	$= 7$

We might conjecture now that if $d_1, d_2, \ldots, d_r$ are the divisors of n, then $\sum_{i=1}^{r} \varphi(d_i) = n$. Let's test this by considering the case when $n = 61 \cdot 2 \cdot 9 = 1{,}098$.

The divisors of 1,098 are

$$1,\ 2,\ 3,\ 6,\ 9,\ 18,\ 61,\ \overset{61\cdot 2}{\overset{\shortparallel}{122}},\ \overset{61\cdot 3}{\overset{\shortparallel}{183}},\ \overset{61\cdot 6}{\overset{\shortparallel}{366}},\ \overset{61\cdot 9}{\overset{\shortparallel}{549}},\ 1098$$

Taking the sums of φ's:

$$1 + 1 + 2 + 2 + 6 + 6 + 60 + 60 + 120 + 120 + 360 + 360$$
$$= 1{,}098$$

We'll prove our conjecture as a series of steps, starting simple and getting more general.

<u>Lemma</u>

If p is prime, then if $d_1, \ldots, d_r$ contains all the divisors of p, $\sum \varphi(d_i) = p$.

<u>Pf</u>

Here the only divisors are 1 and p, hence $\varphi(1) + \varphi(p) = 1 + (p-1) = p$. $\square$

<u>Lemma</u>

If p is a prime and $d_1, \ldots, d_r$ contains all the divisors of p^k, then $\sum \varphi(d_i) = p^k$.

<u>Pf</u>

Note the divisors of p^k are

$$1, p, p^2, p^3, p^4, \ldots, p^k$$

Hence

$$\sum_{i=0}^{k} \varphi(p^i) = \sum_{i=1}^{k} (p^i - p^{i-1}) + \underbrace{1}_{\varphi(p^0)}$$

$$= p^k \quad \text{(this sum is telescoping)}$$

<u>Lemma</u>

If p and q are prime w/ $d_1, \ldots, d_r$ the divisors of pq, then $\sum \varphi(d_i) = pq$

<u>Pf</u>

The divisors are $1, p, q, pq$.

$\varphi(1) + \varphi(p) + \varphi(q) + \varphi(pq) = 1 + \varphi(p) + \varphi(q) +$

$$= 1 + (p-1) + (q-1) + \varphi(p)\varphi(q)$$

$$= p + q - 1 + (p-1)(q-1)$$

$$= p + q - 1 + pq - p - q + 1$$

$$= pq. \qquad \square$$

<u>Lemma</u>

If m and n are relatively prime, then if $d_1, \dots, d_r$ are the divisors of mn, $\sum \varphi(d_i) = \Big[\underbrace{\textstyle\sum_i \varphi(\Delta_i)}_{\text{div. of } m}\Big]\Big[\underbrace{\textstyle\sum_j \varphi(\delta_j)}_{\text{div. of } n}\Big]$

<u>Pf</u>

Say $\Delta_1, \Delta_2, \dots, \Delta_s$ are divisors of m
$\delta_1, \delta_2, \dots, \delta_t$ are divisors of n

Because m, n are relatively prime, the only common elt b/w the Δ_i's and δ_j's is 1. Hence all the divisors of mn are

$$\begin{array}{l}
\Delta_1\delta_1,\ \Delta_1\delta_2,\ \Delta_1\delta_3,\ \dots,\ \Delta_1\delta_t \\
\Delta_2\delta_1,\ \Delta_2\delta_2,\ \Delta_2\delta_3,\ \dots,\ \Delta_2\delta_t \\
\qquad\qquad \vdots \\
\Delta_s\delta_1,\ \Delta_s\delta_2,\ \Delta_s\delta_3,\ \dots,\ \Delta_s\delta_t
\end{array}$$

Note $\varphi(\Delta_i\delta_j) = \varphi(\Delta_i)\cancel{\varphi}\,\varphi(\delta_j)$, so the sum of all divisors is

$$\sum_{j=1}^{t}\sum_{i=1}^{s}\varphi(\Delta_i\delta_j)$$
$$=\sum_{j=1}^{t}\sum_{i=1}^{s}\varphi(\Delta_i)\varphi(\delta_j)$$
$$=\sum_{j=1}^{t}\left[\varphi(\delta_j)\cdot\sum_{i=1}^{s}\varphi(\Delta_i)\right]$$
$$=\left[\sum_{j=1}^{t}\varphi(\delta_j)\right]\left[\sum_{i=1}^{s}\varphi(\Delta_i)\right]$$

$\square$

<u>Thm</u>
Let $d_1,\dots,d_r$ be the divisors of n. Then
$$\sum_{i=1}^{r}\varphi(d_i)=n$$

<u>Pf</u>
Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ be the prime factorization of n. Note $\gcd(p_i^{\alpha_i},p_j^{\alpha_j})=1$. By the previous lemma, the sum of the divisors is the product of the sum of the divisors of each $p_i^{\alpha_i}$ — which we know sums to $p_i^{\alpha_i}$. Hence the sum is $p_1^{\alpha_1}\cdots p_s^{\alpha_s}=n$.