# Continued Fractions

In the last lecture we saw that each real number $x \in \mathbb{R}$ was an equivalence class of Cauchy sequences of rational numbers. Recall that a sequence of rational numbers is a function

$$x : \mathbb{N} \to \mathbb{Q}$$

where we write $x_n$ for $x(n)$, and denote a sequence as $(x_n)_{n \in \mathbb{N}}$. We say the sequence converges to a rational $L \in \mathbb{Q}$ if for every $\varepsilon > 0$ there exists an $N > 0$ such that $|x_n - L| < \varepsilon$ for all $n > N$.

## Fact

Every convergent sequence of rational numbers is Cauchy.

(We say a sequence is Cauchy if for each $\varepsilon > 0$ there exists an $N > 0$ such that $|x_m - x_n| < \varepsilon$ for all $m, n > N$.)

However, there are Cauchy sequences of rational numbers which do not converge to a rational number. E.g.,

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots$$

This sequence is easily seen to be Cauchy

But the sequence <u>does not</u> converge to a rational number. These Cauchy sequences of rat'l #'s which don't converge represent "holes" in $\mathbb{Q}$. We define the real numbers $\mathbb{R}$ by filling in the "holes:" $\mathbb{R}$ is the set of equivalence classes of Cauchy sequences of rat'l #'s where two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are considered equivalent if their difference converges to $0$: $x_n - y_n \to 0$.

We'll now start describing one particular method of representing a given real number as a ~~so~~ "nice" sequence of rational numbers.

Let $a_1, a_2, \ldots, a_n$ be a finite collection of natural numbers and consider the number $x$ defined as

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n}}}}$$

For example, if $a_1 = 3$, $a_2 = 5$, and $a_3 = 2$ then

$$x = \cfrac{1}{3 + \cfrac{1}{5 + \frac{1}{2}}} = \cfrac{1}{3 + \cfrac{1}{\frac{10+1}{2}}} = \cfrac{1}{3 + \frac{2}{11}}$$

$$= \frac{1}{\left(\frac{33+2}{11}\right)} = \frac{11}{35}$$

Instead of taking the space to write out

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_n}}}}$$

we will write the number $x$ as

$x = [a_1, a_2, \ldots, a_n]$. E.g., $\frac{11}{35} = [3, 5, 2]$.

Prop.

For any finite collection of positive integers $a_1, a_2, \ldots, a_n$ the number $x = [a_1, a_2, \ldots, a_n]$ is a positive rat'l # and $0 \leq x \leq 1$. Furthermore, every rational $x \in [0,1]$ can be written as $[a_1, a_2, \ldots, a_n]$.

Pf.

- Note the result is clear if $n = 1$:

$$[a_1] = \frac{1}{a_1}$$

As $a_1 \in \mathbb{N}$, $\frac{1}{a_1}$ is a rational number in $[0,1]$. We now proceed by induction. Suppose any list of $n-1$ ~~rat'l~~ natural numbers gives a rational number $[b_1, b_2, \ldots, b_{n-1}]$ in $[0,1]$. Now consider $[a_1, a_2, \ldots, a_n]$

$$[a_1, a_2, \ldots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}$$

$$= \frac{1}{a_1 + [a_2, a_3, \ldots, a_n]}$$

By our inductive hypothesis, $[a_2, \ldots, a_n]$ — since it has $n-1$ elts — is a rational number in $[0,1]$. Thus

$$[a_2, \ldots, a_n] = \frac{p}{q}$$

where $p, q \in \mathbb{N}$ and $q > p$. Thus

$$\begin{aligned}
[a_1, \ldots, a_n] &= \frac{1}{a_1 + [a_2, \ldots, a_n]} \\
&= \frac{1}{a_1 + \frac{p}{q}} \\
&= \frac{1}{\left(\frac{a_1 q + p}{q}\right)} \\
&= \frac{q}{a_1 q + p}
\end{aligned}$$

This is clearly a rational number and $a_1 q + p > q$, so it is in $[0,1]$.

- Given $\frac{p}{q} \in \mathbb{Q} \cap [0,1]$, we may apply Euclid's algorithm to $p$ and $q$ to obtain

$$\begin{aligned}
q &= a_1 p + b_1 \\
p &= a_2 b_1 + b_2 \\
b_1 &= a_3 b_2 + b_3 \\
b_2 &= a_4 b_3 + b_4 \\
&\vdots \\
b_i &= a_{i+2} b_{i+1} + b_{i+2} \\
&\vdots \\
b_{n-2} &= a_n b_{n-1}
\end{aligned}$$

Dividing each equation by the ~~remainder~~ quotient $b_i$ we have

$$\frac{q}{p} = a_1 + \frac{b_1}{p}$$

$$\frac{p}{b_1} = a_2 + \frac{b_2}{b_1}$$

$$\frac{b_1}{b_2} = a_3 + \frac{b_3}{b_2}$$

$$\vdots$$

$$\frac{b_i}{b_{i+1}} = a_{i+2} + \frac{b_{i+2}}{b_{i+1}}$$

$$\vdots$$

$$\frac{b_{n-2}}{b_{n-1}} = a_n$$

Now we have

$$\frac{p}{q} = \frac{1}{\left(\frac{q}{p}\right)} = \frac{1}{a_1 + \frac{b_1}{p}}$$

$$= \frac{1}{a_1 + \frac{1}{\left(\frac{p}{b_1}\right)}} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{b_2}{b_1}}}$$

$$= \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\left(\frac{b_1}{2}\right)}}}$$

$$\vdots$$

$$= \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots + \frac{1}{a_n}}}}$$

The integers $a_1, a_2, \dots, a_n$ appearing in the expression $\frac{p}{q} = [a_1, a_2, \dots, a_n]$ are "almost" unique. The only ambiguity comes from the last digit:

$$[a_1, a_2, \dots, a_n] = [a_1, a_2, \dots, a_n - 1, 1]$$

For example

$$[3,5,2] = \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{2}}}$$

$$= \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{1 + \frac{1}{1}}}}$$

$$= [3, 5, 1, 1].$$

If we adopt the convention that the last integer can't be a 1, then the $a_i$ are unique. (There is a similar ambiguity for decimal expansions: $0.1234 = 0.1233999999\ldots$)

Prop.
Every $x \in \mathbb{Q}$ may be written as

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots + \cfrac{1}{a_n}}}}$$

In fact, the digits $[a_1, \ldots, a_n]$ giving $x$ are "almost" unique. The only exception comes from the last quotient:

$$\frac{1}{a_n} = \frac{1}{a_n - 1 + 1} = \frac{1}{(a_n - 1) + \frac{1}{1}}$$

E.g.:

$$\frac{1}{7} = \frac{1}{6 + \frac{1}{1}} \qquad \frac{1}{39} = \frac{1}{38 + \frac{1}{1}}$$

so we have two representation for a given rational $x \in (0,1)$:

$$x = [a_1, a_2, \ldots, a_n] = [a_1, a_2, \ldots, a_n - 1, 1]$$

If we adopt the convention that the last # $a_n$ is not 1, then this representation is unique.

<u>Cor</u>
Every $x \in \mathbb{Q}$ may be written as

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \frac{1}{a_n}}}$$

where $a_0 \in \mathbb{Z}$, and $a_1, \ldots, a_n \in \mathbb{N}$. We write

$$[a_0; a_1, a_2, \ldots, a_n]$$

for this number.

Pf
Take $a_0 = \lfloor x \rfloor$, then $\{x\} := x - \lfloor x \rfloor$ is a rat'l # b/w $(0,1)$, so $\{x\} = [a_1, a_2, \ldots, a_n]$

where $a_0 \in \mathbb{Z}$ and $a_1, a_2, \dots, a_n \in \mathbb{N}$.

Pf

Take $a_0 = \lfloor x \rfloor$ and $y = x - \lfloor x \rfloor$ the "fractional part" of $x$. Then let $a_1, a_2, \dots, a_n$ be the integers in $y = [a_1, a_2, \dots, a_n]$. $\square$

These expressions,

$$[a_0; a_1, a_2, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{a_n}}}$$

are called (finite) continued fractions, and we will use these expressions to get "nice" sequences of rational numbers converging to a given real number.

Before doing that, let's make some observation about these finite continued fractions.

Notice if we take an $x = [a_0; a_1, a_2, \dots, a_n]$ and truncate the $a_i$'s we get a list of rat'l numbers which approach $x$:

$$x_0 = [a_0;]$$
$$x_1 = [a_0; a_1]$$
$$x_2 = [a_0; a_1, a_2]$$
$$\vdots$$
$$x_k = [a_0; a_1, a_2, \dots, a_k]$$
$$\vdots$$
$$x_n = x = [a_0; a_1, a_2, \dots, a_n]$$

For example, if $x = [3; 7, 15, 1, 292]$, then

$$x_0 = [3;] = 3$$
$$x_1 = [3;7] = 3 + \frac{1}{7} = \frac{22}{7}$$
$$x_2 = [3;7,15] = 3 + \cfrac{1}{7+\frac{1}{15}} = \frac{333}{106}$$

$$x_3 = [3;7,15,1] = 3 + \cfrac{1}{7+\cfrac{1}{15+\frac{1}{1}}} = \frac{355}{113}$$
$$= [3;7,16]$$

$$x_4 = [3;7,15,1,292] = \frac{103993}{33102}$$

These intermediate values, $x_k = [a_0; a_1, \ldots, a_k]$ are called the convergents of $x$. Since they're rational numbers we may write

$$x_k = \frac{p_k}{q_k}$$

for some integers $p_k$ and $q_k$.

Let's first see if we can express the $p_k$ and $q_k$ in terms of $a_0, a_1, a_2, \ldots, a_k$: To do this, let's compute $x_0, x_1, x_2, x_3, \ldots$ and see if we can find a pattern:

$$x_0 = a_0 = \frac{a_0}{1}$$
$$x_1 = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1}$$
$$x_2 = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2}} = a_0 + \frac{1}{\left(\frac{a_2 a_1 + 1}{a_2}\right)}$$
$$= a_0 + \frac{a_2}{a_2 a_1 + 1} = \frac{a_2 a_1 a_0 + a_2 + a_0}{a_2 a_1 + 1}$$

$$x_3 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}}}$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{\left(\cfrac{a_3 a_2 + 1}{a_3}\right)}}$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{a_3}{a_3 a_2 + 1}}$$

$$= a_0 + \cfrac{1}{\left(\cfrac{a_3 a_2 a_1 + a_3 + a_1}{a_3 a_2 + 1}\right)}$$

$$= a_0 + \frac{a_3 a_2 + 1}{a_3 a_2 a_1 + a_3 + a_1}$$

$$= \frac{a_3 a_2 a_1 a_0 + a_3 a_0 + a_1 a_0 + a_3 a_2 + 1}{a_3 a_2 a_1 + a_3 + a_1}$$

So we've found the following numerators & denominators:

$$p_0 = a_0$$

$$p_1 = a_1 a_0 + 1$$

$$\begin{aligned} p_2 &= a_2 a_1 a_0 + a_2 + a_0 \\ &= a_2 (a_1 a_0 + 1) + a_0 \\ &= a_2 p_1 + p_0 \end{aligned}$$

$$\begin{aligned} p_3 &= a_3 a_2 a_1 a_0 + a_3 a_2 + a_3 a_0 + a_1 a_0 + 1 \\ &= a_3 (a_2 a_1 a_0 + a_2 + a_0) + a_1 a_0 + 1 \\ &= a_3 p_2 + p_1 \end{aligned}$$

Similarly for the denominators:

$$q_0 = 1$$
$$q_1 = a_1$$
$$q_2 = a_2 a_1 + 1$$
$$= a_2 q_1 + q_0$$
$$q_3 = a_3 a_2 a_1 + a_3 + a_1$$
$$= a_3 (a_2 a_1 + 1) + a_1$$
$$= a_3 q_2 + q_1$$

We may conjecture, then, that the $p_k$ and $q_k$ satisfy a recurrence relation — at least if $k \geq 2$:

$$p_k = a_k p_{k-1} + p_{k-2}$$
$$q_k = a_k q_{k-1} + q_{k-2}$$

To prove this we will temporarily adopt the following notation: In the expression $[b_1, b_2, \ldots, b_n]$ we will let $b_i \in \mathbb{Q}$ instead of just $b_i \in \mathbb{N}$. Then we may write

$$x_k = [a_0; a_1, a_2, \ldots, a_{k-2}, a_{k-1}, a_k]$$

$$= [a_0; a_1, a_2, \ldots, a_{k-2}, a_{k-1} + \frac{1}{a_k}]$$

Note this expression for $x_k$ has $k$ values instead of the usual $k+1$.

Now let $x_i'$ be the number obtained from the first $i+1$ terms of $[a_0; a_1, a_2, \ldots, a_{k-1} + \frac{1}{a_k}]$.

Note
$$x_0 = x_0' = [a_0;]$$
$$x_1 = x_1' = [a_0; a_1]$$
$$x_2 = x_2' = [a_0; a_1, a_2]$$
$$\vdots$$
$$x_{k-2} = x_{k-2}' = [a_0; a_1, a_2, \ldots, a_{k-2}]$$

But
$$\begin{aligned} x_{k-1} &= [a_0; a_1, \ldots, a_{k-2}, a_{k-1}] \\ &\neq [a_0; a_1, \ldots, a_{k-2}, a_{k-1} + \frac{1}{a_k}] \\ &= x_{k-1}' \end{aligned}$$

Using this notation we'll write $x_k' = \frac{p_k'}{q_k'}$.

Notice
$$\frac{p_k}{q_k} = \frac{p_{k-1}'}{q_{k-1}'}$$

Now we can easily prove our earlier conjecture by induction.

<u>Proposition</u>

The numerators $p_k$ and denominators $q_k$ of the convergents of the continued fraction $x = [a_0; a_1, a_2, \ldots, a_n]$ satisfy the following recurrences:

$$p_0 = a_0 \qquad q_0 = 1$$
$$p_1 = a_1 a_0 + 1 \qquad q_1 = a_1$$
$$p_k = a_k p_{k-1} + p_{k-2} \qquad q_k = a_k q_{k-1} + q_{k-2} \quad \text{for } k \geq 2.$$

Pf

We've already proven the proposition for $k=0, 1,$ and $2$ above. Suppose now that the proposition holds for $k-1$:

$$p_{k-1} = a_{k-1} p_{k-2} + p_{k-3}$$
$$q_{k-1} = a_{k-1} q_{k-2} + q_{k-3}$$

As noted above,

$$\frac{p_k}{q_k} = \frac{p'_{k-1}}{q'_{k-1}}$$

However, $p'_{k-1}$ and $q'_{k-1}$ are ~~partial fractio~~ convergents of a continued fraction of $k-1$ terms. Hence by the induction hypothesis:

$$p'_{k-1} = \left(a_{k-1} + \frac{1}{a_k}\right) p'_{k-2} + p'_{k-3}$$

$$q'_{k-1} = \left(a_{k-1} + \frac{1}{a_k}\right) q'_{k-2} + q'_{k-3}$$

We've already seen that $p'_{k-2} = p_{k-2}$ and $q'_{k-2} = q_{k-2}$ and likewise for $k-3$. Thus,

$$\frac{p_k}{q_k} = \frac{p'_{k-1}}{q'_{k-1}} = \frac{\left(a_{k-1} + \frac{1}{a_k}\right) p_{k-2} + p_{k-3}}{\left(a_{k-1} + \frac{1}{a_k}\right) q_{k-2} + q_{k-3}}$$

$$= \frac{a_k\left[\left(a_{k-1} + \frac{1}{a_k}\right) p_{k-2} + p_{k-3}\right]}{a_k\left[\left(a_{k-1} + \frac{1}{a_k}\right) q_{k-2} + q_{k-3}\right]}$$

$$= \frac{a_k a_{k-1} p_{k-2} + p_{k-2} + a_k p_{k-3}}{a_k a_{k-1} q_{k-2} + q_{k-2} + a_k q_{k-3}}$$

$$= \frac{a_k (a_{k-1} p_{k-2} + p_{k-3}) + p_{k-2}}{a_k (a_{k-1} q_{k-2} + q_{k-3}) + q_{k-2}}$$

$$= \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}. \qquad \square$$

This recurrence relation has several important consequences:

<u>Prop.</u> Let $x_k = \frac{p_k}{q_k}$ be a convergent of $x = [a_0; a_1, a_2, \ldots, a_3]$. Then

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k \quad \text{if } k \geq 1.$$

<u>Pf</u> Note this holds if $k=1$:

$$q_1 p_0 - p_1 q_0 = a_1 a_0 - (a_1 a_0 + 1) \cdot 1$$
$$= -1$$

Now suppose $q_i p_{i-1} - p_i q_{i-1} = (-1)^i$ for all $1 \leq i < k$. Then

$$q_k p_{k-1} - p_k q_{k-1} = (a_k q_{k-1} + q_{k-2}) p_{k-1} - (a_k p_{k-1} + p_{k-2}) q_{k-1}$$

$$= q_{k-2} p_{k-1} - p_{k-2} q_{k-1} + a_k (q_{k-1} p_{k-1} - p_{k-1} q_{k-1})$$

$$= -(q_{k-1} p_{k-2} - p_{k-1} q_{k-2}) = -(-1)^{k-1} = (-1)^k. \qquad \square$$

Corollary

$$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}}$$

Pf

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k$$

$$\Rightarrow \frac{1}{q_k q_{k-1}} \cdot (q_k p_{k-1} - p_k q_{k-1}) = \frac{(-1)^k}{q_k q_{k-1}}$$

$$\Rightarrow \frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}}$$ $\square$

Cor

$p_k$ and $q_k$ are relatively prime.

Pf

If $k$ is even,

$$q_k p_{k-1} - p_k q_{k-1} = 1$$

$$\Rightarrow q_k x + p_k y = 1 \text{ where } x = p_{k-1},\ y = -q_{k-1}$$

so $\gcd(p_k, q_k) = 1$.

If $k$ is odd,

$$q_k p_{k-1} - p_k q_{k-1} = -1$$

$$\Rightarrow q_k \cdot (-p_{k-1}) + p_k q_{k-1} = 1$$

$$\Rightarrow q_k x + p_k y = 1 \text{ where } x = -p_{k-1},\ y = q_{k-1}$$

so $\gcd(p_k, q_k) = 1$.

$\square$

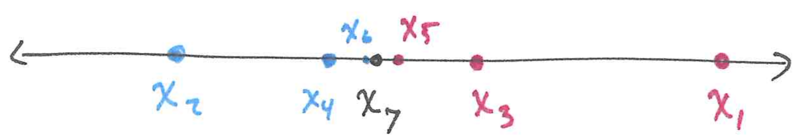
Notice that both $p_k$ and $q_k$ are strictly increasing:

$$0 < p_1 < p_2 < \cdots < p_n$$
$$0 < q_1 < q_2 < \cdots < q_n$$

As a simple example illustrates the sequences $\frac{p_k}{q_k}$ is neither increasing nor decreasing. For example, consider the convergents of $x = [1, 2, 3, 4, 5, 6, 7]$

$$x_1 = [1] = 1$$
$$x_2 = [1,2] = 2/3 \approx 0.66666$$
$$x_3 = [1,2,3] = 7/10 \approx 0.7$$
$$x_4 = [1,2,3,4] = 30/43 \approx 0.69767 4418$$
$$x_5 = [1,2,3,4,5] = 157/225 \approx 0.69777777$$
$$x_6 = [1, \ldots, 6] = 972/1393 \approx 0.69777458$$
$$x_7 = [1, \ldots, 7] = 6961/9976 \approx 0.69777465$$

Notice, however, that the convergents $x_k = \frac{p_k}{q_k}$ alternate between being a little above and a little below $x_7$.




So it seems that if we focus only on the even convergents, $x_{2k}$, we get an increasing seq., but the odd convergents, $x_{2k-1}$ give a decreasing sequence. The next theorem and its corollary justify this.

Thm

For $k \geq 2$,

$$q_k p_{k-2} - p_k q_{k-2} = (-1)^{k-1} a_k$$

Pf

$$\begin{aligned} & q_k p_{k-2} - p_k q_{k-2} \\ &= (a_k q_{k-1} + q_{k-2}) p_{k-2} - (a_k p_{k-1} + p_{k-2}) q_{k-2} \\ &= a_k (q_{k-1} p_{k-2} - p_{k-1} q_{k-2}) \\ &= a_k (-1)^{k-1} \end{aligned}$$

$\square$

Dividing both sides of the equation appearing above by $q_k q_{k-2}$ gives the following

Cor

For $k \geq 2$

$$\frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} = \frac{(-1)^{k-1} a_k}{q_k q_{k-2}}$$

$\square$

So if $k$ is even, then $\frac{(-1)^{k-1} a_k}{q_k q_{k-2}}$ is negative, meaning

$$\frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} < 0$$

$$\Rightarrow \frac{p_{k-2}}{q_{k-2}} < \frac{p_k}{q_k}$$

So the even convergents are increasing.

If $k$ is odd, $\frac{(-1)^{k-1} a_k}{q_k q_{k-2}}$ is positive, and

$$\frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} > 0$$

$$\Rightarrow \frac{p_{k-2}}{q_{k-2}} > \frac{p_k}{q_k}$$

and the odd convergents are decreasing.

With all of these results about finite continued fractions at our disposal, we are now ready to consider infinite continued fractions.

Let $x$ be a real number — say $x = e \approx 2.71828$. Note $x$ is between 2 and 3, so we may write $x = 2 + y$ where $0 \leq y < 1$. In particular, $y = 0.71828\ldots$

$$x = 2 + 0.71828\ldots$$

Since $0 \leq y < 1$, $\frac{1}{y} > 1$. Here, $\frac{1}{y} \approx 1.3922\ldots$
Thus

$$\begin{aligned} x &= 2 + y \\ &= 2 + \frac{1}{1/y} \\ &= 2 + \frac{1}{1.3922} \end{aligned}$$

But we could repeat this process starting from 1.3922...: $1.3922\ldots = 1 + \frac{1}{y}$

$$x = 2 + \cfrac{1}{1 + 0.3922\ldots}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{1/0.3922\ldots}}$$

But $1/0.3922\ldots = 2.54964\ldots$, so

$$x = 2 + \cfrac{1}{1 + \cfrac{1}{2.54964\ldots}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + 0.54964\ldots}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{(1/0.54964\ldots)}}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1.81935\ldots}}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{(1/0.81933\ldots)}}}}$$

Repeating this process ad infinitum we obtain an *(infinite) continued fraction*

$$x = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{8 + \cdots}}}}}}}}}}}$$

which we of course prefer to write as

$$[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$$

The most obvious question to ask now is does this process of infinitely-many additions and divisions actually make sense? If I hand you a list of infinitely-many natural numbers to use in building a continued fraction, can you actually evaluate that expression to get a number? That is, how do you know

$$[a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

actually represents a real number?

Notice that truncating $[a_0; a_1, a_2, \ldots]$ to get a finite list does give us a rational number, $x_n = \frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$ called the $n^{th}$ convergent of $[a_0; a_1, a_2, \ldots]$. Thus we have a natural way to associate a sequence of rational numbers to an infinite continued fraction by considering the convergents. Asking if we can evaluate $[a_0; a_1, a_2, \ldots]$ is essentially asking if the limit of $(x_n)_{n\in\mathbb{N}}$, with $x_n$ the $n^{th}$ convergent, exists. (Compare this to infinite series you learned about in calculus: you say the series $\sum_{k=1}^{\infty} b_k$ converges precisely when the sequence of partial sums $S_n = \sum_{k=1}^{n} b_k$ converges.)

Our first goal, then, is to understand when the sequence of convergents does in fact converge.

All of our previous results about finite continued fractions (e.g., the recurrence relations of the numerators/denominators of convergents, odd convergents decreasing, even convergents increasing, ...) still apply since each convergent is a finite continued fraction, and we obtain more convergents only by adding to the end of the list $[a_0; a_1, \ldots, a_n]$.

Our previous results thus imply the following:

Prop.
Every odd convergent is greater than every even convergent.

Pf
Recall that $\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n-1}}$.
Hence

$$x_{n-1} - x_n > 0 \quad \text{if } n \text{ is even}$$
$$\Rightarrow x_{n-1} > x_n \quad \text{if } n \text{ even}$$

so each even convergent is smaller than the preceding odd convergent. Similarly,

$$x_{n-1} - x_n < 0 \quad \text{if } n \text{ odd}$$
$$\Rightarrow x_{n-1} < x_n \quad \text{if } n \text{ odd}$$

and so each odd convergent is bigger than the preceding even convergent. Recalling the odd convergents are strictly decreasing and the even convergents strictly increasing provides the result. (Exercise: use induction to make this precise.) $\square$

Now we use a fact we won't prove, but can be found in any book on real analysis:

Fact
Every bounded, monotonic (i.e., increasing or decreasing) sequence is Cauchy. $\square$

This implies the even convergents $(x_{2n})_{n\in\mathbb{N}}$ are Cauchy, so converge to a real number and likewise the odd convergents $(x_{2n-1})_{n\in\mathbb{N}}$ are Cauchy and converge to a real number. If both sequences converged to the same real number, then the sequence of convergents converges.

That both sequences converge to the same real number follows from our earlier result that

$$x_{n-1} - x_n = \frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^k}{q_k q_{k-1}}$$

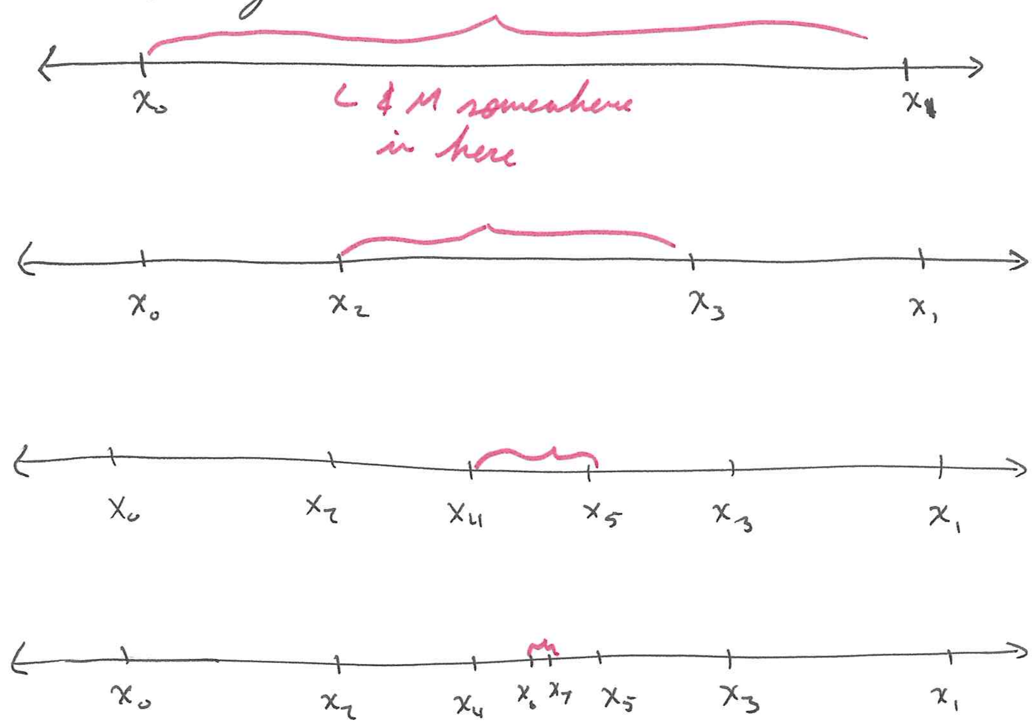
$$\Rightarrow |x_n - x_{n-1}| = \frac{1}{q_n q_{n-1}}$$

Notice $|x_n - x_{n-1}| \to 0$ since $q_n \to \infty$.

Putting all of this together the even convergents converge to some $L$, and the odd convergents converge to some $M$. Note $M$ is smaller than every odd convergent, yet larger than every even convergent. Similarly, $L$ is larger than every even convergent and smaller than every odd convergent. However, the distance b/w even and odd convergents becomes arbitrarily small, so the distance b/w $L$ and $M$ is arbitrarily small:

$$|L-M| < \varepsilon \text{ for all } \varepsilon > 0 \Rightarrow |L-M| = 0 \Rightarrow L = M.$$

Long story short: Every infinite continued fraction converges.




$\vdots$

Let's now take a moment to calculate some continued fractions.

$$\sqrt{2} = 1 + (\sqrt{2}-1)$$

$$= 1 + \cfrac{1}{\left(\frac{1}{\sqrt{2}-1}\right)}$$

$$= 1 + \cfrac{1}{2 + \sqrt{2} - 1}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{\left(\frac{1}{\sqrt{2}-1}\right)}}$$

$$\sqrt{2} - 1 \approx 0.41421356\ldots$$

$$\frac{1}{\sqrt{2}-1} \approx 2.41421356\ldots$$

$$\frac{1}{\sqrt{2}-1} = \frac{2-1}{\sqrt{2}-1} = \frac{2-\sqrt{2}+\sqrt{2}-1}{\sqrt{2}-1}$$

$$= \frac{\sqrt{2}(\sqrt{2}-1)+\sqrt{2}-1}{\sqrt{2}-1}$$

$$= \sqrt{2} + \frac{\sqrt{2}-1}{\sqrt{2}-1}$$

$$= \sqrt{2} + 1$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\left(\frac{1}{\sqrt{2}-1}\right)}}}} = \cdots = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

Thus the continued fraction of $\sqrt{2}$ is $[1; 2, 2, 2, 2, \ldots]$

Recalling that rational numbers give finite continued fractions, we know that infinite continued fractions give irrational numbers. So you can take the above as proof that $\sqrt{2}$ is irrational.

Another example:

$$\sqrt{3} = 1 + (\sqrt{3} - 1) \qquad \approx 1.73205\ldots$$

$$= 1 + \frac{1}{\left(\frac{1}{\sqrt{3}-1}\right)} \qquad \frac{1}{\sqrt{3}-1} \approx 1.366025$$

$$= 1 + \frac{1}{1 + \left[\frac{1}{\sqrt{3}-1} - 1\right]}$$

$$= 1 + \frac{1}{1 + \frac{1}{\sqrt{3}-1} - \frac{\sqrt{3}-1}{\sqrt{3}-1}} \qquad 2-$$

$$= 1 + \frac{1}{1 + \frac{2-\sqrt{3}}{\sqrt{3}-1}}$$

$$= 1 + \frac{1}{1 + \frac{1}{\left(\frac{1}{\frac{2-\sqrt{3}}{\sqrt{3}-1}}\right)}} \qquad \frac{1}{\left(\frac{2-\sqrt{3}}{\sqrt{3}-1}\right)} = 2.73205\ldots$$

(Write $\frac{\sqrt{3}-1}{2-\sqrt{3}}$ as $\sqrt{3}+1$)

$$= 1 + \frac{1}{1 + \frac{1}{2 + 1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{\cdot}}}}}$$

So $\sqrt{3} = [1; 1, 2, 1, 2, 1, 2, \ldots]$

As one final example, let's consider $\sqrt{5}$

$$\sqrt{5} = 2 + \cfrac{1}{\left(\cfrac{1}{\sqrt{5}-2}\right)}$$

$$\sqrt{5} \approx 2.23606\ldots$$

$$\frac{1}{\sqrt{5}-2} = 4.23606\ldots$$

$$= \sqrt{5} + 2$$

$$= 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cdots}}}}$$

So $\sqrt{5} = [2; 4, 4, 4, \ldots]$

So it appears that these square roots have periodic continued fractions. In fact, a more general statement is true. We will show that a number $x^{\in \mathbb{R}}$ is a quadratic irrational number iff the continued fraction expansion $x = [a_0; a_1, a_2, \ldots]$ is eventually periodic.

Let's first define these terms. A real number $x$ is a quadratic irrational if it is the root of a polynomial $\alpha x^2 + \beta x + \gamma$, and is irrational. Saying $[a_0, a_1, \ldots, a_n]$ is eventually periodic means ~~if~~ there exists a number $N$ and a ~~so~~ number $p$ such that for each $k > N$, $a_k = a_{k+p} = a_{k+2p} = a_{k+3p} = \cdots = a_{k+mp} = \cdots$

| Eventually periodic | Not eventually periodic |
|---|---|
| $[1; 1, 1, 1, 1, 1, \cdots]$ | $[1; 2, 3, 4, 5, 6, 7, 8, 9, 10, \cdots]$ |
| $[1; 2, 3, 1, 2, 3, 1, 2, 3, \ldots]$ | $[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots] = e$ |
| $[1; 2, 3, 4, 5, 4, 5, 4, 5, 4, 5, \ldots]$ | |
| $[4; 8; 16; 32; 64; 128; 256; 32; 64; 128; 256; \ldots]$ | |

To prove this we need to introduce one more piece of terminology. Suppose $x \in \mathbb{R}$ has continued fraction expansion $x = [a_0; a_1, a_2, \dots]$. For each $n \geq 0$, define a number $r_n$ by

$$r_n = [a_n; a_{n+1}, a_{n+2}, \dots]$$

For example, if $x = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$, then

$$r_0 = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$$
$$r_1 = [1; 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$$
$$r_2 = [2; 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$$
$$r_3 = [1; 1, 4, 1, 1, 6, 1, 1, 8, \dots]$$

We'll call the $r_n$ the partial remainders of $x$.

Notice

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n + \cfrac{1}{a_{n+1} + \cdots}}}}}}}$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{r_n}}}}}}$$

so we may write

$$x = [a_0; a_1, a_2, a_3, \dots, a_{n-1}, r_n].$$

Note $a_n = \lfloor r_n \rfloor$

Notice each quadratic irrational may be written as

$$x = \frac{A \pm \sqrt{D}}{B}$$

(This is just the quadratic formula. If $x = \alpha x^2 + \beta x + \gamma$, then take $A = -\beta$, $D = \beta^2 - 4\alpha\gamma$, $B = 2\alpha$.)

<u>Thm</u>

The partial remainders of the quadratic irrational

$$x = \frac{A \pm \sqrt{D}}{B}$$

are given by

$$r_n = \frac{A_n + \sqrt{D}}{B_n}$$

where $A_0 = A$, $B_0 = B$ and

$$A_{n+1} = a_n B_n - A_n$$

$$B_{n+1} = \frac{D - A_{n+1}^2}{B_n}$$

($a_n$ are partial quotients of $x = [a_0; a_1, a_2, \dots]$)

Moreover, if $B_0 \mid D - A_0^2$, then $B_n \mid D - A_n^2$ for all $n$, and each $A_n, B_n$ is an integer.

Pf

Note
$$r_{n+1} = \frac{1}{r_n - a_n}$$

$$= \frac{1}{\left(\frac{A_n + \sqrt{D}}{B_n}\right) - a_n}$$

$$= \frac{B_n}{A_n + \sqrt{D} - a_n B_n}$$

Rationalizing the denominator gives

$$r_{n+1} = \frac{B_n}{A_n + \sqrt{D} - a_n B_n}$$

$$= \frac{B_n}{A_n - a_n B_n + \sqrt{D}} \cdot \frac{A_n - a_n B_n - \sqrt{D}}{A_n - a_n B_n - \sqrt{D}}$$

$$= \frac{B_n (A_n - a_n B_n - \sqrt{D})}{(A_n - a_n B_n)^2 - D}$$

$$= \frac{A_n - a_n B_n - \sqrt{D}}{\left[ \frac{(A_n - a_n B_n)^2 - D}{B_n} \right]}$$

$$= \frac{a_n B_n - A_n + \sqrt{D}}{\left[ \frac{D - (A_n - a_n B_n)^2}{B_n} \right]}$$

Hence $A_{n+1} = a_n B_n - A_n$, and $B_{n+1} = \dfrac{D - A_{n+1}^2}{B_n}$.

The second part of the theorem is left as an easy exercise. $\square$

Notice this second part implies if $\dfrac{A + \sqrt{D}}{B}$ is a root of $\alpha x^2 + \beta x + \gamma$, then each $A_n$ and $B_n$ appearing above are integers.

To show the continued fraction of a quadratic irrational is eventually periodic, we simply need to show some $r_n$ repeats — if $r_{m+p} = r_m$, then the partial quotients $a_m, a_{m+1}, a_{m+2}, \dots, a_{m+p-1}$ are forced to repeat as well.

As $A_n, B_n$ are integers, it suffices to show $\{A_0, B_0, A_1, B_1, \dots\}$ is finite — this will force the $A_n$'s & $B_n$'s to repeat (by the pigeonhole principle), hence the $r_n$'s must repeat.

To show this we need to use the conjugate of a quad. irrat. If $y = \alpha + \beta\sqrt{\delta}$, we call $\bar{y} = \alpha - \beta\sqrt{\delta}$ its *conjugate*

<u>Lemma</u>

Suppose $y = \alpha + \beta\sqrt{\delta}$ is greater than 1, and $-1 < \bar{y} < 0$. If $y = a + \frac{1}{z}$ where $a = \lfloor y \rfloor$, then $z > 1$ and $-1 < \bar{z} < 0$.

Pf

As $y > 1$, $a = \lfloor y \rfloor \geq 1$, and $\frac{1}{z} < 1 \Rightarrow z > 1$.

Note $\bar{y} = a + \frac{1}{\bar{z}}$. Hence $-1 < a + \frac{1}{\bar{z}} < 0$,

$$\Rightarrow -1 - a < \frac{1}{\bar{z}} < -a$$

$$\Rightarrow \frac{1}{\bar{z}} < 0$$

$$\Rightarrow \bar{z} < 0.$$

Since $a \geq 1$, $\frac{1}{\bar{z}} < -a$ means $\frac{1}{\bar{z}} < -1 \Rightarrow 1 > -\bar{z} \Rightarrow -1 < \bar{z}$.

$\square$

For the next lemma we'll express $x$ in terms of its partial remainders

## Lemma

Let $x = [a_0; a_1, \ldots]$; $x_n = [a_0; a_1, \ldots, a_n] = \frac{p_n}{q_n}$ and $r_n = [a_n; a_{n+1}, \ldots]$. Then for each $n \geq 2$,

$$x = \frac{r_n p_{n-1} + p_{n-2}}{r_n q_{n-1} + q_{n-2}}$$

Pf

For $n = 2$ we have

$$x = [a_0; a_1, r_2]$$

$$= a_0 + \frac{1}{a_1 + \frac{1}{r_2}} = a_0 + \frac{1}{\left(\frac{r_2 a_1 + 1}{r_2}\right)}$$

$$= a_0 + \frac{r_2}{r_2 a_1 + 1} = \frac{a_0(r_2 a_1 + 1) + r_2}{r_2 a_1 + 1}$$

$$= \frac{r_2 a_1 a_0 + a_0 + r_2}{r_2 a_1 + 1} = \frac{r_2(a_1 a_0 + 1) + a_0}{r_2 a_1 + 1}$$

$$= \frac{r_2 p_1 + p_0}{r_2 q_1 + q_0}$$

Now suppose the formula holds for $n-1$: $x = \frac{r_{n-1} p_{n-2} + p_{n-3}}{r_{n-1} q_{n-2} + q_{n-3}}$.

Then

$$x = \frac{r_{n-1} p_{n-2} + p_{n-3}}{r_{n-1} q_{n-2} + q_{n-3}} = \frac{(a_{n-1} + \frac{1}{r_n}) p_{n-2} + p_{n-3}}{(a_{n-1} + \frac{1}{r_n}) q_{n-2} + q_{n-3}}$$

$$=$$

Now we can show the following:

<u>Lemma</u>

The partial remainders of a quadratic irrational $x$ satisfy $-1 < \overline{r_k} < 0$ and $r_k > 1$ for all sufficiently large $k$.

Pf

Note

$$x = \frac{r_k p_{k-1} + p_{k-2}}{r_k q_{k-1} + q_{k-2}}$$

$$\Rightarrow \overline{x} = \frac{\overline{r_k} p_{k-1} + p_{k-2}}{\overline{r_k} q_{k-1} + q_{k-2}}$$

$$\Rightarrow (\overline{r_k} q_{k-1} + q_{k-2}) \overline{x} = \overline{r_k} p_{k-1} + p_{k-2}$$

$$\Rightarrow \overline{r}_k (\overline{x} q_{k-1} - p_{k-1}) = -\overline{x} q_{k-2} + p_{k-2}$$

$$\Rightarrow \overline{r}_k = \frac{-\overline{x} q_{k-2} + p_{k-2}}{\overline{x} q_{k-1} - p_{k-1}}$$

$$= -\frac{q_{k-2}}{q_{k-1}} \left( \frac{\overline{x} - \frac{p_{k-2}}{q_{k-2}}}{\overline{x} - \frac{p_{k-1}}{q_{k-1}}} \right)$$

as $k \to \infty$, this approaches $-\frac{q_{k-2}}{q_{k-1}}$, hence $\overline{r_k}$ is $< 0$ for large $k$.

Note $\overline{r_k} < 0 \Longrightarrow -1 < \overline{r_{k+1}} < 0$ b/c $\overline{r_{k+1}} = \frac{1}{\overline{r_k} - a_k}$ and $a_k \geq 1$

To show only fin. many $A_k$, $B_k$ nov, note for large $k$,

$$r_k > 1$$

$$-1 < \bar{r}_k < 0$$

That is

$$\frac{A_k + \sqrt{D}}{B_k} > 1 \qquad -1 < \frac{A_k - \sqrt{D}}{B_k} < 0$$

Note

$$\frac{A_k + \sqrt{D}}{B_k} - \frac{A_k - \sqrt{D}}{B_k} = \frac{2\sqrt{D}}{B_k} > 0$$

so $B_k > 0$. Also

$$A_{k+1} = a_k B_k - A_k < r_k B_k - A_k = \sqrt{D}$$

$$\Rightarrow A_{k+1} < \sqrt{D}$$

and $\dfrac{A_k + \sqrt{D}}{B_k} > 1 \Rightarrow B_k < A_k + \sqrt{D} < 2\sqrt{D}$.

As $-B_k < A_k - \sqrt{D}$, $A_k > -B_k + \sqrt{D} > -\sqrt{D}$. So

$-\sqrt{D} < A_k < \sqrt{D}$ and $0 < B_k < 2\sqrt{D}$.

$\therefore$ only fin. many $A_k$ & $B_k$

This proves the following theorem of Lagrange:

**Thm**

The continued fraction expansion of any quadratic irrational is eventually periodic.