Single Variable Calculus II

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Spring 2024

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Introduction to the Course

The reason a lot of people do not recognize opportunity is because it usually goes around wearing overalls looking like hard work.

THOMAS EDISON

Welcome to Math 255, the second course of the calculus sequence at Western Carolina University. This course is meant to continue where a typical Calculus I course (such as WCU's Math 153) leaves off. In particular, we will assume that students are comfortable with the material covered in Calculus I, such as limits, derivatives, and basic integrals, as well as material that is prerequisite to Calculus I, namely basic algebra and trigonometry. There is a short set of appendices to these notes reviewing some of the material from algebra, trig, and Calculus I that students will need in this course, but may have forgotten.

Over the course of the semester we will continue to develop our understanding of integration by learning more sophisticated integration techniques and applications of integrals, and we will also introduce the ideas of series, Taylor polynomials, parametric curves, and polar coordinates.

The remainder of this introductory chapter is meant to prepare you for what to expect in Calculus II this semester, including the format of the course, the types of assignments that will be assigned, and some tips for how to do well in the class.

Format of the course

This course will be taught as a traditional lecture-style course, though hopefully one with lots of interaction between the instructor and the students. Class will typically start with reminders about upcoming assignments and a brief summary of the previous class, followed by time for questions from students. After this we will begin the day's lecture in earnest, introducing the main ideas of the lecture and mentioning any relevant theorems before working through examples. Sometimes examples will be left as exercises for students to work on in class, and students are welcome to work with others in class at this time, while the instructor walks around answering any individual questions students may have before discussing the solutions to the exercise with the class as a whole.

Assignments and grades

Your final letter grade in this course is determined by a weighted average of your grades on the assignments you'll complete throughout the semester. These assignments come in several forms: online "labs" on Canvas, out-of-class written homework, three midterm exams, one final exam, and attendance & class participation.

Canvas Labs and Written Homework

It's sometimes said that the only way to learn math is to do math, which means that in order to truly understand the material in this class you will need to regularly work on problems. These problems should not simply be "cookie cutter" problems solved by a procedure learned in class, but they should also be problems that require you to deeply engage and think about the material.

In order to have you solve problems on a regular basis, we will regularly have online labs through Canvas. You will typically have one lab due at noon each day that we have class concerning material discussed in the last class. The intent with these labs is mostly to get you into the habit of thinking about the material and solving some simple problems on your own. That is, the lab problems are not meant to be overly difficult or challenging, but are meant to give you practice with the basic concepts from each week's reading. In order to truly understand the material, however, you will need to also work on more difficult problems. To facilitate this, I will also post a set of more difficult homework problems each week. These homework problems will be normally be posted at the beginning of the week, and will be due the following Monday.

The homework assignments will be challenging, and it is up to you to manage your time wisely and start on assignments early. You have about a week to complete a homework assignment, and the intention is that you will work on the assignment regularly during that week. *Waiting until the day before the assignment is due to begin is a bad idea.*

Important: Extensions to labs and homeworks will not be given except in extreme circumstances. You have about a week to complete these assignments, so waiting until the last minute to start the assignment is not a valid excuse!

Exams

We will have three midterm exams and a final this semester. The exact dates of the midterms are subject to change, but tentatively the midterms will be held on Thursday February 8, Thursday March 14, and Thursday April 24.

The final exam will be held on the Monday of finals week (May 6) from 12pm until 2:30pm. The final exam will be cumulative and is required of all students.

Expectations

Students in this class are expected to be mature and conduct themselves in a professional manner. In terms of this classroom this means

- students are expected to come to class each day;
- be in class prepared with pencil and paper at the start of class
- students should have completed the assigned reading before coming to class;
- pay active attention during class and have any computers, phones, or tablets put away (students *may* take notes on a tablet, however);
- and be ready to participate in class by asking questions about examples from the previous lecture, problems from homework assignments, or any concepts discussed in class or the assigned reading.

Students are expected to spend a *minimum* of eight hours per week working on material for Math 153 (working on homework, reading the textbook, studying notes, etc.). Keep in mind eight hours is the minimum: each additional hour spent working outside of class will have been well-invested come exam time.

Students are strongly encouraged to take advantage of the various studying resources provided by the university and the mathematics department, such as the MTC.

Online notes

In addition to our textbook, I will be typing up my lecture notes for the course and posting them online in Canvas. Students are expected to read both the online lecture notes as well as the OpenStax textbook. The readings for each week will be posted to Canvas.

How to succeed in this course

Calculus II is a difficult course which is simultaneously more conceptual and more applied than Calculus I. There is no denying that succeeding in this course will require that you put significant time and energy into the course by regularly reading the assigned materials, thinking carefully and deeply about the material, and investing time to work through all of the assigned problems.

Every student is different and you have to figure out what study methods and schedule will work best for you, but here are some pointers you might helpful.

- Start on material early. Check Canvas for the assigned reading, labs, and homework for the upcoming week on Sunday evening. Start reading as soon as you can, and once you finish the reading, start working on the homework. You might get stuck, and that's okay: that's just something for you to ask about during class or office hours.

- Work regularly. Instead of trying to complete your homework in one sitting, spread it out over the week. Skip around, working on the problems you think you know how to do first, but make a point to at least think about harder problems. Sometimes a problem that seems incomprehensible or very difficult at first will become easier after you've thought about it a little and given your brain time to digest it.

- Ask questions. When you read something you don't understand or get stuck on a problem, look for help. Keep a record of the questions you have and ask them in class, during office hours, through email, etc. Make an effort to get your questions answered and don't let them linger hoping they magically get cleared up.

- Work hard. You *will* need to work hard to understand the material in this course and get a good grade. Unfortunately there are no short cuts here. Just be aware from the beginning that you'll need to make a serious effort to commit time to this class. This will be a lot easier if you get into the habit of working on material in the class regularly. For example, you might make a point to set aside an hour of your day each day to focus on Calculus II. (You do have other classes and other responsibilities, so you may not be able to literally spend one hour every single day on Calculus, but that should be the goal you shoot for.)

- Don't stress out. Working through difficult material in a class like Calculus II can be stressful and frustrating. Take breaks when you're getting tired and frustrated and do something fun. Even when you are doing something else your subconscious mind may have still been thinking through what you were working on and you might find that something that seemed difficult earlier doesn't seem as bad after you've taken a break. Remember that this is only one class in your college career and it doesn't define you. A bad grade here or there really isn't as big of a deal as you may think at the time.

Chris Johnson
Spring 2024

Review of Calculus I Material

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If you have an important point to make, don't try to be subtle or clever. Use a pile driver. Hit the point once. Then come back and hit it again. Then hit it a third time – a tremendous whack.

WINSTON CHURCHILL

Knowledge of the material from Calculus I, especially concerning integration, antiderivatives, and the fundamental theorem of calculus, is vital for success in Calculus II. As students in this class are expected to have seen the material in this chapter before, we only give a brief summary of some of the pertinent material from a typical Calculus I course. If you would like to see more details about the formulas, tools, and "tricks" that are only quickly reviewed here, see the Appendix B which fills in some of the gaps, or consult a Calculus I textbook such as the free online OpenStax book,

https://openstax.org/details/books/calculus-volume-1.

1.1 The definite integral

Recall that the ***definite integral*** of a function $f(x)$ over an interval $[a, b]$ is defined as a limit of Riemann sums. In particular, the Riemann sum using n subintervals of equal width and the left-hand endpoints of the subintervals to determine the height of the corresponding rectangle is given by

$$\sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right) \frac{b-a}{n}.$$

Taking the limit as n goes to infinity of this quantity then gives us the integral,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right) \frac{b-a}{n}.$$

Though the expression above seems overwhelming at first glance, the underlying idea has as very simple interpretation. If we wish to find the area under the graph of a function $y = f(x)$ over the interval $[a, b]$ on the

x-axis, then we can approximate that area with rectangles. The Riemann sum above is simply adding up the areas of these individual rectangles. Using a finite number of rectangles, this quantity is generally only an approximation of what we're interested in. We get better approximations by using more rectangles, so we consider what happens when we use more and more and more rectangles by letting n (the number of rectangles) go off to infinity. See Figure 1.1.

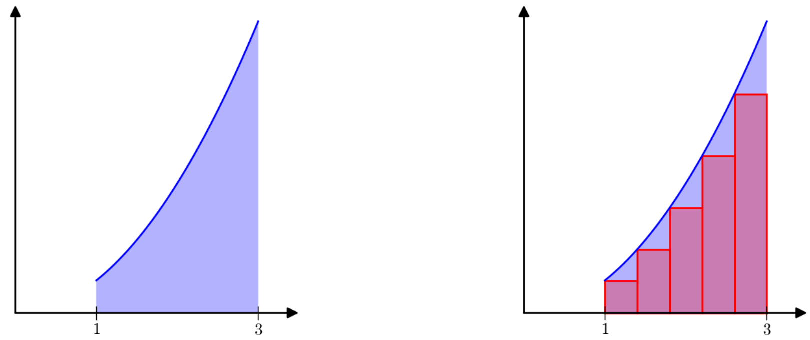


Figure 1.1: The area under $y = x^2$ between $x = 1$ and $x = 3$ on the left is approximated by the sums of areas of rectangles on the right.

Remark.
A very natural question to ask at this point is how do we know the limit actually exists? In Calculus I you saw many examples of limits that blew up to infinity, or oscillated infinitely-often, or had other undesirable properties that prevented the limit from existing. In general, any time you see a limit the first question you should ask yourself is whether the limit exists or not. It will turn out that the limits described above will *always* exist, provided the function $f(x)$ is continuous.

Most of the time in this class we will only consider continuous functions, so the limit defining the integral will exist, and we won't bother to explicitly discuss the existence of the limit. There will be exceptions to this when we discuss "improper integrals" later in the course.

Calculating an integral as a limit is possible, but it is often a tedious calculation. For the sake of completeness, we'll go ahead and compute one integral this way.

Example 1.1.

Compute $\int_1^3 x^2\,dx$ as a limit of Riemann sums.

Comparing this with the expression above,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right)\frac{b-a}{n},$$

in this example we have $a = 1$, $b = 3$, and $f(x) = x^2$. Thus the integral becomes

$$\begin{aligned}
\int_1^3 x^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1 + \frac{(3-1)i}{n}\right)\frac{3-1}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\frac{2}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^2\frac{2}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right)\frac{2}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left(\frac{2}{n} + \frac{8i}{n^2} + \frac{8i^2}{n^3}\right)
\end{aligned}$$

At this point we may pause to just quickly fill in all the details of what just happened. First, we simply wrote out the integral as a limit of Riemann sums by replacing the a, b, and $f(x)$ in our earlier expression with 1, 3, and x^2. Note that since we are plugging $1 + \frac{2i}{n}$ in for x in $f(x) = x^2$, we must square this entire expression. We can compute

$$\left(1 + \frac{2i}{n}\right)^2 = 1 + \frac{4i}{n} + \frac{4i^2}{n^2}$$

by either manually "FOILing" the expression, or using the binomial theorem. After doing this we simply distributed $\frac{2}{n}$ to each of the terms in our summation.

To proceed in calculating this limit, we need to recall a couple of basic properties of summations. First, we can always break sums of multiple terms up into multiple sums. E.g., $\sum_{i=1}^{n}(g(i)+h(i)) = \sum_{i=1}^{n} g(i) + \sum_{i=1}^{n} h(i)$. Applying this to the terms in our sum above we have

$$\begin{aligned}\int_1^3 x^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2}{n}+\frac{8i}{n^2}+\frac{8i^2}{n^3}\right)\\ &= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{2}{n}+\sum_{i=1}^{n}\frac{8i}{n^2}+\sum_{i=1}^{n}\frac{8i^2}{n^3}\right)\end{aligned}$$

The next property of sums we need to recall is that we can always factor constants out of summations. E.g., $\sum_{i=1}^{n} kg(i) = k\sum_{i=1}^{n} g(i)$. Here the "constant" is actually anything which does not depend on the index i of the summation. So, n, for example, can be factored out as well. This then gives us

$$\begin{aligned}\int_1^3 x^2\,dx &= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{2}{n}+\sum_{i=1}^{n}\frac{8i}{n^2}+\sum_{i=1}^{n}\frac{8i^2}{n^3}\right)\\ &= \lim_{n\to\infty}\left(\frac{2}{n}\cdot\sum_{i=1}^{n}1+\frac{8}{n^2}\cdot\sum_{i=1}^{n}i+\frac{8}{n^3}\cdot\sum_{i=1}^{n}i^2\right)\end{aligned}$$

Now we recall a few simple formulas you learned in Calculus I which will help us simplify these summations:

$$\sum_{i=1}^{n}1=n,\quad \sum_{i=1}^{n}i=\frac{n(n+1)}{2},\quad \text{and}\quad \sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}.$$

Applying these to the expression above we have

$$\begin{aligned}
\int_1^3 x^2\,dx &= \lim_{n\to\infty}\left(\frac{2}{n}\cdot\sum_{i=1}^{n}1+\frac{8}{n^2}\cdot\sum_{i=1}^{n}i+\frac{8}{n^3}\cdot\sum_{i=1}^{n}i^2\right)\\
&= \lim_{nto\infty}\left(\frac{2}{n}\cdot n+\frac{8}{n^2}\cdot\frac{n(n+1)}{2}+\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right)\\
&= \lim_{n\to\infty}\left(2+\frac{4n^2+4n}{n^2}+\frac{8n^3+12n^2+4n}{3n^2}\right).
\end{aligned}$$

At this point we are basically home free. If we recall that when you take a limit as n goes to infinity of a ratio of polynomials of the same degree you simply get the ratio of coefficients, we then easily see

$$\begin{aligned}
\int_1^3 x^2\,dx &= \lim_{n\to\infty}\left(2+\frac{4n^2+4n}{n^2}+\frac{8n^3+12n^2+4n}{3n^3}\right)\\
&= 2+4+\frac{8}{3}\\
&= \frac{6+12+8}{3}\\
&= \frac{26}{3}
\end{aligned}$$

While it is possible to compute definite integrals this way, it is very tedious. Luckily there is a simpler way, using the fundamental theorem of calculus.

1.2 The fundamental theorem of calculus and antiderivatives

The fundamental theorem of calculus establishes a relationship between definite integrals and antiderivatives. We *really* like this theorem because it replaces something that is generally difficult to do (compute a limit of Riemann sums) and replaces it with something that is generally easier to do (compute an antiderivative). There are some caveats to this that we will worry about later, but the basic idea is that we want to convert a difficult problem into an easier problem.

To be more precise we have the following:

Theorem 1.1 (The fundamental theorem of calculus)**.**
If $F(x)$ is an antiderivative of $f(x)$ defined on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

That is, provided we can find the antiderivative $F(x)$, computing definite integrals becomes very simple – again, assuming we can actually compute our antiderivative.

Recall that the ***antiderivative*** of a function $f(x)$ is a function $F(x)$ with the property that $F'(x) = f(x)$. That is, differentiating the antiderivative of a function gives us back the original function; antidifferentiating is a kind of opposite (or inverse) of differentiating, hence the name.

Remark.
A natural question to ask at this point is "how do you know that a function $f(x)$ has an antiderivative?" There are actually two versions of the fundamental theorem of calculus, and the version we did not state above answers this question: if $f(x)$ is continuous on an interval $[a, b]$, then it has an antiderivative $F(x)$ defined on the same interval. To prove this you have to describe a method of constructing an antiderivative of any arbitrary continuous function, and that boils down to using the definition of the definite integral as a limit of Riemann sums. See Appendix B if you're interested in seeing the details.

For the fundamental theorem of calculus to be useful, though, we have to actually be able to compute antiderivatives. For some simple functions this isn't too terribly difficult and follows from basically performing our derivative rules "in reverse." The basic antiderivative rules you should remember from Calculus I are summarized in the following theorem. Recall that $\int f(x)\,dx$, sometimes called the ***indefinite integral of*** $f(x)$, is just notation that means the antiderivative of $f(x)$.

Theorem 1.2.
Suppose $f(x)$ and $g(x)$ are two continuous functions and k is a constant. Then,

- $\displaystyle\int (f(x) \pm g(x))\ dx = \int f(x)\,dx + \int g(x)\,dx,$
- $\displaystyle\int kf(x)\,dx = k\int f(x)\,dx,$
- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ *provided* $n \neq -1$,
- $\displaystyle\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C,$
- $\displaystyle\int e^x\,dx = e^x + C,$
- $\displaystyle\int \cos(x)\,dx = \sin(x) + C,$
- $\displaystyle\int \sin(x)\,dx = -\cos(x) + C,$
- $\displaystyle\int \sec^2(x)\,dx = \tan(x) + C,$
- $\displaystyle\int \csc^2(x)\,dx = -\cot(x) + C,$
- $\displaystyle\int \sec(x)\tan(x)\,dx = \sec(x) + C,$ *and*
- $\displaystyle\int \csc(x)\cot(x)\,dx = -\csc(x) + C.$

Notice that many of the antiderivative rules above have a "$+C$" at the end. The reason for this is simply that the derivative of a constant is zero, and so we can always add any constant onto an antiderivative and get another perfectly valid antiderivative. For example, $\frac{x^3}{3}$, $\frac{x^3}{3}+3$, and $\frac{x^3}{3}-7$ are all perfectly fine antiderivatives of the function x^2. Since the choice of constant we add onto the end of our antiderivative can be any arbitrary value, we usually just refer to it as C and sometimes call it a ***constant of integration***.

Remark.
When we are just finding *an* antiderivative of a function, any choice of C is perfectly valid. However, in some applications we may want to find a particular antiderivative and then need to compute the correct value of C. In order to do this we need to have more information about the antiderivative we desire, for example we may want to find the antiderivative $F(x)$ of $f(x)$ satisfying $F(x_0) = y_0$. This additional condition we wish to satisfy is called an ***initial condition***, and determining the antiderivative together with the value of C that will solve our initial condition is called an ***initial value problem***.

These rules are our bread and butter for computing antiderivatives: they are the most basic of our antiderivative rules. You should be familiar with all of these rules, but let's have a few examples of computing antiderivatives using these rules just to see how powerful they are.

Example 1.2.
Compute the antiderivative of $6x^2 + 8x - 5$.

For the sake of this example, let's very clearly walk through every detail of computing this antiderivative using the rules above. (We won't usually be this verbose in explaining antiderivatives, and you aren't expected to write down *every* detail like this in work you turn in, but we'll do it in this example just to be as clear as possible.)

We wish to compute

$$\int \left(6x^2 + 8x - 5\right)\, dx.$$

We begin by applying the first rule of Theorem 1.2 which allows us to break our antiderivative problem up as

$$\int 6x^2\, dx + \int 8x\, dx - \int 5\, dx.$$

The point of applying this rule is that we took a "hard" problem and split it up into "easier" problems. (Perhaps the original problem isn't truly hard, but the strategy here, as in many problems, is to keep breaking the problem into simpler and simpler pieces until we arrive at something we can easily solve.) Now we can apply the second rule in Theorem 1.2 to

$$6\int x^2\,dx + 8\int x\,dx - 5\int 1\,dx.$$

Now we are virtually done because we can simply apply the third rule from Theorem 1.2 to each of these terms to obtain

$$6\frac{x^3}{3} + C_1 + 8\frac{x^2}{2} + C_2 - 5\left(x + C_3\right).$$

Here we should pause for just a moment to discuss the last step in a little bit of detail. In the very last term we were integrating the constant function 1, $\int 1\,dx$. To apply the third rule from Theorem 1.2, we should think of 1 as x^0. The rule then tells us

$$\int 1\,dx = \int x^0\,dx = \frac{x^1}{1} + C = x + C.$$

Let's also notice that we have three different constants of integration above, which we denoted C_1, C_2, and C_3, in the last step of our calculation. Theorem 1.2 says that each time we apply the third rule we should pick up a $+C$. In principle these are different C's, one for each integration. Keep in mind, however, these C's are completely arbitrary. So we may add the completely arbitrary C_1 to the completely arbitrary C_2, we can just call this some new arbitrary constant. Similarly, when we distribute the -5 of the last term to obtain $-5C_3$, we have the constant -5 times some arbitrary constant C_3 and this is still just as equally arbitrary. Thus we can always combine multiple arbitrary C's together into one constant, which we'll keep denoting C. Our last step can thus be rewritten as

$$6\frac{x^3}{3} + 8\frac{x^2}{2} - 5x + C.$$

We won't typically bother to write down all of these different C's as C_1, C_2, C_3, and so on – usually we'll just write one $+C$ at the very end, but it's worth pointing out that there really are different C's that we're just adding together.

Finally, we can do a tiny bit of simplification to write down our antiderivative,

$$\int \left(6x^2 + 8x - 5\right)\, dx = 2x^3 + 4x^2 - 5x + C.$$

Recall that one of the nice things about antiderivatives is that we can always check our answer: if $2x^3 + 4x^2 - 5x + C$ really is an antiderivative of $6x^2 + 8x - 5$, then we should be able to differentiate $2x^3 + 4x^2 - 5x + C$ and get back $6x^2 + 8x - 5$, which is easy to verify:

$$\frac{d}{dx}\left(2x^3 + 4x^2 - 5x + C\right) = 3 \cdot 2x^2 + 2 \cdot 4x - 5 + 0 = 6x^2 + 8x - 5.$$

With Theorem 1.2 and the fundamental theorem of calculus available, problems that would otherwise be very difficult become very easy. For instance, the tedious limit of Riemann sums calculation from Example 1.1 above now becomes much easier.

Example 1.3.
Compute $\int_1^3 x^2\, dx$ using the fundamental theorem of calculus.

To apply the fundamental theorem of calculus we need an antiderivative of x^2, but this is easy to compute using the third rule of Theorem 1.2:

$$\int x^2\, dx = \frac{x^3}{3} + C.$$

Now, by the fundamental theorem of calculus, we simply need to evaluate our antiderivative at 3 and at 1 then subtract to obtain

$$\begin{aligned} \int_1^3 x^2\, dx &= \left(\frac{3^3}{3} + C\right) - \left(\frac{1^3}{3} + C\right) \\ &= \frac{27}{3} + C - \frac{1}{3} - C \\ &= \frac{26}{3}. \end{aligned}$$

Notice this is the same value we computed in Example 1.1, but it requires *much* less work!

Remark.
Notice that the $+C$'s from our antiderivative in Example 1.3 cancelled out. This will *always* happen when we evaluate a definite integral, and for this reason we often don't both to write down the $+C$ when we are evaluating a definite integral as it will just cancel out with a $-C$ anyway. However, this is specifically for definite integrals: you still need a $+C$ for indefinite integrals (aka, general antiderivative problems)!

Recall that as a notation convenience we often write $F(x)|_a^b$ as a short-hand for $F(b) - F(a)$ in definite integral problems. For example, the integral in Example 1.3 may be written as

$$\int_1^3 x^2\,dx = \left.\frac{x^3}{3}\right|_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \frac{26}{3}.$$

Though the rules described in Theorem 1.2 should be relatively familiar and easy to use, they are certainly not enough to integrate many of the functions that we will care about. For example, it's not at all clear how to compute $\int \ln(x)\,dx$ based on the rules above. That is, our list of rules is far from complete.

We will spend a fair bit of time this semester extending our list of antiderivative rules, but let's first recall one more rule that you learned in Calculus I which is not mentioned in the theorem above.

1.3 Substitution

Given an indefinite integral of the the form

$$\int f'(g(x))g'(x)\,dx$$

we can simplify this complicated-looking integral by introducing a new variable. If we were to introduce a variable u which we define to be

$u = g(x)$, then its differential (see the "Detailed Review of Calculus I Material" for a reminder about differentials) is $du = g'(x)\,dx$. Notice that this means we can rewrite the integral above as

$$\int f'(u)\,du.$$

Now, since we're looking for the antiderivative of $f'(u)$ – the derivative of $f(u)$ – we just need a function whose derivative is $f'(u)$. Of course this is simply $f(u)$ (plus an arbitrary constant C):

$$\int f'(u)\,du = f(u) + C.$$

So our complicated-looking integral from before actually became extremely easy when we introduced our variable u. However, we started off by asking for an antiderivative of a function of x, and currently have a function of u. Since the antiderivative of a function of x should also be a function of x, we need to rewrite $f(u) + C$ in terms of x, but this is easy: we'll just replace u with $g(x)$ since those are equal.

Putting all of this together we have determined

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + C.$$

Since this is an antiderivative problem, we can easily check our answer by seeing if the derivative of $f(g(x))$ is $f'(g(x))g'(x)$, but this is really just the chain rule:

$$\frac{d}{dx}\left(f(g(x)) + C\right) = f'(g(x))g'(x).$$

That is, the substitution $u = g(x)$ we introduced and the corresponding antiderivative we computed is really just doing the chain rule "in reverse."

Example 1.4.
Find the antiderivative of $\sqrt{x^3 + x - 2}\,(3x^2 + 1)$.

We wish to compute

$$\int \sqrt{x^3 + x - 2}\,\left(3x^2 + 1\right)\,dx.$$

If we were to let $u = x^3 + x - 2$, then we would have $du = (3x^2 + 1)\ dx$, and the complicated integral we started with becomes simply $\int \sqrt{u}\,du$.

Recalling that taking a square root is the same as raising to the $1/2$ powever, we can now compute

$$\int \sqrt{u}\, du = \int u^{1/2}\, du = \frac{u^{3/2}}{3/2} = \frac{2}{3}u^{3/2} + C.$$

Keeping in mind we started with a function of x and so we need to end with a function of x, we convert our u back into $x^2 + x - 2$ to obtain

$$\int \sqrt{x^3 + x - 2}\left(3x^2 + 1\right)\, dx = \frac{2}{3}\left(x^3 + x - 2\right)^{3/2} + C.$$

Exercise 1.1.
Verify that the antiderivative computed in Example 1.4 is correct by differentiating $\frac{2}{3}\left(x^3 + x - 2\right)^{3/2} + C$.

Recall that when performing a substitution like this, we want to choose our u so that the differential du is inside the original integral. However, sometimes we may have to do a little bit of work to see that du is hiding in the integral.

Example 1.5.
Compute $\displaystyle\int \frac{4x^2}{1 + 8x^3}\, dx$.

We want to introduce a u so that this integral becomes simpler (this process isn't very helpful if it makes our problem harder), and so that du is "essentially" in the integral. If we set $u = 1 + 8x^3$, $du = 24x^2\, dx$. We don't have a $24x^2\, dx$ in our integral, but we can turn $4x^2\, dx$ into $24x^2\, dx$ by multiplying by 6. However, this changes our interal, so we'll need to divide by 6 as well to compensate. I.e., what we're really doing is just multiplying by 1, but writing 1 in a convenient way.

$$\begin{aligned}\int \frac{4x^2}{\sqrt{1+8x^3}}\,dx &= 1 \cdot \int \frac{4x^2}{\sqrt{1+8x^3}}\,dx \\ &= \frac{6}{6} \cdot \int \frac{4x^2}{\sqrt{1+8x^3}}\,dx \\ &= \frac{1}{6} \cdot \int \frac{24x^2}{\sqrt{1+8x^3}}\,dx.\end{aligned}$$

Now performing the substitution $u = 1 + 8x^3$, $du = 24x^2\,dx$, the integral becomes

$$\frac{1}{6}\sqrt{1}\sqrt{u}\,du = \frac{1}{6}\int u^{-1/2}\,du = \frac{1}{6} \cdot \frac{u^{1/2}}{1/2} + C = \frac{1}{3}\sqrt{u} + C.$$

Putting this back in terms of x we have

$$\int \frac{4x^2}{\sqrt{1+8x^3}}\,dx = \frac{1}{3}\sqrt{1+8x^3} + C.$$

Example 1.6.

Compute $\displaystyle\int \frac{x}{\sqrt{1+2x}}\,dx$.

Warning: In this example we will address a common misconception with substitutions and in doing so we will intentionally show incorrect work to highlight the misconception. *Read the text of the example very carefully so that you do not confuse the incorrect work we're highlighting with correct work!*

If we let $u = 1 + 2x$, then $du = 2\,dx$. We now have two problems: there is not a corresponding 2 in our integral, and there is an extra x that does not immediately get "swallowed up" by u or du. Both of these issues can be easily fixed, however. The issue with the missing 2 is fixed by using the same kind of trick from Example 1.5: we will simply multiply and divide by 2:

$$\begin{aligned}\int \frac{x}{\sqrt{1+2x}}\,dx &= \frac{2}{2}\int \frac{x}{\sqrt{1+2x}}\,dx\\ &= \frac{1}{2}\int \frac{2x}{\sqrt{1+2x}}\,dx\\ &= \frac{1}{2}\int \frac{x}{\sqrt{1+2x}}2\,dx.\end{aligned}$$

Now let's tackle the problem of the extra x that was mentioned earlier. At this point in the process you may be tempted to go ahead and put u in for the $1+2x$ that appears in the square root, rewriting the integral as

$$\frac{1}{2}\int \frac{x}{\sqrt{u}}\,du.$$

While this seems like a pretty reasonable thing to do *it is incorrect*, but the reason why it's incorrect is a little bit subtle. When we have an integral the differential at the end (the du or dx or "d-whatever-the-variable is") tells us which variable we're integrating *and everything else is treated as a constant*. That is, if you were to just hand the integral $\frac{1}{2}\displaystyle\int \frac{x}{\sqrt{u}}\,du$ to a random person on the street (well, a random person that has taken calculus...), then when they computed the antiderivative they would pull the x out of the integral and leave it alone, just like any constant because of the du that appears. This, however, is definitely not going to give us the right answer to our problem.

To see more precisely exactly why the integral $\frac{1}{2}\int \frac{x}{\sqrt{u}}\,du$ is incorrect, let's actually work it out and see if we get an antiderivative to our initial $\int \frac{x}{\sqrt{1+2x}}\,dx$ or not. Because of the du that appears we would pull the x out (since we'd be treating it like a constant) to get

$$\frac{1}{2}\int \frac{x}{\sqrt{u}}\,du = \frac{x}{2}\int u^{-1/2}\,du = \frac{x}{2}\frac{u^{1/2}}{1/2} + C = x\sqrt{u} + C.$$

At this point you'd like to replace u with $1+2x$ to get the (incorrect) antiderivative, $x\sqrt{1+2x}+C$. As mentioned, though, this is incorrect, which we can easily verify by differentiating:

$$\frac{d}{dx}\left(x\sqrt{1+2x}+C\right) = \frac{x}{\sqrt{1+2x}} + \sqrt{1+2x}.$$

Notice this *is not* what we want, our antiderivative is incorrect!

The issue with the above is that when we mixed u's and x's in our integral, we treated one of these as a constant and one as a variable but both should have been variables! The way around this problem is to only have one variable in the integral. That is, **we always want only x's or only u's in our integrals and we should never mix variables**!

Keeping this in mind, let's go back to our original problem which we have rewritten as

$$\int \frac{x}{\sqrt{1+2x}}\,dx = \frac{1}{2}\int \frac{x}{\sqrt{1+2x}} 2\,dx.$$

We want to replace the $1+2x$ in the denominator with just u, but we need to replace the x in the numerator with some expression involving u's as well. To do this, keep in mind we're making the substitution $u = 1+2x$. Let's notice that we could solve this equation for x:

$$\begin{aligned} u &= 1+2x \\ \implies u-1 &= 2x \\ \implies \frac{u-1}{2} &= x. \end{aligned}$$

That is, we can take the x in the numerator and rewrite it in terms of u as $x = \frac{u-1}{2}$. Doing this our integral now becomes

$$\frac{1}{2}\int \frac{\left(\frac{u-1}{2}\right)}{\sqrt{u}} = \frac{1}{4}\int \frac{u-1}{\sqrt{u}}\,du.$$

Now, before we can start applying our rules from Theorem 1.2, we need to do just a tiny bit of algebra to the integrand:

$$\frac{1}{4}\int \frac{u-1}{\sqrt{u}}\,du = \frac{1}{4}\int (u-1)u^{-1/2}\,du = \frac{1}{4}\int \left(u^{1/2} - u^{-1/2}\right)\,du.$$

At this point we can easily compute our integral:

$$\begin{aligned}\frac{1}{4}\int\left(u^{1/2}-u^{-1/2}\right)\,du &= \frac{1}{4}\left(\frac{u^{3/2}}{3/2}-\frac{u^{1/2}}{1/2}\right)+C\\ &= \frac{1}{4}\left(\frac{2}{3}u^{3/2}-2u^{1/2}\right)+C\\ &= \frac{1}{6}u^{3/2}-\frac{1}{2}u^{1/2}+C.\end{aligned}$$

Now rewriting this in terms of x we see that our original antiderivative problem is solved by

$$\int\frac{x}{\sqrt{1+2x}}\,dx = \frac{1}{6}(1+2x)^{3/2}-\frac{1}{2}(1+2x)^{1/2}+C.$$

Exercise 1.2.
Verify the antiderivative computed in Example 1.6 is correct.

The examples of substitution that we have seen so far were all indefinite integrals – i.e., just computing antiderivatives. What about definite integrals where we want to compute a number in the end? One thing we could do is find the antiderivative as before, then plug in our original limits of integral as in Example 1.7 below.

Example 1.7.
Compute $\displaystyle\int_0^2 (6x-4)^5\,dx$.

We could first find the antiderivative in terms of x by computing the indefinite integral $\int(6x-4)^5\,dx$. This would require the substitution $u = 6x-4$, $du = 6\,dx$. Multiplying and dividing by 6 to account for the 6 we have picked up in the dx, the integral would become

$$\frac{1}{6}\int u^5\,du = \frac{1}{6}\cdot\frac{u^6}{6} + C = \frac{u^6}{36} + C.$$

Converting this back to x's we see

$$\int (6x-4)^5\,dx = \frac{(6x-4)^6}{36} + C.$$

Now that we have an antiderivative we could apply the fundamental theorem of calculus to obtain

$$\begin{aligned}\int_0^2 (6x-4)^5\,dx &= \left.\frac{(6x-4)^6}{36}\right|_0^2 \qquad = \frac{8^6}{36} - \frac{(-4)^6}{36}\\ &= \frac{262,144-4096}{36}\\ &= 7168\end{aligned}$$

Remark.
Notice that you could actually compute $\int(6x-4)^5\,dx$ *without* using a substitution, but you'd have to expand $(6x-4)^5$. While this is possible, it's certainly very tedious to do by hand. Thus substitution can sometimes be used to convert tedious problems into easier ones!

While the work appearing in Example 1.7 is perfectly valid, there is another way. Instead of performing a u-substitution in an indefinite integral, converting back to x, and then plugging in the original limits of integration, we could actually change the limits of integration when we perform the substitution, as showin Example 1.8

Example 1.8.
Compute $\displaystyle\int_0^2 (6x-4)^5\,dx$ by changing the limits of integration during the substitution.

Just as before we will perform the substitution $u = 6x-4$, $du =$

$6\,dx$. Notice that this gives us an integral in terms of u. When we change our definite integral in terms of x to a definite integral in terms of u, we need to change the limits of integration from x-values to the corresponding u-values. That is, when $x = 0$, we will have the corresponding u-value

$$u = 6x - 4 = 6 \cdot 0 - 4 = -4.$$

Similarly, when $x = 2$, the corresponding u-value is

$$u = 6x - 4 = 6 \cdot 2 - 4 = 8.$$

Changing our limits of integration from $x = 0$ and $x = 2$ to the corresponding u-values $u = -4$ and $u = 8$ we have

$$\begin{aligned}\int_0^2 (6x-4)^5\,dx &= \frac{1}{6}\int_{-4}^{8} u^5\,du \\ &= \frac{u^6}{36}\bigg|_{-4}^{8} \\ &= \frac{8^6 - (-4)^6}{36} \\ &= 7168.\end{aligned}$$

Notice that we never needed to convert "back" to x's and we still computed the same value as before.

Sometimes we can use knowledge of an integral to determine the value of another integral *without even knowing the function being integrated*!

Example 1.9.
Suppose that $f(x)$ is a continuous function and $\int_0^9 f(x)\,dx = 4$. What is $\int_0^3 xf(x^2)\,dx$?

Let's perform the substitution $u = x^2$, $du = 2x\,dx$. Then our integral may be rewritten as

$$\begin{aligned}\int_0^3 x f(x^2)\,dx &= \frac{1}{2}\int_0^9 f(u)\,du \\ &= \frac{4}{2} \\ &= 2\end{aligned}$$

The above example might seem a little bit silly, but tricks like this are often very useful. In many applications you may not know the function modelling something you're interested in, but are still able to determine various properties of the function, and tricks like the one above can then be used to tell you information about related functions. (This is perhaps hard to imagine or understand when you're first learning this material, but many functions we are interested in aren't explicitly known and we often have to work with functions based solely on their properties.)

If the necessity of changing limits of integration is confusing, it might help to think of this as changing the units being used in the problem. For example, imagine taking a problem that's described in minutes and converting it into a problem in terms of seconds. If your original problem in terms of minutes took place on the interval $[0, 1/2]$, when you convert the problem into seconds this interval has to change to $[0, 30]$.

Example 1.10.
Suppose the velocity of a particle t minutes after the particle is first observed is given by $v(t) = 30t\frac{\text{meters}}{\text{minute}}$. How far does the particle travel over the course of half a minute?

Recall that velocity is the derivative of position, so the change in position (the distance the particle travelled) is given by integrating velocity (this is really just the fundamental theorem of calculus).

When we compute the problem in terms of minutes we have

$$\int_0^{1/2} 30t\,dt = 15t^2\Big|_0^{1/2} = \frac{15}{4} = 3.75$$

and so the particle travelled 3.75 meters.

There's nothing "magical" about measuring time in minutes, of course: we could convert this into a problem where time is measured in seconds. Our t before was measured in minutes, so if we let u denote the same amount of time measured in seconds we would have $u = 60t$ since there are sixty seconds in each minute. Thus we will perform the substitution $u = 60t$, $du = 60dt$. Of course, when we integrate we will need to multiply and divide by 60 because of the 60 that appears in $du = 60dt$. However, we also need to change our limits of integration. The problem we are interested in takes place over half a minute which is thirty seconds. In terms of our substitution, when $t = 0$ the corresponding u-value is

$$u = 60t = 60 \cdot 0 = 0;$$

and when $t = 1/2$, the corresponding u is

$$u = 60t = 60 \cdot \frac{1}{2} = 30.$$

Our integral is thus

$$\begin{aligned} \frac{1}{60} \int_0^{30} \frac{u}{2}\, du &= \frac{1}{60} \cdot \frac{u^2}{4} \bigg|_0^{30} \\ &= \frac{30^2}{240} = \frac{900}{240} = \frac{90}{24} = \frac{30}{8} \\ &= \frac{15}{4} = 3.75. \end{aligned}$$

And, of course, we computed the same distance travelled regardless of whether we measured time in minutes or seconds, but we had to change our integrals appropriately when doing the conversion from minutes to seconds.

Remark.
One minor "trick" may have slipped by you in Example 1.10. When we wrote the integral in terms of u, notice that our integrand changed from $30t$ to $\frac{u}{2}$, and you may wonder where the $\frac{u}{2}$ came from. Keep in mind we were doing the substitution $u = 60t$, and this could

be rewritten as $t = \frac{u}{60}$. When we replaced t with $\frac{u}{60}$ our integrand switched from $30t$ to $30 \cdot \frac{u}{60} = \frac{30u}{60} = \frac{u}{2}$.

One more little remark about the previous example: sometimes when students first learn about substitution, they don't understand why the dx has to change, or has to change to something more complicated than just du. For example, if we performed the substitution $u = x^2 + 2x$ we would have $du = (2x + 2)\, dx$, but why do we care about this $2x + 2$ that appeared? Example 1.10 gives us some rationale for why we need this. The dt that appeared in the original integral essentially represents a change of time *measured in minutes*, but we are rewriting this as an integral measured in seconds, so our du (since u is measured in seconds) needs to represent the change in time *measured in seconds*. Notice that changing time by one minute is the same as changing time by sixty seconds, and this is what $du = 60dt$ (or equivalently, $dt = \frac{1}{60}du$) represents. One tick of a clock that measures only in minutes corresponds to sixty ticks of a clock that measures only in seconds.

1.4 Practice problems

Problem 1.1. Compute the antiderivative of $x^2 \sin(x^3 - 1)$.

Problem 1.2. Compute $\displaystyle\int \frac{\ln(x)}{x}\, dx$.

Problem 1.3. Compute $\displaystyle\int \frac{1}{x\ln(x)}\, dx$.

Problem 1.4. Compute $\displaystyle\int \cos(4x)\sin(4x)\ dx$.

Problem 1.5. Compute $\displaystyle\int \frac{\sin(\tan(\theta))}{\cos^2(\theta)}\, d\theta$.

Problem 1.6. Compute the antiderivative of $\sec(x)$ by first multiplying and dividing by $\sec(x) + \tan(x)$, and then performing an appropriate u-substitution.

Problem 1.7. Compute the definite integral $\displaystyle\int_3^9 2x^3\sqrt{x^2 - 3}\, dx$.

Problem 1.8. Compute the definite integral $\int_{-2}^{3} \frac{14}{23+7x}\,dx$.

Problem 1.9. Suppose $f(x)$ is a continuous function satisfying

$$\int_0^2 f(x) = 5.$$

What is $\int_0^1 f(2x)\,dx$?

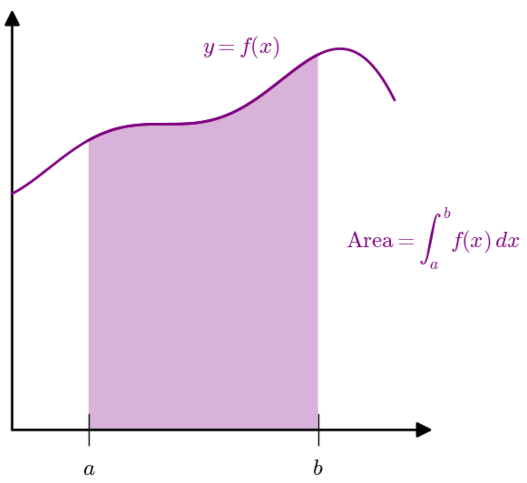
2 Applications of Integrals

Knowledge is not power, it is only potential.
Applying that knowledge is power.

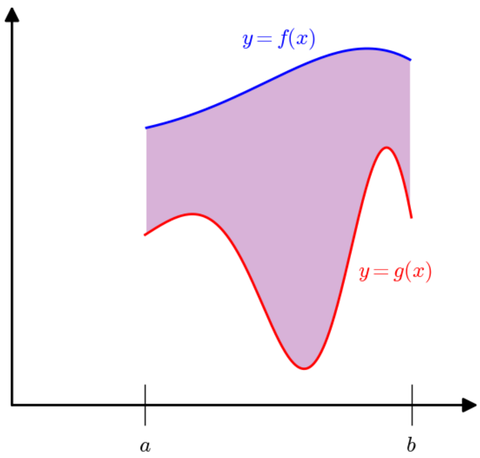
TAKEDA SHINGEN

2.1 Area between curves

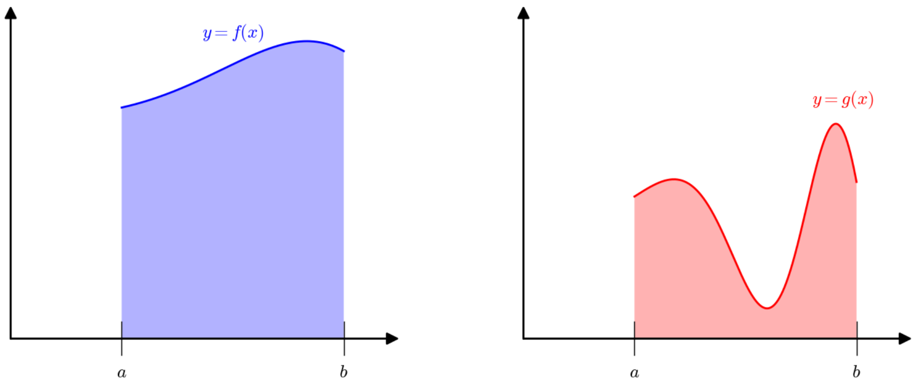
Recall that one interpretation of the quantity $\int_a^b f(x)\,dx$ is that it represents the area between the graph $y = f(x)$ and the x-axis between $x = a$ and $x = b$ – at least when $f(x) \geq 0$ for $a \leq x \leq b$.



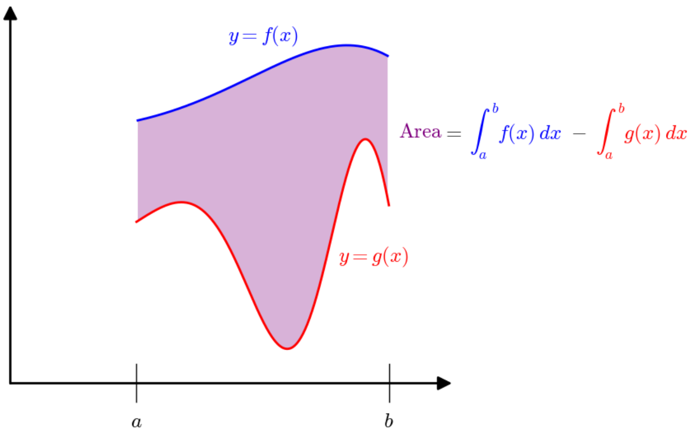
Suppose that we wanted to find the area between the graphs of two functions instead of the area between a graph and the x-axis? How can we do this using integrals? For example, suppose $f(x)$ and $g(x)$ are the graphs in the blue and red below, and we want to find the area of the region between these graphs, shaded in purple.



Notice that the area we care about is obtained by first taking the area between the graph of $y = f(x)$ and the x-axis (the blue region below), and then removing the area between the graph of $y = g(x)$ and the x-axis (red below).



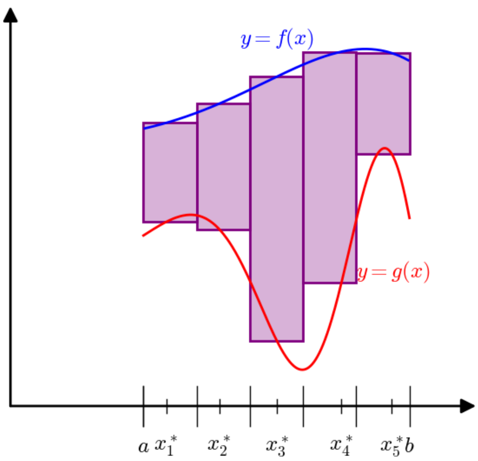
Each of these areas we can compute as an integral, and so to find the area of the purple region we're interested in, we'll simply subtract the area under $y = g(x)$ from the area under $y = f(x)$:



By one of the basic properties of integrals, we could rewrite this as

$$\text{Area} = \int_a^b (f(x) - g(x))\ dx.$$

Notice we subtract the function whose graph is the curve on the bottom from the function whose graph is the curve on the top. If we were to think about estimating the area between the curves by adding the areas of rectangles between the curves, notice that the heights of these rectangles would be given by an expression of the form $f(x_i^*) + g(x_i^*)$ for some chosen "sample points" x_i^*:

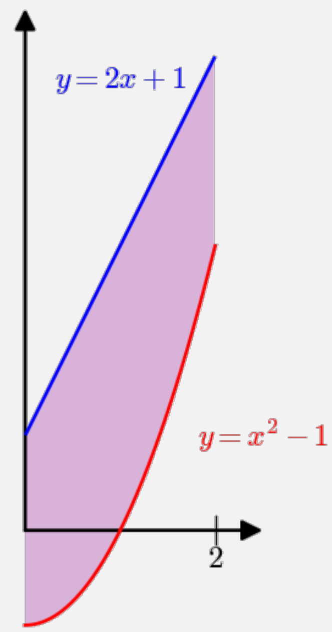


Adding up the areas of these rectangles, using some partition $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and chosen points x_i^* on $[x_{i-1}, x_i]$, would give us the sum

$$\sum_{n=1}^{n} \left(f(x_i^*) - g(x_i^*)\right) \Delta x_i$$

Taking the limit of such sums of course gives us the integral $\int_a^b \left(f(x) - g(x)\right)\, dx$.

Example 2.1.
Fine the area between $y = x^2 - 1$ and $y = 2x + 1$, between $x = 0$ and $x = 2$.

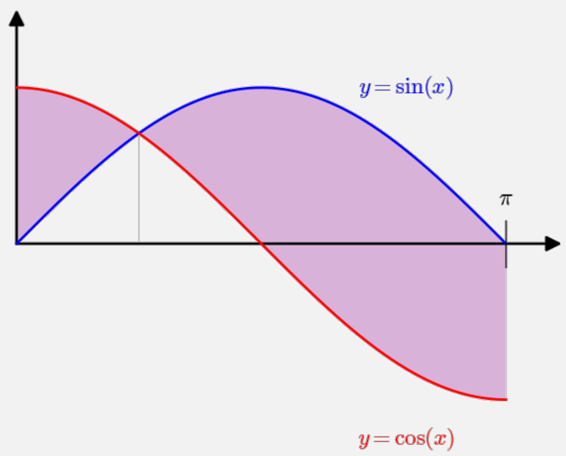


Noting that $y = 2x + 1$ is above $y = x^2 - 1$, we see that our $f(x)$ and $g(x)$ in this example are $f(x) = 2x + 1$ and $g(x) = x^2 - 1$, and so the area between the two curves is

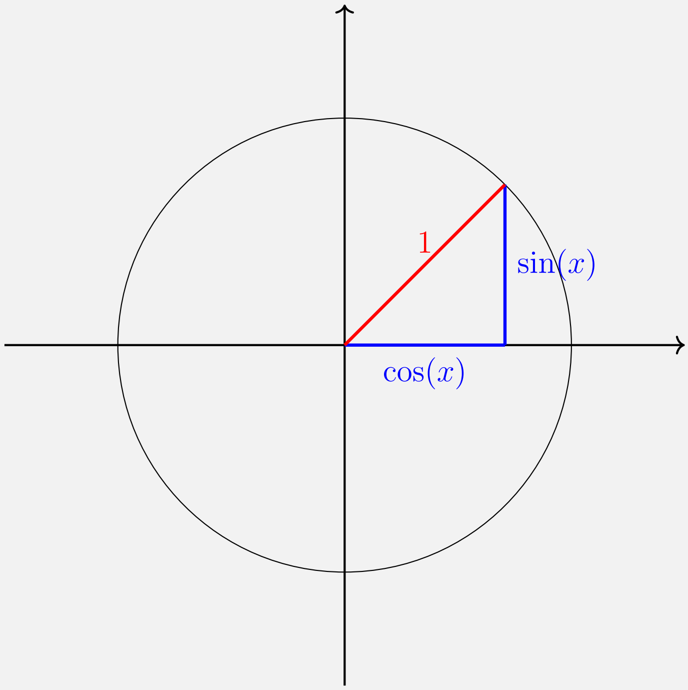
$$\begin{aligned}\int_0^2 \left((2x+1)-\left(x^2-1\right)\right) &= \int_0^2 \left(-x^2+2x+2\right)\,dx\\ &= \left(\frac{-x^3}{3}+x^2+2x\right)\Bigg|_0^2\\ &= \left(\frac{-8}{3}+4+4\right)-0\\ &= 8-\frac{8}{3}=\frac{24-8}{3}\\ &= \frac{16}{3}\end{aligned}$$

In the example above, it was evident from the given graph that $y = 2x + 1$ was on the top and $y = x^2 - 1$ was on the bottom. A very reasonable question to ask now would be how do you know which is the "top" and which is the "bottom" if you aren't given a graph? We will address this in a moment, but let's first do an example where the roles of top and bottom switch.

Example 2.2.
Find the area between $y = \sin(x)$ and $y = \cos(x)$ for $0 \le x \le \pi$:

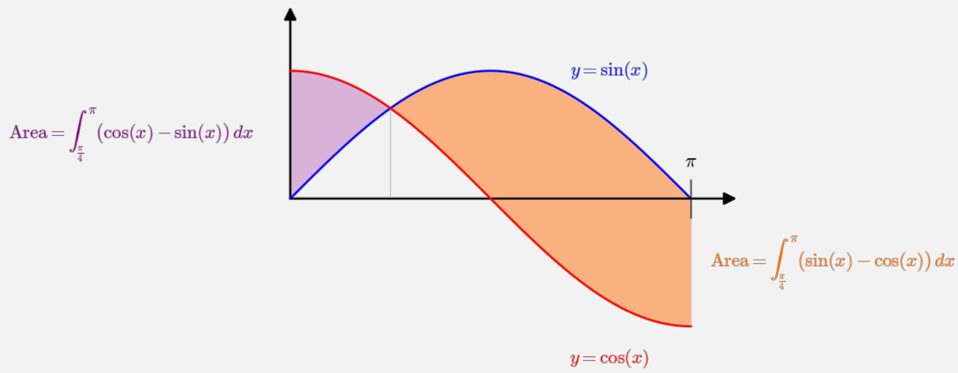


Notice that the roles of top and bottom change when the graphs intersect. So, we should first determine where this point of intersection occurs. I.e., we want to find the value of x between 0 and π such that $\cos(x) = \sin(x)$. Recalling the definition of $\cos(x)$ and $\sin(x)$ in terms of the unit circle, we see that this occurs when $x = \pi/4$.



In particular, we consider the triangle of hypotenuse 1 whose sides are $\cos(x)$ and $\sin(x)$ for the special value of x giving us $\cos(x) = \sin(x)$. We simply note that since the horizontal and vertical sides of this triangle have the same length, we must have a $45°$ angle, and so $x = \pi/4$.

Thus, from our graph, we see that for $0 \leq x \leq \pi/4$, $y = \cos(x)$ is the top curve and $y = \sin(x)$ is the bottom curve. For $\pi/4 \leq x \leq 4$, however, the curve $y = \sin(x)$ is on the top and $y = \cos(x)$ is on the bottom. We'll thus compute the area in two steps by breaking the integral up at $\pi/4$.



Now we simply compute each of these integrals separately and add the areas together to obtain the total area enclosed by the two curves.

$$\begin{aligned}
\int_0^{\pi/4} (\cos(x) - \sin(x))\ dx &= (\sin(x) - (-\cos(x)))\Big|_0^{\pi/4} \\
&= (\sin(\pi/4 - \cos(\pi/4) - (\sin(0) + \cos(0)) \\
&= (\sqrt{2}/2 + \sqrt{2}/2) - (0 + 1) \\
&= \sqrt{2} - 1
\end{aligned}$$

$$\begin{aligned}
\int_{\pi/4}^{\pi} (\sin(x) - \cos(x))\ dx &= (-\cos(x) - \sin(x))\Big|_{\pi/4}^{\pi} \\
&= (-\cos(\pi) - \sin(\pi)) - (-\cos(\pi/4) - \sin(\pi/4)) \\
&= (-(-1) - 0) - (-\sqrt{2}/2 - \sqrt{2}/2) \\
&= 1 + \sqrt{2}
\end{aligned}$$

Finally, adding these together we see that the total area between $y = \cos(x)$ and $y = \sin(x)$ is $2\sqrt{2}$.

So, back to our earlier concern about determining the top and bottom curves without a graph. Since we have to worry that the roles of the top and bottom can switch, let's first determine where all of these possible switches could concern. For simplicity, let's suppose $f(x)$ and $g(x)$ are both continuous. Then we want to find all all the points where $y = f(x)$ intersects $y = g(x)$, meaning we need to solve the equation $f(x) = g(x)$.

Once we have all of the intersection points, we'll break the interval we're integrating over up at these points. For example, if we were concerned with the interval $[a,b]$ and $f(x) = g(x)$ at $x = c$, $x = d$, and $x = e$, where $a < c < d < b < e$, then we would want to consider integrating over the intervals $[a,c]$, $[c,d]$, and $[d,b]$. (Notice the point e here did not actually matter since it was outside of the interval we care about.)

On each of these intervals we would then check to see which curve was on the top and the bottom by simply evaluating each of the functions at some point inside the intervals (but *not* the endpoints, since we know the functions are equal there).

Example 2.3.
Find the area of the region enclosed by $y = x^2 - 1$ and $y = 1 - x^2$ on the interval $[-2, 2]$.

First we find where the roles of top and bottom may switch:

$$\begin{aligned} & x^2 - 1 = 1 - x^2 \\ \implies & 2x^2 = 2 \\ \implies & x^2 = 1 \\ \implies & x = \pm 1 \end{aligned}$$

Now we consider the intervals $[-2,-1]$, $[-1,1]$, and $[1,2]$ and evaluate both $x^2 - 1$ and $1 - x^2$ at some point inside of each interval.

Interval	**Point**	$x^2 - 1$	$1 - x^2$	**Conclusion**
$[-2,-1]$	$-3/2$	$5/4$	$-5/4$	$x^2 - 1$ is on top
$[-1,1]$	0	-1	1	$1 - x^2$ is on top
$[1,2]$	$3/2$	$5/4$	$-5/4$	$x^2 - 1$ is on top

Now we compute the area of the closed region as

$$\begin{aligned}
&\int_{-2}^{-1} \left(\left(x^2-1\right)-\left(1-x^2\right)\right)\,dx \\
&\quad+\int_{-1}^{1} \left(\left(1-x^2\right)-\left(x^2-1\right)\right)\,dx \\
&\quad+\int_{1}^{2} \left(\left(x^2-1\right)-\left(1-x^2\right)\right)\,dx \\
&=\int_{-2}^{-1} \left(2x^2-2\right)\,dx+\int_{-1}^{1} \left(2-2x^2\right)\,dx+\int_{1}^{2} \left(2x^2-2\right)\,dx \\
&=\left(\frac{2x^3}{3}-2x\right)\Bigg|_{-2}^{-1}\left(2x-\frac{2x^3}{3}\right)\Bigg|_{-1}^{1}+\left(\frac{2x^3}{3}-2x\right)\Bigg|_{1}^{2} \\
&=\left[\left(\frac{-2}{3}+2\right)-\left(\frac{-16}{3}+4\right)\right] \\
&\quad+\left[\left(2-\frac{2}{3}\right)-\left(\frac{-2}{3}+2\right)\right] \\
&\quad+\left[\left(\frac{16}{3}-4\right)-\left(\frac{2}{3}-2\right)\right] \\
&=\left[\frac{4}{3}-\left(-\frac{4}{3}\right)\right]+\left[\frac{4}{3}-\left(-\frac{4}{3}\right)\right]+\left[\frac{4}{3}-\left(-\frac{4}{3}\right)\right] \\
&=\frac{8}{3}+\frac{8}{3}+\frac{8}{3} \\
&=8
\end{aligned}$$

Let's just notice that the two graphs x^2-1 and $1-x^2$ are curves which are easy to graph (even by hand), just to confirm our conclusion about which curves were on the top and the bottom.

Exercise 2.1.
It is possible to express the the area between the graphs $y=f(x)$ and $y=g(x)$ as one integral, regardless of the number of times the roles of top and bottom switch. How could this be done? (Hint: It may not be easy to actually evaluate that integral without splitting it back up.)

Exercise 2.2.
Above we assumed the functions $f(x)$ and $g(x)$ were continuous. How would we modify the strategy outlined above if $f(x)$ and $g(x)$ have discontinuities?

In some problems the interval of x-values may not be given to us and we have to determine what the interval is. In particular, if two curves intersect at exactly two points, and we want to find the area of the region enclosed by the curves, their intersection points will tell us the interval.

Example 2.4.
Find the area of the region enclosed by $y = x$ and $y = x^2 - 2$.

For this example we will do all of our calculations without viewing a graph of the regions, even though these two curves could easily be graphed by hand, just to emphasize that we don't strictly *need* a graph: we can determine everything we need with some basic algebra.

First we must determine where these two curves intersect:

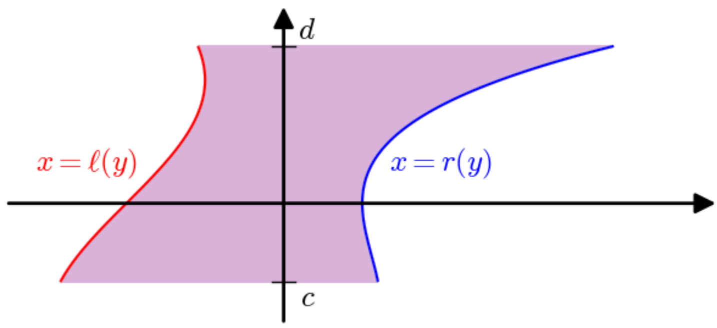
$$\begin{aligned} & x = x^2 - 2 \\ \implies & x^2 - x - 2 = 0 \\ \implies & (x+1)(x-2) = 0. \end{aligned}$$

The curves intersect at $x = -1$ and $x = 2$, so we'll integrate over $[-1, 2]$. Now we still need to determine which curve is on the top and which is on the bottom. To do this we'll simply evaluate each function at some point inside $[-1, 2]$, such as $x = 0$. At $x = 0$ the function $y = x$ gives us 0, and the function $y = x^2 - 2$ gives us -2, thus $y = x$ is on the top and $y = x^2 - 2$ is on the bottom. Now we can compute the enclosed area as

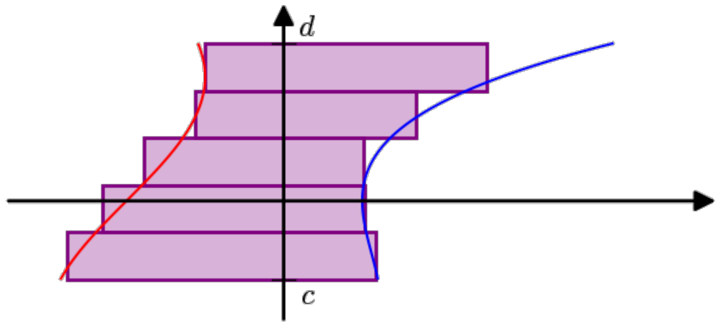
$$\begin{aligned}\int_{-1}^{2}\left(x-\left(x^2-2\right)\right)\,dx &= \int_{-1}^{2}\left(-x^2+x+2\right)\,dx \\ &= \left(\frac{-x^3}{3}+\frac{x^2}{2}+2x\right)\Big|_{-1}^{2} \\ &= \left(\frac{-8}{3}+2+4\right)-\left(\frac{1}{3}+\frac{1}{2}-2\right) \\ &= \frac{10}{3}-\left(\frac{-7}{3}\right)=\frac{20+7}{6}=\frac{27}{6} \\ &= \frac{9}{2}\end{aligned}$$

In all of the examples we have seen thus far, our curves have both been graphs of a function of x, but this need not always be the case. If our graphs are functions of y, then the same sort of procedure applies, except we integrate with respect to y instead of x. Before jumping to the integrals, let's sketch what's happening in terms of Riemann sums.

Suppose that $x = \ell(y)$ and $x = r(y)$ are two graphs as indicated in the figure below, with $r(y) > \ell(y)$ for all y-values between c and d, and we want to find the area between these curves for $c \leq y \leq d$.



We can of course approximate this area using rectangles. However, now the change in the y-values determines the heights of the rectangles, and the widths of the rectangles are given by evaluating $r(y)$ and $\ell(y)$ are values y_i^* in our i-th subinterval of $[c, d]$ along the y- axis, then subtracting $\ell(y_i^*)$ from $r(y_i^*)$.



The area of this region is then approximated by the sums of these rectangles,

$$\text{Area} \approx \sum_{i=1}^{n} [r(y_i^*) - \ell(y_i^*)]\, \Delta y_i,$$

and taking the limit as the rectangles become arbitrarily skinny, we have

$$\text{Area} = \int [r(y) - \ell(y)]\ dy.$$

Exercise 2.3.
Why is $\ell(y_i^*)$ subtracted from $r(y_i^*)$? Why not $r(y_i^*)$ subtracted from $\ell(y_i^*)$, or some other expression?

Example 2.5.
Find the area of the region bounded by $x = y^2 - 2y - 1$ and $x = y\sqrt{y^2+1}$ with $0 \le y \le 3$.

To compute this area we need to calculate

$$\int_0^3 \left(y\sqrt{y^2+1} - \left(y^2 - 2y - 1\right)\right)\ dy$$
$$= \int_0^3 y\sqrt{y^2+1}\, dy - \int_0^3 \left(y^2 - 2y - 1\right)\ dy.$$

For the first term, we can perform the substitution $u = y^2 + 1$, $du =$

$2y\,dy$. Notice that $ydy = \frac{1}{2}du$, and so our first integral evaluates to

$$\begin{aligned}\int_0^3 y\sqrt{y^2+1}\,dy &= \frac{1}{2}\int_1^{10}\sqrt{u}\,du \\ &= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^{10} \\ &= \frac{1}{3}\left(10^{3/2}-1^{3/2}\right) \\ &= \frac{10^{3/2-1}}{3}.\end{aligned}$$

The second integral is even easier to compute:

$$\int_0^3 \left(y^2-2y-1\right)\,dy = \left(\frac{y^3}{3}-y^2-y\right)\Big|_0^3 = -3.$$

Together these tell us the area of the enclosed region is

$$\int_0^3 \left[y\sqrt{y^2+1}-\left(y^2-2y-1\right)\right]\,dy = \frac{10^{3/2-1}}{3}-(-3) = \frac{10^{3/2}+8}{3}.$$

We'll end our discussion of the area between curves by making two observations and having one final example.

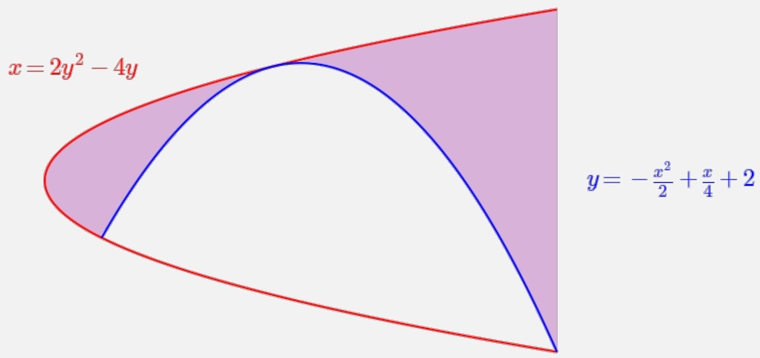
1. If a region is divided into two non-overlapping pieces, the area of the entire region can be computed by adding the areas of the pieces together.

2. When dividing a region into pieces, it may be convenient to integrate one piece with respect to x, but integrate another piece with respect to y.

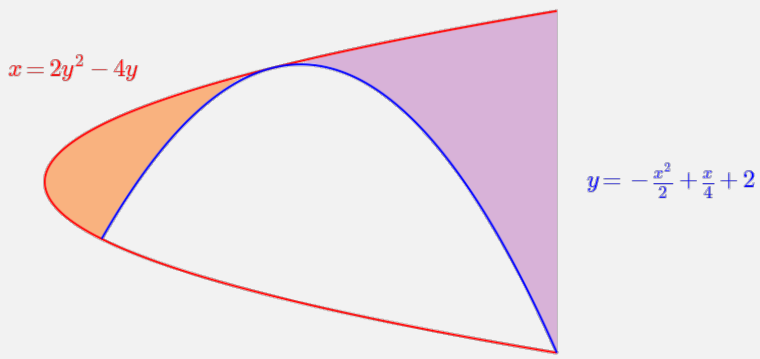
Let's have one example which applies both of these observations.

Caution: The example below requires a fair bit of work, including some tedious algebra. The most important part of the problem, though, is the setup. You should feel free to skim this example the first time you look at it, focusing on the big ideas in the setup. If the algebra seems overly tedious, don't worry too much about it; don't worry about getting bogged down in the details.

Example 2.6.
Compute the area of the shaded region below.



One way of doing this is to divide the region into two pieces as shown below,



We'll compute the areas of the orange and purple regions separately. Notice that for the purple piece the boundary curves (the top and bottom) are graphs of functions of x since they pass the vertical line test. For this reason we will perform that integral with respect to x. The boundaries of the orange piece (the left- and right-hand sides) are functions of y since they pass the horizontal line test. Thus we'll compute the area of the orange piece by integrating with respect to y.

In both cases we need to determine where the curves intersect, so first we need to d a little bit of algebra.

Points on the red curve satisfy $x = 2y^2 - 4y$, and points on the blue curve satisfy $y = {-x^2}/2 + x/4 + 2$. At the intersection points, both equations are satisfied, and so we have a system of equations,

$$\begin{aligned} x &= 2y^2 - 4y \\ y &= \frac{-x^2}{2} + \frac{x}{4} + 2. \end{aligned}$$

Plugging in $y = {-x^2}/2 + x/4 + 2$ in for the y that appears in the first equation gives us

$$\begin{aligned} x &= 2\left(\frac{-x^2}{2} + \frac{x}{4} + 2\right)^2 - 4\left(\frac{-x^2}{2} + \frac{x}{4} + 2\right) \\ &= 2\left[\frac{x^4}{4} - x^2\left(\frac{x}{4} + 2\right) + \left(\frac{x}{4} + 2\right)^2\right] + 2x^2 - x - 8 \\ &= 2\left[\frac{x^4}{4} - \frac{x^3}{4} - 2x^2 + \frac{x^2}{16} + x + 4\right] + 2x^2 - x - 8 \\ &= \frac{x^4}{2} - \frac{x^3}{2} - 4x^2 + \frac{x^2}{8} + 2x + 8 + 2x^2 - x - 8. \end{aligned}$$

Multiplying through by 8 to remove the denominators we then have

$$8x = 4x^4 - 4x^3 - 32x^2 + x^2 + 16x + 16x^2 - 8x$$

Moving all of the terms to the left-hand side of the equation and combining like-terms, the equation then becomes

$$4x^4 - 4x^3 - 15x^2 = 0.$$

Conveniently, the polynomial on the left factors as

$$4x^4 - 4x^3 - 15x^2 = x^2\left(4x^2 - 4x - 15\right) = x^2(2x - 5)(2x + 3).$$

This tells us that our curves intersect at three points: $x = 0$, $x = 5/2$, and $x = {-3}/2$.

For the integral which gives the area of the orange region, we need to know the y-values bounding this region since we will integrate with respect to y. Notice that if we plug in $x = -3/2$ and $x = 0$ into $y = {-x^2}/2 + x/4 + 2$, then we obtain the y-values $y = 1/2$ and $y = 2$. Before we can write out our integral, we need to write the portion of the blue curve (giving the right-hand side of the orange region) as a function of y. This requires just a little bit of algebra:

$$\begin{aligned} y &= \frac{-x^2}{2} + \frac{x}{4} + 2 \\ \implies -2y &= x^2 - \frac{x}{2} - 4 \\ \implies 4 - 2y &= x^2 - \frac{x}{2} \end{aligned}$$

Completing the square on the right-hand side gives us the following:

$$\begin{aligned} 4 - 2y &= x^2 - \frac{x}{2} \\ &= x^2 - \frac{x}{2} + \frac{1}{16} - \frac{1}{16} \\ &= \left(x - \frac{1}{4}\right)^2 - \frac{1}{16} \end{aligned}$$

Adding $1/16$ to both sides of the equation we have

$$\frac{65}{16} - 2y = \left(x - \frac{1}{4}\right)^2,$$

and solving for x this gives us

$$x = \pm\sqrt{\frac{65}{16} - 2y} + \frac{1}{4}.$$

Notice this means there are two possible functions which give x in terms of y satisfying the equation $y = \frac{-x^2}{2} + \frac{x}{4} + 2$, namely $x = \sqrt{\frac{65}{16} - 2y} + \frac{1}{4}$ and $x = -\sqrt{\frac{65}{16} - 2y} + \frac{1}{4}$. To determine which of these two roots we should use in our problem, let's simply notice that when these functions are evaluated at $y = 1$ we obtain one positive value and one negative value:

$$\sqrt{\frac{65}{16}-2}+\frac{1}{4}=\sqrt{\frac{33}{16}}+\frac{1}{4}=\frac{1+\sqrt{33}}{4}>0,$$
$$-\sqrt{\frac{65}{16}-2}+\frac{1}{4}=-\sqrt{\frac{33}{16}}+\frac{1}{4}=\frac{1-\sqrt{33}}{4}>0.$$

We don't really care what the exact value of these expressions might be, we just care about the sign for now. In particular, for the portion of the cure in our picture, we see that the x value that occurs when $y=1$ is negative, and so we will use the negative root.

That is, for computing the area of the orange region, we will integrate with respect to y, and the right-hand curve is given by $x=-\sqrt{65/16-2y}+1/4$,and the left-hand curve is given by $x=2y^2-4y$. Thus the area of the orange region is given by the integral

$$\int_{1/2}^{2}\left[-\sqrt{\frac{65}{16}-2y}+\frac{1}{4}-\left(2y^2-4y\right)\right]\,dy.$$

We will compute this integral by breaking it into two parts as follows:

$$\begin{aligned}&\int_{1/2}^{2}\left[-\sqrt{\frac{65}{16}-2y}+\frac{1}{4}-\left(2y^2-4y\right)\right]\,dy\\ &=\int_{1/2}^{2}\left[\frac{1}{4}-2y^2+4y-\sqrt{\frac{65}{16}-2y}\right]\,dy\\ &=\int_{1/2}^{2}\left(\frac{1}{4}-2y^2+4y\right)\,dy-\int_{1/2}^{2}\sqrt{\frac{65}{16}-2y}\,dy.\end{aligned}$$

The first integral is straight-forward:

$$\begin{aligned}\int_{1/2}^{2}\left(\frac{1}{4}-2y^2+4y\right)\,dy&=\left(\frac{y}{4}-\frac{2y^3}{3}+2y^2\right)\Big|_{1/2}^{2}\\ &=\left(\frac{1}{2}-\frac{16}{3}+8\right)-\left(\frac{1}{8}-\frac{1}{12}+\frac{1}{2}\right)\\ &=\frac{21}{8}.\end{aligned}$$

For the second integral we'll perform the substitution $u = \frac{65}{16} - 2y$, $du = -2\,dy$ and obtain

$$\begin{aligned}
\int_{1/2}^{2} \sqrt{\frac{65}{16} - 2y}\,dy &= \frac{-1}{2}\int_{49/16}^{1/16} u^{1/2}\,du \\
&= \frac{1}{2}\int_{1/16}^{49/16} u^{1/2}\,du \\
&= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_{1/16}^{49/16} \\
&= \frac{1}{3}\left(\left[\frac{49}{16}\right]^{3/2} - [1/16]^{3/2}\right) \\
&= \frac{1}{3}\cdot\left(\frac{343}{64} - \frac{1}{64}\right) \\
&= \frac{1}{3}\cdot\frac{342}{64} = \frac{114}{64} \\
&= \frac{57}{32}.
\end{aligned}$$

Thus the total area of the orange region is

$$\frac{21}{8} - \frac{57}{32} = \frac{84-57}{32} = \frac{27}{32}.$$

To find the area of the purple region, we'll integrate the top (red curve) minus the bottom (blue curve) from $x = 0$ to $x = 5/2$. As the red curve was given as the graph of a function of y, however, we first need to write it as the graph of a function of x, and this is accomplished with a little bit of algebra.

$$\begin{aligned}
x &= 2y^2 - 4y \\
\implies \frac{x}{2} &= y^2 - 2y
\end{aligned}$$

We now complete the square on the right-hand side,

$$y^2 - 2y = y^2 - 2y + 1 - 1 = (y-1)^2 - 1.$$

Thus we have

$$
\begin{aligned}
\frac{x}{2} &= (y-1)^2 - 1 \\
\implies (y-1)^2 &= \frac{x}{2} + 1 \\
\implies y - 1 &= \pm\sqrt{\frac{x}{2} + 1} \\
\implies y &= 1 \pm \sqrt{\frac{x}{2} + 1}.
\end{aligned}
$$

The top curve we have in our picture is above the line $y = 1$, and so we will need to use the expression with the positive root; our top curve is given by $y = 1 + \sqrt{x/2 + 1}$. Keeping in mind the bottom curve is given by $y = -x^2/2 + x/4 + 2$, we simply compute the following integral to determine the area of the purple region,

$$
\begin{aligned}
&\int_0^{5/2} \left[1 + \sqrt{\frac{x}{2} + 1} - \left(\frac{-x^2}{2} + \frac{x}{4} + 2\right)\right] dx \\
&= \int_0^{5/2} \left(\frac{x^2}{2} - \frac{x}{4} - 1\right) dx + \int_0^{5/2} \sqrt{\frac{x}{2} + 1}\, dx.
\end{aligned}
$$

The first integral is compute easily,

$$
\begin{aligned}
\int_0^{5/2} \left(\frac{x^2}{2} - \frac{x}{4} - 1\right) dx &= \left(\frac{x^3}{6} - \frac{x^2}{8} - x\right)\Bigg|_0^{5/2} \\
&= \frac{125}{48} - \frac{25}{32} - \frac{5}{2} \\
&= \frac{-65}{96}.
\end{aligned}
$$

The second integral requires an easy substitution $u = \frac{x}{2} + 1$, $du =$

$\frac{1}{2}dx$,

$$\begin{aligned}\int_0^{5/2}\sqrt{\frac{x}{2}+1}\,dx &= 2\int_1^{9/4}u^{1/2}\,du\\ &= \frac{4}{3}u^{3/2}\Big|_1^{9/4}\\ &= \frac{4}{3}\left(\left(\frac{9}{4}\right)^{3/2}-1^{3/2}\right)\\ &= \frac{4}{3}\left(\frac{27}{8}-1\right)=\frac{4}{3}\cdot\frac{19}{8}\\ &= \frac{19}{6}.\end{aligned}$$

The area of the purple region is thus

$$\frac{19}{6}-\frac{65}{96}=\frac{239}{96}.$$

Adding the areas of the orange and purple regions together, we see that the area of the entire shaded region is thus

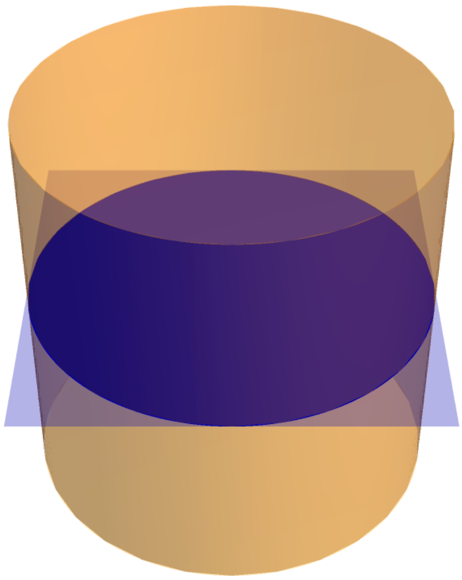
$$\frac{141}{32}+\frac{19}{6}.$$

2.2 Volume by slicing

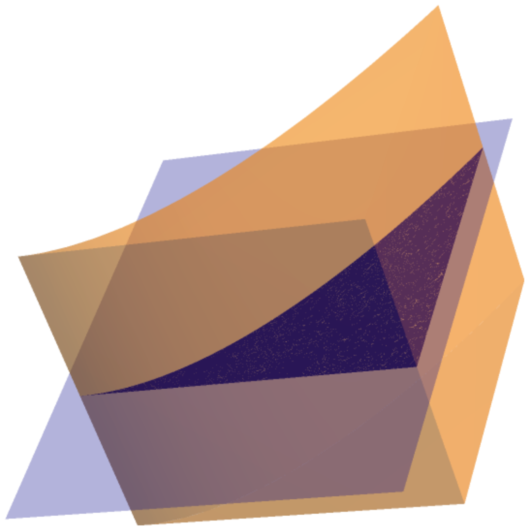
When you first learn about integrals, you're often motivated by the problem of finding the area under the graph of a function. However, integrals are used to compute all sorts of quantities besides area. For example, in your first semester calculus class you likely saw that integrals can be used to compute displacement by integrating velocity. Now we will show how integrals can be used to compute another quantity of interest: volume.

There are a few special ways that we can use integrals to compute volumes in this class, and if you take multivariable calculus you will see a more general technique.

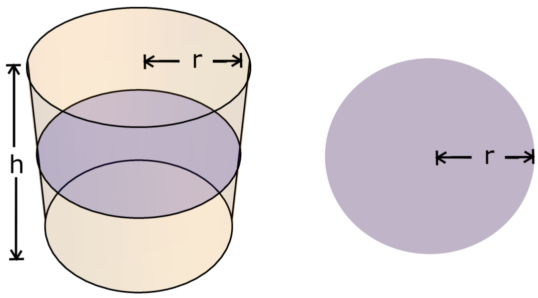
Before describing how to compute volume using integrals, though, we will need to discuss some preliminaries. We say that a three-dimensional solid is ***cylindrical*** if all of its cross sections along some axes give the same two-dimensional shape. In the case of traditional round cylinders these cross sections are circles.



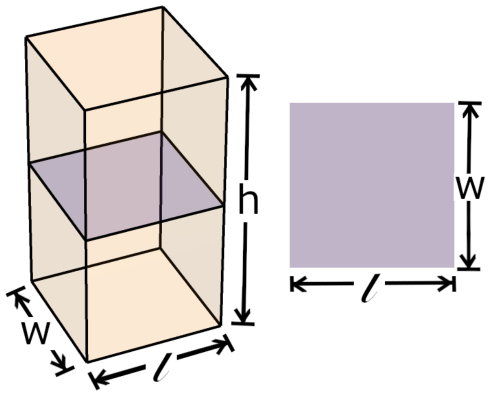
These cross sections can be more complicated than circles, however; in principle they can be any arbitrarily complicated two-dimensional shape.



One of the nice properties of these cylindrical solids is that their volumes are very easy to compute: if the cross sections of a cylindrical solid have area A and the length of the axis perpendicular to the cross sections (e.g., the height of the solids shown above) is h, then the volume of the solid is simply Ah. From this we can recover a few families of volume formulas. For example, the volume of a round cylinder of radius r and height h is $\pi r^2 h$ since this is just the area of the cross-sectional circle times the height of the cylinder.



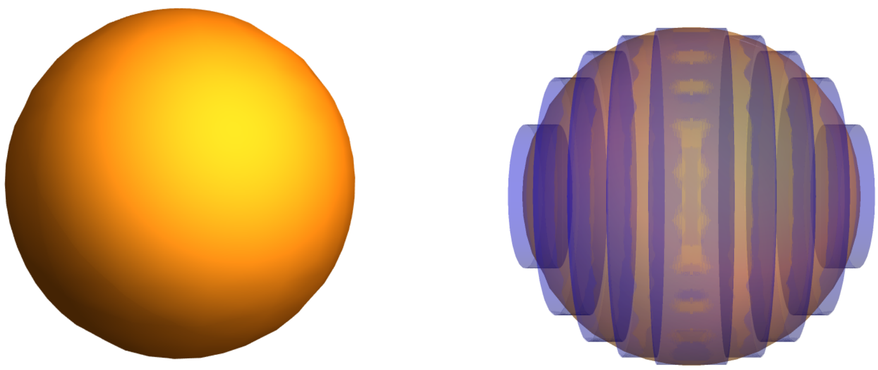
The volume of a rectangular prism of dimensions ℓ, w and h is also ℓwh: the area of the cross-sectional square is ℓw and the height is h.



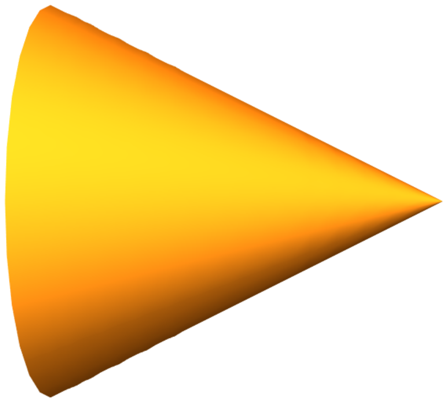
Now suppose that we have a solid which *is not* cylindrical. How can we use our knowledge of volumes of cylindrical solids to help us find the volume of this more complicated solid?

At first glance this seems like a hard problem, so let's do what we always do in calculus when we want to compute something but aren't sure how: let's approximate the quantity we care about with something we can actually compute. The idea in general is that we'll imagine taking a non-cylindrical solid and slicing it up into smaller pieces which we will approximate with cylindrical solids.

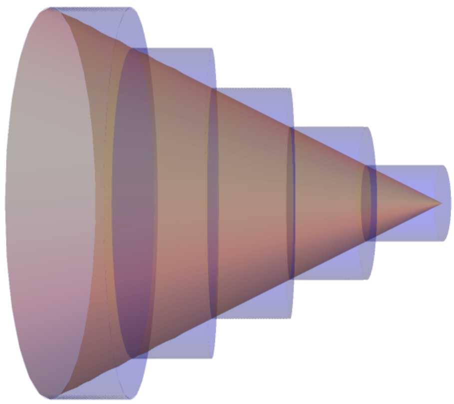
In the picture below we have a sphere which is a non-cylindrical solid, and an approximation of that sphere by cylinders. By adding up the volumes of these approximating cylinders (which we can compute with the $V = Ah$ formula mentioned above), we have an approximation to the volume of the solid.



Just to illustrate the idea, let's try to approximate the volume of a cone. We'll imagine the cone is on its side with the x-axis going through the center of the base and out through the tip of the cone. Say the radius of the cone's base is r and its height (which because we're laying the cone on its side runs along the x-axis) is h.



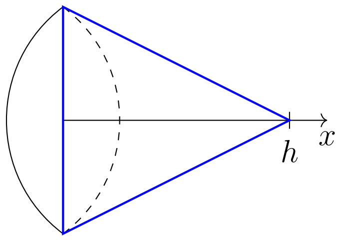
To estimate the volume of this cone, let's first imagine that we use five cylinders of equal height but varying circumferences. For concreteness, say we use five cylinders of height $^h/_5$.



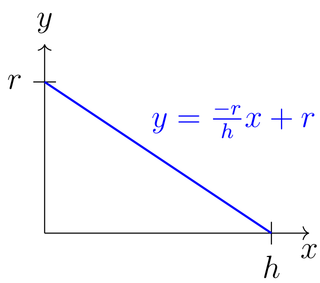
Each of these cylindrical pieces has an easy to compute volume. If we let r_1 denote the radius of the base of the first piece, r_2 the radius of the base of the second piece, and so on, then we'll estimate the volume of the cone to be

$$\sum_{i=1}^{5} \pi r_i^2 \frac{h}{5}.$$

If we could figure out what these radii r_i are, then we could actually compute this number. To do this, it's helpful to think about the following cross-section of our cone where we intersect the cone with a plane containing the x-axis.

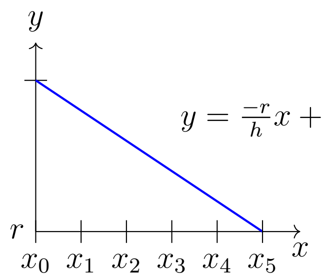


This gives us a triangle lying on its side with the x-axis running through the center. Note the vertical part of this triangle gives us the radius of our pieces. Given a value of x (distance from the base of the cone), the corresponding point on the line over the x-axis tells us the radius of the circular cross section of the cone at that point. We can easily see that the slope of this line is $-r/h$ and the y-intercept is r (the radius of the bottom of the cone), so the this line is given by $y = -rx/h + r$.



Since we're chopping our cone up into five pieces, this means we are interested in the following x-values (these correspond to the bases of the cylindrical pieces we're interested in),

$$
\begin{aligned}
x_0 &= 0 \\
x_1 &= \frac{h}{5} \\
x_2 &= \frac{2h}{5} \\
x_3 &= \frac{3h}{5} \\
x_4 &= \frac{4h}{5}
\end{aligned}
$$



And so the corresponding radii are

$$\begin{aligned}
r_1 &= \frac{-r}{h}x_0 + r = r\\
r_2 &= \frac{-r}{h}x_1 + r = \frac{4r}{5}\\
r_3 &= \frac{-r}{h}x_2 + r = \frac{3r}{5}\\
r_4 &= \frac{-r}{h}x_3 + r = \frac{2r}{5}\\
r_5 &= \frac{-r}{h}x_4 + r = \frac{r}{5}
\end{aligned}$$

Notice the radii here are given by $r_i = \frac{5-i}{5}r$.

Using these five cylindrical pieces, our estimate to the volume of the cone is

$$\begin{aligned}
\sum_{i=1}^{5} \pi r_i^2 \frac{h}{5} &= \sum_{i=1}^{5} \pi \left(\frac{5-i}{5}r\right)^2 \frac{h}{5}\\
&= \sum_{i=1}^{5} \pi \cdot \frac{25 - 10i + i^2}{25} \cdot r^2 \frac{h}{5}\\
&= \pi r^2 \frac{h}{5} \left(\sum_{i=1}^{5} 1 - \frac{2}{5} \sum_{i=1}^{5} i + \frac{1}{25} \sum_{i=1}^{5} i^2 \right)\\
&= \pi r^2 \frac{h}{5} \left(5 - 6 + \frac{330}{6} \right)\\
&= \pi r^2 \frac{h}{5} \cdot 54.
\end{aligned}$$

This is only an approximation to our volume, and we get better approximations by using more pieces. For n cylindrical pieces of the same height $\frac{h}{n}$, the i-th radius will be given by

$$\frac{-r}{h} \cdot \frac{hi}{n} + r,$$

and so the estimated volume will be

$$\sum_{i=1}^{n} \pi \left(\frac{-r}{h} \cdot \frac{hi}{n} + r \right)^2 \cdot \frac{h}{n}.$$

After looking at this for a minute you may realize this is a Riemann sum for $\pi \left(\frac{-r}{h}x + r \right)^2$ on the interval $[0, h]$ using n rectangles of equal width. The limit of these Riemann sums thus gives us the integral

$$\int_0^h \pi \left(\frac{-r}{h}x + r \right)^2 \, dx.$$

Evaluating this integral gives us the standard formula for the volume of a cone of height h and radius r:

$$\begin{aligned}
\text{Volume} &= \int_0^h \pi \left(\frac{-r}{h}x + r \right)^2 \, dx \\
&= \pi \int_0^h \left(\frac{r^2}{h^2}x^2 - \frac{2r^2}{h}x + r^2 \right) \, dx \\
&= \pi \left(\frac{r^2x^3}{3h^2} - \frac{r^2x^2}{h} + r^2x \right) \Bigg|_0^h \\
&= \pi \left(\frac{r^2h}{3} - r^2h + r^2h \right) \\
&= \frac{\pi r^2 h}{3}
\end{aligned}$$

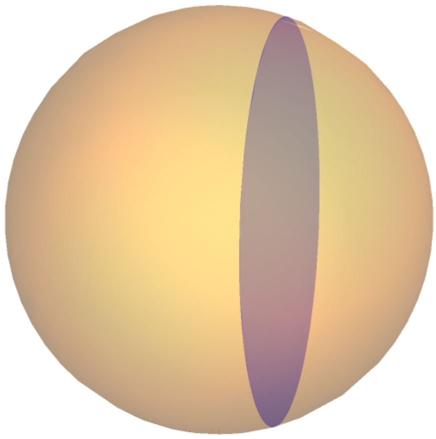
Notice that in the above we estimated volume by slicing our solid into pieces which we approximated with cylindrical solids whose volumes were easy to compute. Adding these volumes of cylinders and taking a limit resulted in integrating the cross-sectional area of "infinitely-thin" slices of our solid.

Computing volume in the way described above (integrating cross-sectional area) is often called ***volume by slicing***, and we'll see many special cases of these calculations soon, but in general we have the following proposition which summarizes the above discussion.

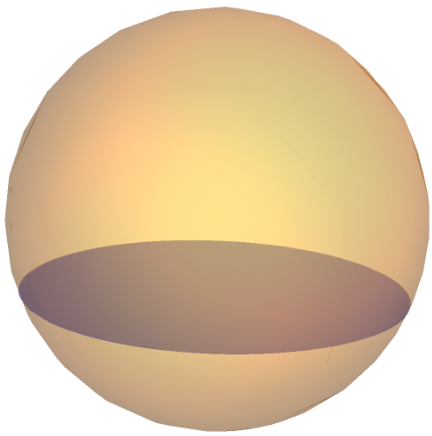
Proposition 2.1.
Suppose a three-dimensional solid is positioned so the x-axis goes through it from $x = a$ to $x = b$. If the cross-sectional area of the slice of the solid located at position x is denoted $A(x)$, then the volume of the solid may be

computed as

$$\text{Volume} = \int_a^b A(x)\,dx.$$

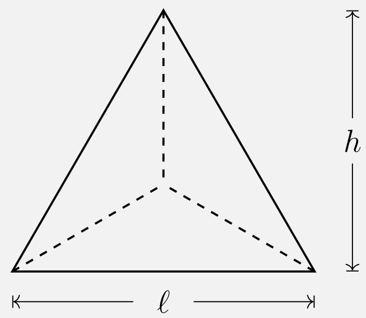


Though we stated the proposition above with the cross-sectional area being a function of x, this isn't strictly necessary. In some problems it may be more natural for the cross-sectional area to be a function of y. Regardless, the take-away is that we can compute volume by integrating cross-sectional area.

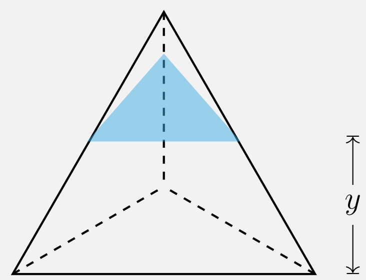


As another example of calculating volume in this way, let's determine the volume of a triangular pyramid.

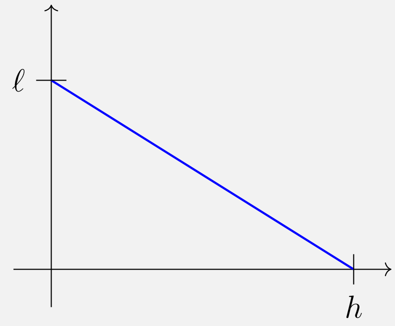
Example 2.7.
Find the volume of a triangular pyramid of height h whose base is an equilateral triangle of side length ℓ.



Let's first determine the cross-sectional area obtained by slicing the pyramid with a horizontal plane at height y from the base to get an equilateral triangle.



Notice the side lengths start at ℓ when $y = 0$ (the base of the pyramid) and decrease linearly to 0 when $y = h$ (the top of the pyramid). That is, the side lengths are a function of y whose graph is a line through $(0, \ell)$ and $(h, 0)$.



The side length for the cross-section at y is thus $\frac{-\ell}{h}y+\ell$. The area of a triangle of side length s is $\frac{\sqrt{3}}{4}s^2$, so the area of our cross-sections is

$$A(y) = \frac{\sqrt{3}}{4}\left(\frac{-\ell}{h}y+\ell\right)^2.$$

Now we can compute the volume of the pyramid as

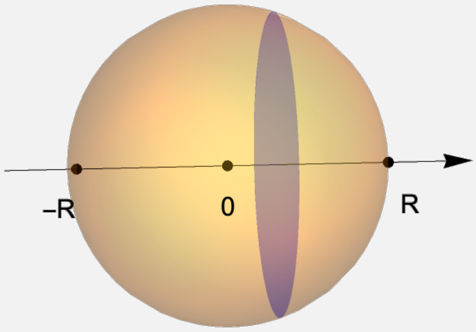
$$\int_0^h \frac{\sqrt{3}}{4}\left(\frac{-\ell}{h}y+\ell\right)^2 \, dy.$$

Using the substitution $u = \frac{-\ell}{h}y+\ell$, $du = \frac{-\ell}{h}dy$, this integral becomes

$$\begin{aligned}
\text{Volume} &= \int_0^h \frac{\sqrt{3}}{4}\left(\frac{-\ell}{h}y+\ell\right)^2 \, dy \\
&= \frac{-\sqrt{3}h}{4\ell}\int_\ell^0 u^2\,du \\
&= \frac{\sqrt{3}h}{4\ell}\int_0^\ell u^2\,du \\
&= \left.\frac{\sqrt{3}hu^3}{12\ell}\right|_0^\ell \\
&= \frac{\sqrt{3}h\ell^2}{12}.
\end{aligned}$$

Example 2.8.
Compute the volume of a sphere of radius R by integrating cross-sectional area.

If we imagine that the sphere is placed so that the x-axis runs through its center in three-dimensional space, then the sphere intersects the x-axis along the interval $[-R, R]$. Intersecting the sphere with planes perpendicular to the x-axis gives us discs of radius that start at 0, increase up to R at the center of the sphere, then decrease back down to 0.



To find the volume of the sphere, we need the radius of each disc. The radii of the discs are given by the vertical distance between a point of the sphere (on the edge of the disc) and the x-axis (center of the disc).

We see these radii are functions of x, say $r(x)$ is the radius of the disc centered at x in $[-R, R]$ on the x-axis. The graph of $r(x)$ is then a semi-circle of radius R centered at the origin, and from this we see that

$$r(x) = \sqrt{R^2 - x^2}.$$

Thus the area of the associated disc is $\pi r(x)^2 = \pi\left(R^2 - x^2\right)$, and we can now compute the volume of the sphere as follows.

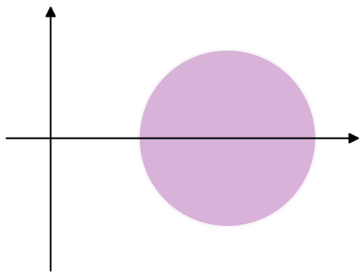
$$\begin{aligned}
\text{Volume} &= \int_{-R}^{R} \pi\left(R^2 - x^2\right)\, dx \\
&= \pi\left(R^2 x - \frac{x^3}{3}\right)\Bigg|_{-R}^{R} \\
&= \pi\left(\left[R^2 \cdot R - \frac{R^3}{3}\right] - \left[R^2 \cdot (-R) - \frac{(-R)^3}{3}\right]\right) \\
&= \pi\left[\frac{2R^3}{3} - \left(\frac{-2R^3}{3}\right)\right] \\
&= \frac{4}{3}\pi R^3
\end{aligned}$$

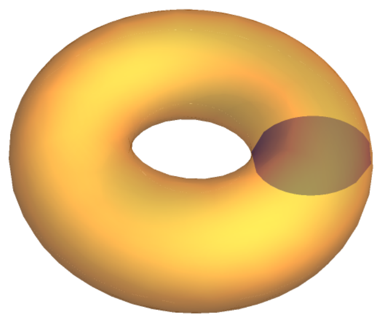
2.3 Solids of revolution

In the last section we saw that the volume of a solid could be obtained by integrating cross-sectional area. For this to be useful, though, we must be able to compute these cross-sectional areas, and in general this can be difficult. In this section we will restrict ourselves to special types of solids whose cross-sectional areas are easier to compute called *solids of revolution*.

A ***solid of revolution*** is a three-dimensional solid obtained by revolving a region in the plane 360° around a line (usually the x-axis, y-axis, or a line parallel to one of these). Notice this produces two-dimensional shapes in every plane that contains that axis of rotation. By gluing together all of these two-dimensional slices we obtain a solid three-dimensional object.

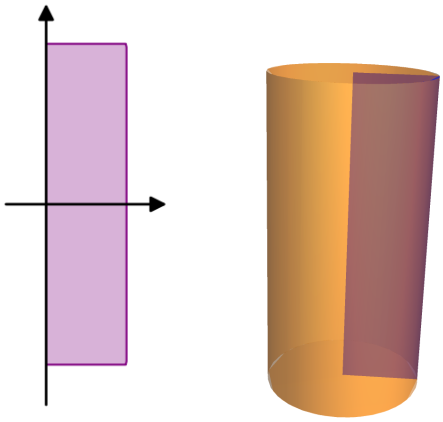
For example, imagine that we rotate the disc of radius 1 centered at $(2, 0)$ around the y-axis. This means the disc will come out of the page (or screen) representing the plane and spin around that axis. We attach all of the two-dimensional discs obtained by this process together and we obtain a three-dimensional solid.



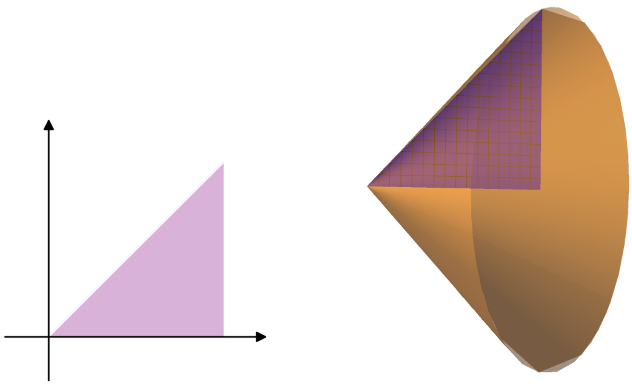


In this particular case the solid we obtain is shaped like a doughnut. (The mathematical term for this object is a *torus*.)

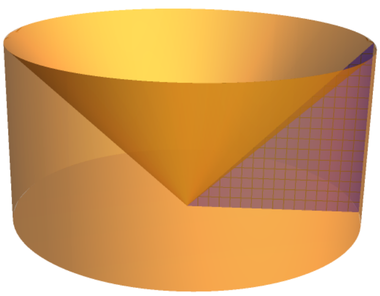
As another example, imagine rotating a rectangle which is flush against the y-axis around the axis. This gives a cylinder.



Rotating the right triangle presented below around the x-axis gives a cone:



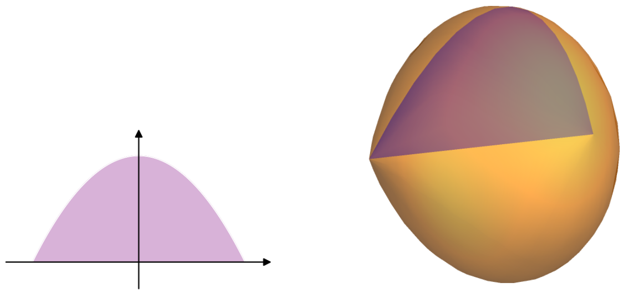
Notice that if we change the axis of rotation from the x-axis to the y-axis in the example above we obtain a very different solid.



The disc and washer methods

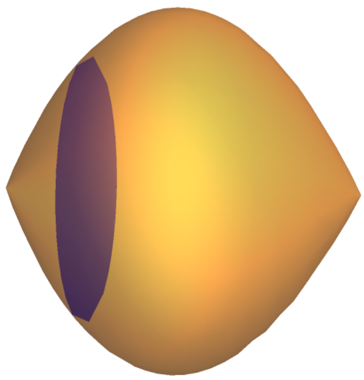
Once we have one of these solids of revolution, the question we are interested in is what is the cross-sectional area of that solid, since that's what we need to compute the solid's volume.

To determine the cross-sectional area, let's first suppose the region we are rotating around an axis is flush against that axis. For example, the region below is bounded by the x-axis and the graph $y = 1 - x^2$, and we can rotate that around the x-axis to obtain a solid.



Intersecting this solid with planes sticking out perpendicularly from the plane containing our original two-dimensional figure gives us a cross-section. If we can determine the area of all the cross sections that arise in this way, then we can integrate those areas to determine the volume of the solid.

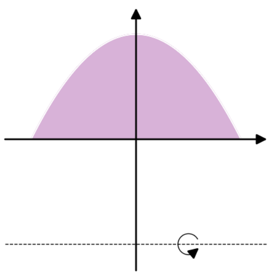
Notice, though, that each of these cross sections can be obtained as follows. Take one vertical line segment from the x-axis to the graph $y = 1 - x^2$ and rotate that single line segment around the x-axis in three-dimensional space. This carves out our cross section, and we can now easily see that cross section must be a disc.

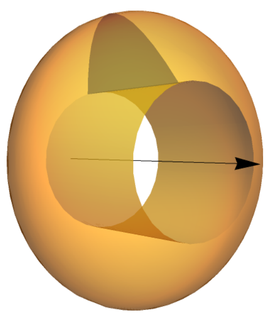


The area of a disc is easily computed as πr^2 where r is the disc's radius, so all that we need to do is determine that radius. The radius, though, is simply the height of the line segment from above! At a point x between -1 and 1 this height is just $1 - x^2$. That is, the area of the cross-section corresponding to x is $\pi(1 - x^2)^2$. Now that we have a formula for the area of our cross section, we can easily integrate the cross-sectional areas to determine the volume of our solid:

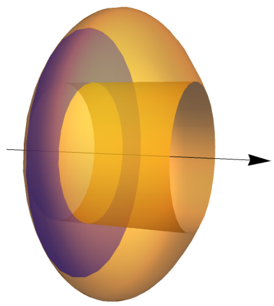
$$\begin{aligned}
\text{Volume} &= \int_{-1}^{1} \pi \left(1 - x^2\right)^2 \, dx \\
&= \pi \int_{-1}^{1} \left(1 - 2x^2 + x^4\right) \, dx \\
&= \pi \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right) \Bigg|_{-1}^{1} \\
&= \pi \left[\left(1 - \frac{2}{3} + \frac{1}{5} \right) - \left((-1) + \frac{2}{3} - \frac{1}{5} \right) \right] \\
&= \pi \left[2 - \frac{4}{3} + \frac{2}{5} \right] \\
&= \pi \left[\frac{30 - 20 + 6}{15} \right] \\
&= \frac{16}{15}\pi
\end{aligned}$$

Suppose that instead of rotating our region above around the x axis, we rotate around the line $y = -1$. This will certain change our solid:





Instead of having cross sections which are discs, we now have *annuli*. (An annulus is a "ring" obtained by removing a small, concentric disc from a larger disc.)



Notice that if the outer radius of an annulus is R and the inner radius is r, then the area of the annulus is

$$\pi R^2 - \pi r^2 = \pi(R^2 - r^2).$$

In our cross sections, we thus need to find both the outer and inner radii.

Note that our outer and inner radii are just the distances from the axis of rotation to the top or bottom curve enclosing the figure we rotate. In particular, the outer radius comes from the semicircle $y = 1 - x^2$ just as before, but because we're rotating around the line $y = -1$ instead of $y = 0$ (aka the x-axis), we are one unit further away from the axis of rotation. That is, the outer radius is

$$1 + (1 - x^2) = 2 - x^2.$$

The inner radius, however, is always just 1 since the bottom of our figure is at the x axis which is one unit above the axis of rotation in our problem. Putting these pieces of information together, we see that the cross-sectional area of our figure (corresponding to the cross section at x) is

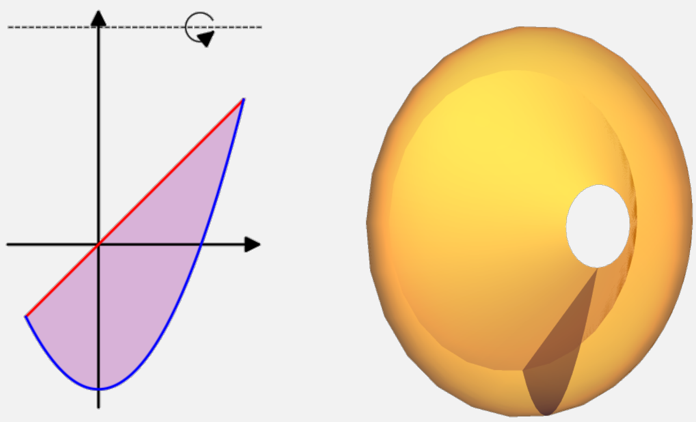
$$\pi\left[\left(2 - x^2\right)^2 - 1\right] = \pi\left[4 - 4x^2 + x^4 - 1\right] = \pi\left[3 - 4x^2 + x^4\right]$$

and so the volume of the solid is

$$\begin{aligned}
\int_{-1}^{1} \pi\left(3-4x^2+x^4\right)\,dx &= \pi\left(3x-\frac{4}{3}x^3+\frac{1}{5}x^5\right)\Big|_{-1}^{1} \\
&= \pi\left(\left[3-\frac{4}{3}+\frac{1}{5}\right]-\left[-3+\frac{4}{3}-\frac{1}{5}\right]\right) \\
&= \pi\left(6-\frac{8}{3}+\frac{2}{5}\right) \\
&= \pi\left(\frac{90-40+6}{15}\right) \\
&= \frac{56}{15}\pi
\end{aligned}$$

Notice that the important details in computing a volume of a solid of revolution like this are not simply the "top" and "bottom" curves, but rather the distances of those curves from the axis of rotation. This can take on various forms depending on, for instance, if the axis of rotation is below or above the figure, as the next example shows.

Example 2.9.
Find the volume of the solid of revolution obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2 - 2$ around the line $y = 4$. (We had seen in in the previous section these curves intersect at $x = -1$ and $x = 2$.)



The cross sections are again annuli, but now the outer radius is given by the bottom curve and the inner radius is given by the top curve. Since our axis of rotation is above the curve, we compute that the outer radius is $3-(x^2-2)=5-x^2$ and the inner radius is $3-x$. Notice here we have the y-value of the line minus the curve, whereas in our earlier calculation we had the curve minus the y-value of the line. The distinction between these two is that the axis of rotation is above the figure in this example, but below the figure in our earlier calculation. (For this reason I think it's better to think in terms of "inner" and "outer" instead of getting fixated on "top" and "bottom.")

Now that we have the inner and outer radii, we see the cross-sectional area is given by

$$\begin{aligned}\pi\left[\left(5-x^2\right)^2-(3-x)^2\right] &= \pi\left[25-10x^2+x^4-\left(9-6x+x^2\right)\right]\\ &= \pi\left[25-10x^2+x^4-9+6x-x^2\right]\\ &= \pi\left[16+6x-11x^2+x^4\right]\end{aligned}$$

With the cross-sectional areas computed, we are now ready to compute the volume of the solid:

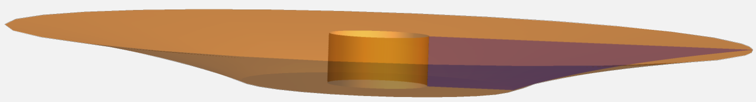
$$\begin{aligned}\text{Volume} &= \int_{-1}^{2}\pi\left[16+6x-11x^2+x^4\right]\,dx\\ &= \pi\int_{-1}^{2}\left[16+6x-11x^2+x^4\right]\,dx\\ &= \pi\left(16x+3x^2-\frac{11}{3}x^3+\frac{1}{5}x^5\right)\Big|_{-1}^{2}\\ &= \pi\left[\left(32+12-\frac{88}{3}+\frac{32}{5}\right)-\left(-16+3+\frac{11}{3}-\frac{1}{5}\right)\right]\\ &= \pi\left[57-\frac{77}{3}+\frac{33}{5}\right]\\ &= \pi\left[\frac{855-385+99}{15}\right]\\ &= \frac{569}{15}\pi\end{aligned}$$

The process of computing volumes of solids of revolution by integrating these cross-sectional areas where the cross sections are discs or annuli is sometimes called the ***disc method*** and the ***washer method***.

Remark.
Notice the disc and washer methods are really the same thing. In particular, the disc method is the same as the washer method where the inner radius is zero.

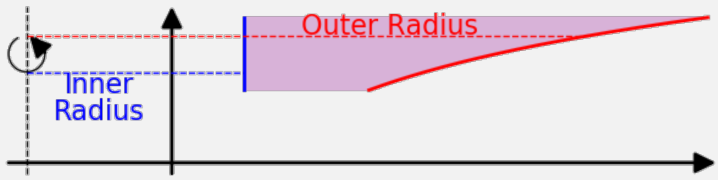
In all of the examples thus far we have rotated a figure around a line parallel to the x-axis. We could of course rotate around a line parallel to the y-axis instead. Mathematically this isn't really any different, though psychologically it may feel different.

Example 2.10.
Rotate the region bounded by the curves $y = \ln(x)$, $y = 1$, $y = 2$, and $x = 0$ around the line $x = -2$.



In order to obtain our annuli (washers) we will consider intersecting the solid with planes perpendicular to the y-axis. (We *could* continue to intersect with planes perpendicular to the x-axis as we did before; there is nothing mathematically wrong about doing that. However, the cross sections we would obtain would be more complicated than discs or annuli.)

The cross-sections occurring at a given y-value are annuli whose outer radius correspond to $y = \ln(x)$ and inner radius correspond to $x = 0$. In order to use the washer method here, though, we will want to integrate with respect to y. Thus we need to rewrite $y = \ln(x)$ as a graph of a function of y, namely $x = e^y$. With this in mind we see the outer radius is $e^y - (-2) = e^y + 2$, and the inner radius is simply 2.



Thus our cross-sectional area is simply

$$\pi\left((e^y+2)^2 - 2^2\right) = \pi\left(e^{2y} + 4e^y + 4 - 4\right) = \pi\left(e^{2y} + 4e^y\right).$$

We can thus compute our volume as

$$\text{Volume} = \int_1^2 \pi\left(e^{2y} + 4e^y\right)\, dy.$$

To compute this integral we will break it into two parts,

$$\int_1^2 \pi\left(e^{2y} + 4e^y\right)\, dy = \pi\int_1^2 e^{2y}\, dy + 4\pi\int_1^2 e^y\, dy.$$

The first integral requires a simple u-substitution with $u = 2y$, $du = 2\, dy$, and so we compute

$$\pi\int_1^2 e^{2y}\, dy = \frac{\pi}{2}\int_2^4 e^u\, du = \frac{\pi}{2}\left(e^4 - e^2\right).$$

The second integral is slightly simpler,

$$4\pi\int_1^2 e^y\, dy = 4\pi e^y\Big|_1^2 = 4\pi\left(e^2 - e\right).$$

Putting these together we see the volume of our solid is

$$\begin{aligned} &\frac{\pi}{2}\left(e^4 - e^2\right) + 4\pi\left(e^2 - e\right) \\ =&\frac{\pi e^4 - \pi e^2 + 8\pi e^2 - 8\pi e}{2} \\ =&\frac{\pi}{2}\left(e^4 + 7e^2 - 8e\right). \end{aligned}$$

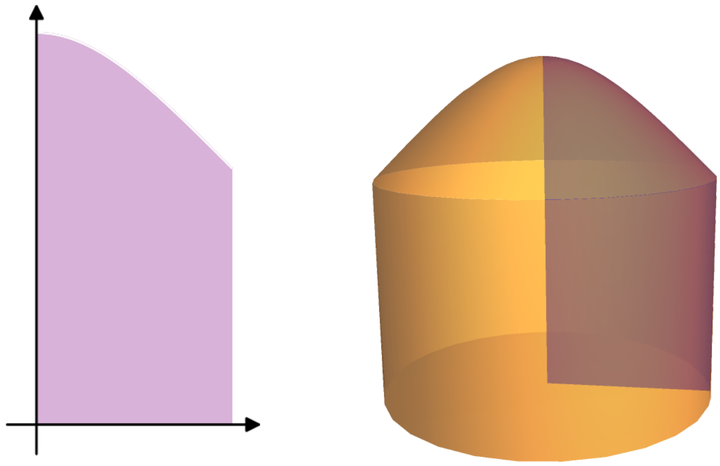
To summarize what we have seen thus far:

- The volume of the solid of revolution obtained by rotating the region bounded by $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$, assuming $f(x) \geq g(x)$ for all $a \leq x \leq b$, around the axis $y = k$ is given by
  - $\int_a^b \pi \left((f(x) - k)^2 - (g(x) - k)^2\right) dx$ if $y = k$ is below the region, and
  - $\int_a^b \pi \left((k - g(x))^2 - (k - f(x))^2\right) dx$ if $y = k$ is above the region.
- The volume of the solid bounded by $x = r(y)$, $x = \ell(y)$, $y = c$, $y = d$, assuming $r(y) \geq \ell(y)$ for all $c \leq y \leq d$, around the axis $x = k$ is
  - $\int_c^d \pi \left((r(y) - k)^2 - (\ell(y) - k)^2\right) dy$ if $x = k$ is to the left of the region, and
  - $\int_c^d \pi \left((k - \ell(y))^2 - (k - r(y))^2\right) dy$ if $x = k$ is to the right of the region.

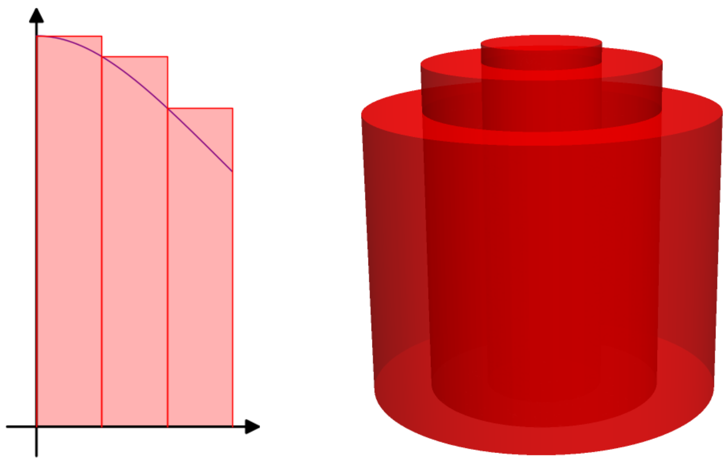
Remark.
I personally do not think it's worthwhile to memorize the four formulas above. Instead, I would recommend that you understand the ideas (volume is computed by integrating cross-sectional area; cross sections for solids of revolution are discs or washers; the radius of a disc or radii of a washer depend on the axis of rotation) and derive the formulas as necessary.

The shell method

We now describe another method for computing volumes of solids of revolution. Instead of integrating cross-sectional areas as before, we will integrate areas of "shells" centered at the axis of rotation. Let's illustrate the idea with an example: find the volume of the solid obtained by rotating the region bounded by $y = f(x)$, $y = 0$, $x = 0$, and $x = 1$ around the y-axis.



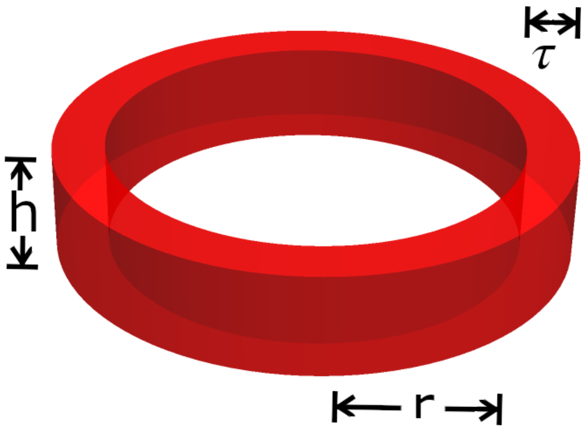
Let's imagine that that instead of rotating the region described above around the axis, we instead approximate the region with rectangles (just as when approximating area under a curve) and instead rotate those rectangles around the y-axis.



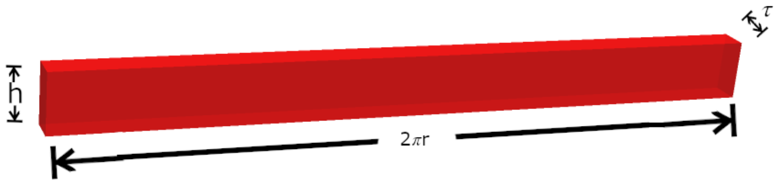
Each rectangle we rotate around the axis gives a solid cylindrical shell. We could approximate the volume of the original solid by adding the volumes of these shells. So, we need to determine the volume of the shell.

Here we will actually cheat a little bit and instead of computing the volume exactly, we will only approximate the volume of the shell. (You *could* compute the exact volume, but what we are about to describe will result in a simpler integral.)

Let's suppose the cylindrical shell we have has an outer radius of r, a height of h, and a thickness of τ.



Imagine that we cut the shell along one side and unroll it to give a "rectangular prism,"



Notice the height of this prism is h, the width of the prism is τ, but the length of the prism is $2\pi r$ since this corresponds to the circumference of the original shell. Thus the volume of the prism is $2\pi rh\tau$. We are cheating a little bit here because when we unroll the cylindrical shell we don't quite have a rectangular prism since one side of our "prism" will be shorter than the opposite side (corresponding to the inner and outer radii of the shell). Despite this, we will use $2\pi rh\tau$ as an approximation to the volume of the shell. Notice this approximation gets better and better as the shell gets thinner and thinner (i.e., as τ goes to zero).

In our example above, let's suppose the i-th shell is given by an x-coordinate we'll write as x_i^*. The height of the shell would then correspond to $f(x_i^*)$, and the thickness of the shell is the width of the rectangle, Δx_i. Putting all of this together, the volume of our solid of revolution is approximately

$$\text{Volume} \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x_i.$$

As our shells get thinner (the corresponding rectangles we rotated around the axis also becoming thinner), we get better and better approximations. In the limit we get the true volume, and the limit of the Riemann sums above becomes an integral. That is, we may compute the volume as

$$\text{Volume} = \int_0^1 2\pi x f(x)\,dx.$$

More generally, if the region bounded by $y = f(x)$, $x = a$, $x = b$ and the x-axis is rotated around the y-axis, the volume of the resulting solid can be computed as

$$\text{Volume} = \int_a^b 2\pi x f(x)\,dx.$$

Computing the volume this way is called ***the shell method*** since it comes from approximating the region with cylindrical shells.

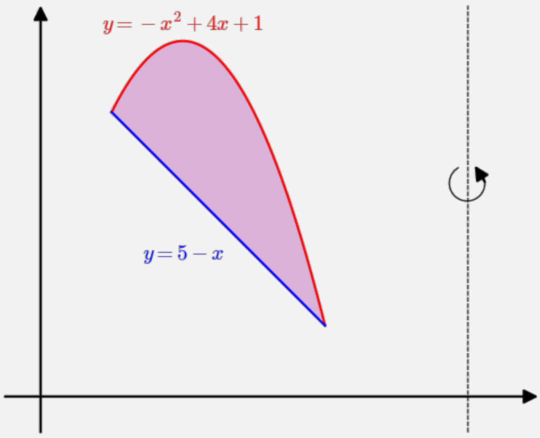
Notice that we could have computed the volume described above using the disc or washer method, but because we rotated around the y-axis we would have to integrate with respect to y, and this would require us to rewrite the $y = f(x)$ above as a function of y which could very well be difficult depending on what the function was. With the shell method, though, we integrate with respect to x and get to sidestep that very annoying algebra, although possibly at the price of having a more involved integral.

Example 2.11.
Use the shell method to compute the volume of the cone obtained by rotating the region bounded by the line $y = h - \frac{h}{r}x$ and the axes, rotated around the y-axis. (This results in a cone of height h and base radius r.)

$$\begin{aligned}
\text{Volume} &= \int_0^r 2\pi x \left(h - \frac{h}{r}x\right)\,dx \\
&= 2\pi \int_0^r \left(hx - \frac{h}{r}x^2\right)\,dx \\
&= 2\pi \left(\frac{hx^2}{2} - \frac{hx^3}{3r}\right)\Bigg|_0^r dx \\
&= 2\pi \left(\frac{hr^2}{2} - \frac{hr^3}{3r}\right) \\
&= 2\pi \left(\frac{hr^2}{2} - \frac{hr^2}{3}\right) \\
&= 2\pi \left(\frac{3hr^2 - 2hr^2}{6}\right) \\
&= 2\pi \cdot \frac{hr^2}{6} \\
&= \frac{\pi r^2 h}{3}
\end{aligned}$$

Of course, we could also slide the axis of rotation around and this will modify our integral.

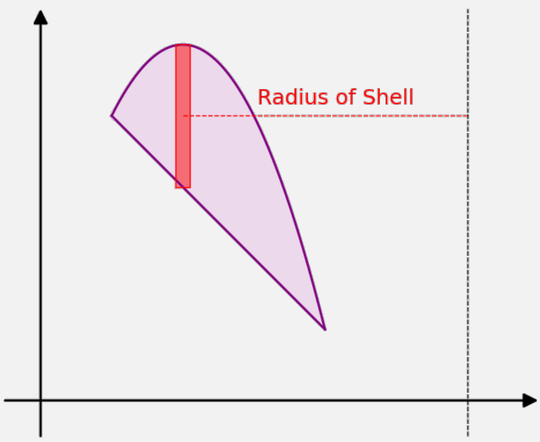
Example 2.12.
Find the volume of the solid obtained by rotating the region bounded by $y = -x^2 + 4x + 1$ and $y = 5 - x$ around the axis $x = 6$.



Before jumping straight to an integral, let's spend a moment thinking about the shells that are involved. If we choose an x-value our region to determine the shell, notice the top of the shell occurs with y-value $-x^2 + 4x + 1$ and the bottom of the shell has y-value $5 - x$. That is, the height of the shell is

$$-x^2 + 4x + 1 - (5 - x) = -x^2 + 4x + 1 - 5 + x = -x^2 + 5x - 4.$$

The radius of the shell is given by the distance from the shell to the axis of rotation. Since the axis is $x = 6$ and this is to the right of our region, the shell corresponding to our x has radius $6 - x$.



Letting Δx denote the width of our rectangle (aka the thickness of our shell), the volume of the shell is approximately

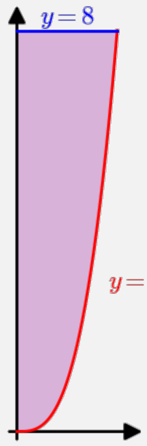
$$2\pi(6-x)\left(-x^2+5x-4\right)\Delta x$$

Summing up these volumes and taking the limit as the shells become thinner and thinner we are left with the following integral:

$$\begin{aligned}
\text{Volume} &= \int_1^5 2\pi(6-x)\left(-x^2+5x-4\right)\,dx \\
&= 2\pi\int_1^5 \left(-6x^2+30x-24+x^3-5x^2+4x\right)\,dx \\
&= 2\pi\int_1^5 \left(x^3-11x^2+34x-24\right)\,dx \\
&= 2\pi\left(\frac{x^4}{4}-\frac{11x^3}{3}+17x^2-28x\right)\Bigg|_1^5 \\
&= \frac{80\pi}{3}
\end{aligned}$$

Of course, we could rotate a region around the x-axis instead of the y-axis. The idea is exactly the same, but now our shells will sit on their sides, and their heights and radii will be functions of y instead of x, so we will integrate with respect to y.

Example 2.13.
Find the volume of the solid obtained by rotating the region below around the x-axis.



Here, since we are rotating around the x-axis, our integral will have the form $\int_a^b 2\pi y f(y)\,dy$, and so we must write our curves as functions of y. This, however, is simply $x = y^{1/3}$. We can then compute the volume as

$$\begin{aligned}
\text{Volume} &= \int_0^8 2\pi y \cdot y^{\frac{1}{3}}\,dy \\
&= 2\pi \int_0^8 y^{\frac{2}{3}}\,dy \\
&= 2\pi \frac{3}{5} y^{\frac{5}{3}} \Big|_0^8 \\
&= \frac{6\pi}{5}\left(8^{\frac{5}{3}} - 0^{\frac{5}{3}}\right) \\
&= \frac{6\pi}{5} \cdot 32 \\
&= \frac{192\pi}{5}
\end{aligned}$$

2.4 Work

In physics, work is defined as force times distance. That is, the work done by applying a force F across a distance d is by definition $W = Fd$. At least, this is the case if the force is constant and applied in the direction of motion. But what if the force changes? For example, suppose we being pushing a heavy object with a force of 100 pounds, but over time grow tired and push with less force. How can we determine the work done in this situation when our force changed?

To solve this, we'll suppose the distance we're traversing is broken into small segments, assume the force is constant on each segment, and add up the work done on each of those segments to get an approximation. This gives us an expression of the form

$$W \approx \sum_{i=1}^{n} F(x_i^*)\,\Delta x_i.$$

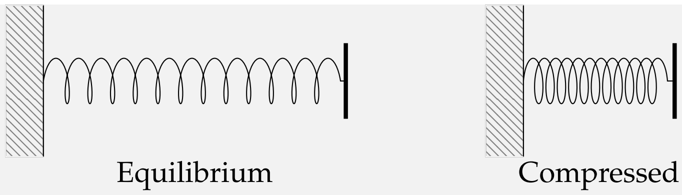
Of course, we get better estimates by using smaller segments, and in the limit the sum above turns into an integral,

$$W = \int_a^b F(x)\,dx.$$

Example 2.14.
Hooke's law states that the force exerted by a spring stretched or compressed by a distance of x from its natural equilibrium position is $-kx$, where k depends on the stiffness of the spring.

What this is really means is the following: When you push the spring in by compressing it a distance of x, the spring pushes back, and how hard it pushes back depends on both the constant k and how much you've compressed the spring. (The constant k simply tells us how "stiff" the spring is. For example, for a spring made of soft plastic k might be a very small number; for a heavy spring made of cast iron the k might be very large.) Imagine a stiff spring made of some kind of metal. If you attached one end of the spring to a wall and then pushed the other end with your hand, you would feel the spring pushing back, trying to get back to its equilibrium position. The more you compress the spring, the stronger it pushes back, and in particular if you compress the spring twice as much, the spring will push back twice as hard. That is all that Hooke's law is saying. (The negative that appears is because the spring is pushing against whichever direction you are stretching/compressing it.)

Determine the work required to compress a spring a distance of ℓ meters from its natural equilibrium position.



We simply integrate the force required to compress the spring. When the spring is compressed by a distance of x, the spring supplies a force of $-kx$, so we have to supply a force of kx to balance this out. As we continually push the spring in, though, the x constantly changes, so the force we're supplying is constantly changing too. The total work required is the integral of this force,

$$W = \int_0^\ell kx\,dx = \left.\frac{kx^2}{2}\right|_0^\ell = \frac{k\ell^2}{2}.$$

Remark.
You may recognize $k\ell^2/2$ as the elastic potential energy of the spring.

Example 2.15.
If we were to build a concrete column by pouring cement to the current top of the column (the top rises as the column is built), the force required to lift the concrete to the top increases as the column rises. How much work is done in building a cylindrical column of concrete with radius three feet and height ten feet, if the concrete being used has a density of 95 pounds per cubic foot?

First notice that the weight of the concrete is given by its density (which we're told is 95 pounds per cubic foot) times its volume. For example, a $2 \times 2 \times 2$ block of this concrete would weigh 760 pounds as the volume of the block is 8 cubic feet.

The weight of the concrete tells us the force required to lift it. To lift the 760 lb concrete block described a moment ago, for instance, you need to supply a force of 760 lb. When we lift the block some distance h the work done is the force times the distance. Lifting our 760 lb block three feet, for instance, means we have to perform 2280 foot-pounds of work.

In our situation we are raising our concrete higher and higher amounts as the column is constructed, and this makes the problem seem hard, so let's consider a simplification. Suppose we imagine our 10 foot tall cylindrical column is constructed from five prefabricated cylindrical blocks of concrete with radius 3 feet and height 2 feet. Now imagine we build our column by simply stacking these blocks on top of one another. Notice that each block weighs

$$95\frac{\text{lb}}{\text{ft}^3} \cdot \pi\,(3\text{ft})^2 \cdot 2\text{ft} = 1710\pi\text{lb}$$

. As we build our column we have to raise each block higher than the previous blocks to put it on the top of the column.

Our first block will be the base of the column, and it requires zero work to leave it on the ground.

The second block will go on top of the first block. This means it must be lifted a height of 2 ft. Since the block weighs 1710π pounds, the work required is 3420π foot-pounds.

The third block goes on top of the second, but since the second block is already on top of the first, it must be lifted 4 ft. This means the required work is 6840π foot-pounds.

The fourth block must be lifted 6 feet, so 10260π foot-pounds of work is required.

Finally, the fifth block must be lifted 8 feet and requires 13680π foot-pounds of work.

Putting all of this together, the total work required to build our column from these pre-made blocks is

$$0 + 3420\pi + 6840\pi + 10260\pi + 13680\pi = 34200\pi \text{ foot-pounds.}$$

The process just described is an approximation of what we want to do, except since we are pouring the concrete we should think the height changes continually – not just the height of the column, but the height of the individual blocks. That is, using five blocks of height 2 we approximated the work to be 34200π foot-pounds. We could repeat this process using ten blocks of height 1 or twenty blocks of height 0.5 feet, or 120 blocks of height $1/12$-foot. As we make the blocks thinner and thinner, we get better and better approximations to what we are interested in.

In the limit we will take more and more blocks of thinner and thinner heights, and the resultant sum becomes an integral. In particular, using n blocks of height $\frac{10}{n}$, the work is approximately

$$\sum_{i=1}^{n} \underbrace{\pi 3^2 \frac{10}{n}}_{\text{volume}} \underbrace{95}_{\text{density}} \underbrace{\frac{10(i-1)}{n}}_{\text{distance}}$$

Notice the distance we move the i-th block is one-blocks-height less than the distance we moved the previous, $(i-1)$-st, block.

Let's slightly rewrite this as

$$\sum_{i=1}^{n} 855\pi \frac{10(i-1)}{n} \frac{10}{n}.$$

Notice that since $\frac{10(i-1)}{n}$ is a height, we could think of it as a y-value; say $y_i^* = \frac{10(i-1)}{n}$. Note too that $\frac{10}{n}$ is the change in heights from one height to the next, so we can write $\Delta y_i = \frac{10}{n}$, and the sum becomes

$$\sum_{i=1}^{n} 855\pi y_i^* \Delta y_i.$$

As n goes to infinity this becomes the integral

$$\int_0^{10} 855\pi y \, dy.$$

Thus our work to build the concrete column can be computed as

$$\begin{aligned} \text{Work} &= \int_0^{10} 855\pi y \, dy \\ &= 855\pi \frac{y^2}{2}\bigg|_0^{10} \\ &= 42750\pi \end{aligned}$$

and so 42750π foot-pounds of work is required.

In general, our integral for the work done in moving a substance of density δ with cross sections of area $A(y)$ moved by a height of y is given by

$$\int_{\text{shortest distance}}^{\text{largest distance}} \delta A(y) y \, dy$$

There is one issue here we must pay attention to when discussing "density." Ultimately we must integrate a force, and so if our density has units

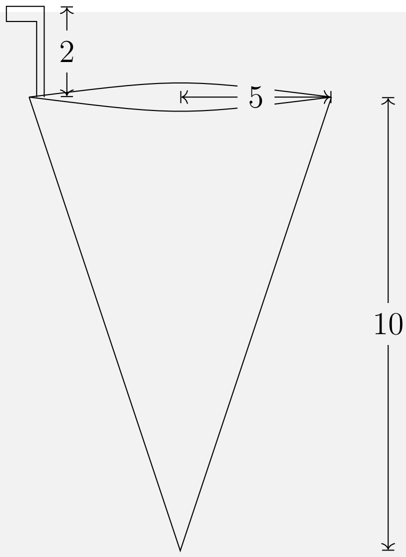
$$\frac{\text{Force}}{\text{Volume}}$$

then we don't need to do anything special. However, if the units for our density were

$$\frac{\text{Mass}}{\text{Volume}}$$

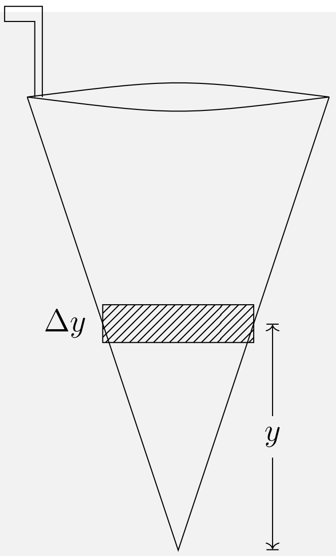
we need to convert the mass into weight (force) by multiplying by the acceleration due to gravity. This will mostly be an issue when we use metric units where we may say the density is given in kilograms per cubic meter. As kilogram is a unit of mass, not force, we must multiply this by the acceleration due to gravity to "upgrade" from kilograms to Newtons. When using English units like pounds, however, this is already a unit of force with gravity already being incorporated.

Example 2.16.
Suppose a tank shaped like an inverted cone of height $10m$ and radius of $5m$ has a pump attached to a spigot $2m$ above the top of the tank.

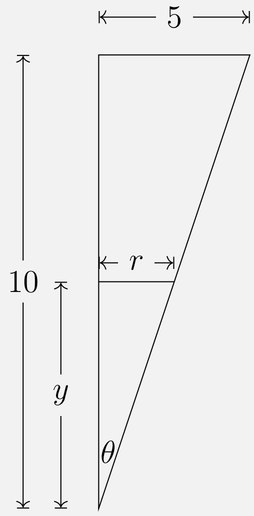


If the tank is filled with water which has a density of $1000\frac{\text{kg}}{\text{m}^3}$, how much work is done to pump the water out of the tank through the spigot?

Let's imagine that our water comes in little (approximately) cylindrical chunks. (E.g., maybe the water was frozen and we slice it in layers.) Let's try to find the work done in moving just a single chunk. To do this, let y denote the height of the chunk from the bottom of the tank and let Δy denote its thickness. We need to compute the volume of the chunk, but this requires us to know its radius. Note that the radius changes as we consider different chunks, however.



In order to determine the radius, we will use some basic trigonometry. In particular, consider the right triangle whose height is the center of the conical tank and whose top is the radius of the tank. Now consider the similar triangle obtained by looking at the (unknown) radius of our chunk distance y from the bottom of the cone.



The bottom tip of this triangle has some angle θ, whatever it happens to be. Notice that we can compute $\tan(\theta)$ in two ways using the fact tangent is opposite over adjacent. Using the large triangle we have

$$\tan(\theta) = \frac{5}{10},$$

but using the smaller triangle we have

$$\tan(\theta) = \frac{r}{y}.$$

However, these are the same θ in each case, so these expressions must be equal, and from this we can solve for r:

$$\frac{5}{10} = \frac{r}{y}$$
$$\implies \frac{y}{2} = r.$$

That is, the radius is a function of our height from the bottom of the cone.

Given that $r = \frac{y}{2}$, we can now compute the volume of the little chunk as

$$\pi \left(\frac{y}{2}\right)^2 \Delta y = \frac{\pi}{4} y^2 \Delta y.$$

The weight of this chunk is thus given by its volume times its density times acceleration due to gravity (since the given density is only a mass and not a force). Since acceleration due to gravity near the surface of the Earth is $9.8 \frac{\text{m}}{\text{s}^2}$, we have the force is

$$9.8 \cdot 1000 \cdot \frac{\pi}{4} y^2 \Delta y = 2450\pi y^2 \Delta y.$$

Now we need to determine the distance this chunk of water moves. As we are moving to the spigot which is $2m$ above the top of the $10m$ cone, the distance from the top of the spigot to the bottom of the cone is $12m$. However, our chunk is already distance y from the bottom of the tank, and so it needs to be pumped the remaining $12 - y$ meters.

Putting all of this together, the approximate work done in pumping one of our little chunks of water in the tank is

$$2450\pi y^2 (12 - y) \Delta y.$$

The units here can be determined by thinking through each part of our integral: $2450\pi y^2 \Delta y$ was measured in Newtons and $12 - y$

is measured in meters. Thus each term has units "Newton-meter", Nm, also known as "Joule," J.

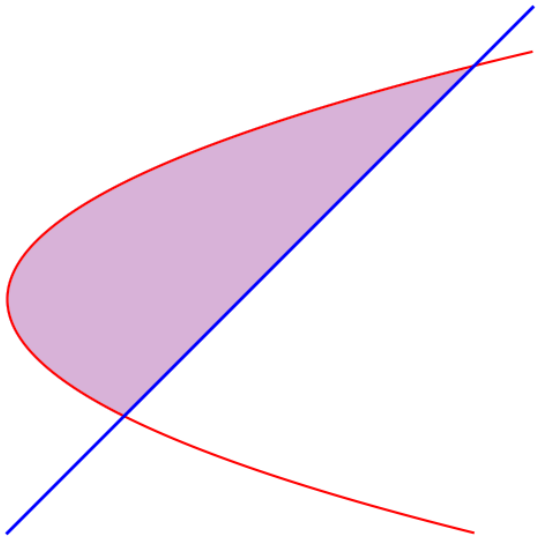
Summing up these approximations to work for each chunk gives us a Riemann sum, and in the limit we obtain the following integral:

$$\begin{aligned}\text{Work} &= \int_0^{10} 2450\pi y^2(12-y)dy \\ &= 2450\pi \int_0^{10} \left(12y^2 - y^3\right)\, dy \\ &= 2450\pi \left(4y^3 - \frac{y^4}{4}\right)\Big|_0^{10} \\ &= 2450\pi \left(4\cdot 10^3 - \frac{10^4}{4}\right) \\ &= 2450\pi\,(4000-2500) \\ &= 3675000\pi \\ &\approx 1.55\times 10^7\end{aligned}$$

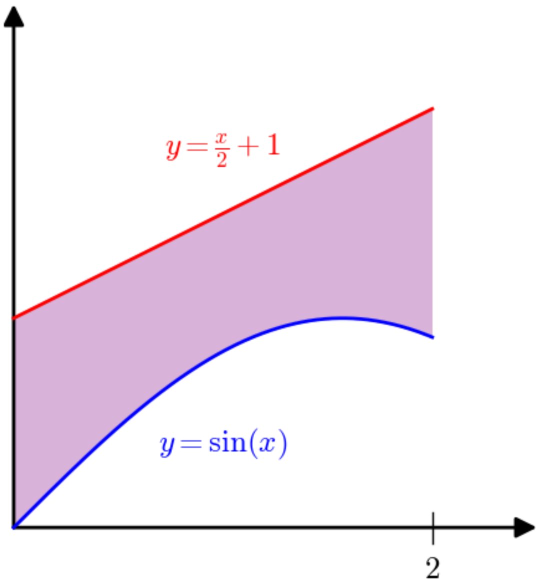
2.5 Practice problems

Problems about area between curves

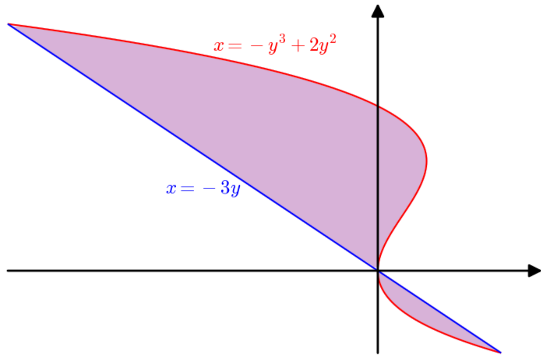
Problem 2.1. Find the area of the region bounded by $y = x$ and $x = (y-2)^2$.



Problem 2.2. Find the area between the graphs $y = \frac{x}{2} + 1$ and $y = \sin(x)$ between $x = 0$ and $x = 2$.



Problem 2.3. Find the area enclosed by $x = -y^3 + 2y^2$ and $x = -3y$.

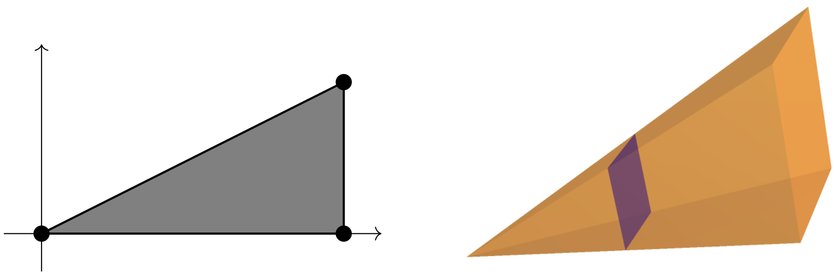


Problem 2.4. Find the area enclosed by by $x = y^2 - 2$ and $x = y$.

Problem 2.5. Find the area enclosed by $y = x^4$ and $y = 2 - x^2$.

Problems about volume

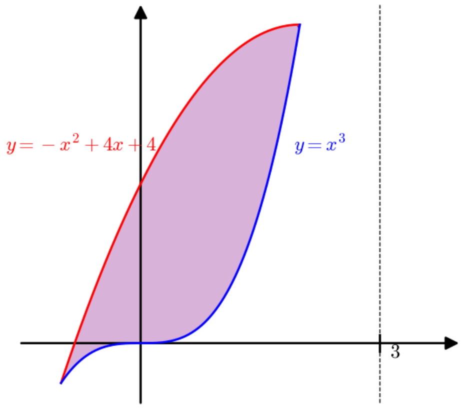
Problem 2.6. Determine the volume of the three-dimensional solid constructed as follows: the base of the solid is the right triangle with vertices $(0, 0)$, $(2, 1)$, and $(2, 0)$ in the xy-plane, and the cross sections perpendicular to the x-axis sitting in three-dimensional space are squares.



Problem 2.7. Using the washer method, compute the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = x^2$ and $y = 3x$ around the x-axis.

Problem 2.8. Using the shell method, compute the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = x^2$ and $y = 3x$ around the x-axis.

Problem 2.9. Using the shell method, compute the volume of the solid obtained by rotating the region bounded by $y = x^3$ and $y = -x^2 + 4x + 4$ between $x = -1$ and $x = 2$ around the line $x = 3$.



Problem 2.10. Repeat Problem 2.9, but using the washer method instead of the shell method

Problems about work

Problem 2.11. How much work is done if a crane lifts a stone block weighing 900 pounds upwards ten feet, assuming the weight of the chain connecting the block to the crane is negligible?

Problem 2.12. Suppose a 900 pound block is attached to a crane via a chain that weighs 50 pounds per foot. If the chain is initially 30 feet long but is then reeled in to bring the block up 10 feet, how much work has the crone done?

Problem 2.13. Imagine a cylindrical tank of liquid has a spigot at its top, and attached to the spigot is a pump which is used to pump all of the liquid out of the tank. If the tank has a radius of 10 feet and a height of 30 feet, and if the density of the liquid in the tank is 65 pounds per cubic foot, how much work is done in pumping all of the liquid out of the tank?

3 Integration Techniques

The beauty of mathematics only shows itself to more patient followers.

MARYAM MIRZAKHANI

3.1 Integration by parts

In this section we will see a rule for integration that can be thought of as a sort of product rule for integrals. Let's begin by recalling the product rule for derivatives which tells us

$$\frac{d}{dx} f(x) \cdot g(x) = f'(x)g(x) + f(x)g'(x)\, dx.$$

Notice this means the antiderivative of $f'(x)g(x) + f(x)g'(x)$ is $f(x)g(x) + C$:

$$\int \left(f'(x)g(x) + f(x)g'(x)\right)\, dx = f(x)g(x) + C.$$

Let's rewrite this as follows:

$$\begin{aligned} &\int \left(f'(x)g(x) + f(x)g'(x)\right)\, dx = f(x)g(x) + C \\ \implies &\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx + C \end{aligned}$$

Noting the antiderivative $\int f'(x)g(x)\, dx$ may be written as $\int g(x)f'(x)\, dx$ and that this integration will supply its own "$+C$", we may further rewrite this as

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx.$$

This way of integrating products of functions is often called ***integration by parts*** and you can think of it as a sort of product rule for integrals.

Notice, though, that this expresses one integral in terms of another integral. Thus this procedure is only helpful if the other integral is easier for us to compute.

Example 3.1.
Compute $\int xe^x\,dx$.

Suppose that $f(x) = x$ and $g'(x) = e^x$ in our formula. Notice we need to compute $f'(x)$ and $g(x)$, but both of these are easy: $f'(x) = 1$ and

$$g(x) = \int g'(x)\,dx = \int e^x\,dx = e^x + C.$$

In fact, since there will be a second integral involved, we can drop this "$+C$" and rely on our second integral (written below) to supply us with the constant of integration.

Using the integration by parts formula above we thus have

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

We can easily verify this is the correct antiderivative:

$$\frac{d}{dx}\left(xe^x - e^x + C\right) = 1 \cdot e^x + xe^x - e^x + 0 = xe^x.$$

Example 3.2.
Compute $\int \ln(x)x^2\,dx$.

Letting $f(x) = \ln(x)$ and $g'(x) = x^2$, we compute $g(x) = \int x^2\,dx = {}^{x^3}\!/_3$ and $f'(x) = {}^1\!/_x$. Thus our integration by parts formula gives us

$$\begin{aligned}\int \ln(x)x^2\,dx &= \ln(x)\frac{x^3}{3} - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx \\ &= \frac{x^3\ln(x)}{3} - \int \frac{x^2}{3}\,dx \\ &= \frac{x^3\ln(x)}{3} - \frac{x^3}{9} + C \\ &= \frac{3x^3\ln(x) - x^3}{9} + C.\end{aligned}$$

Often when people discuss integration by parts the introduce two new variables to simplify the formula above. Letting $u = f(x)$ and

$v = g(x)$ in our earlier formula, notice $du = f'(x)dx$ and $dv = g'(x)dx$. We may then rewrite

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

as

$$\int u\,dv = uv - \int v\,du.$$

This is entirely equivalent to what we had earlier, just expressed in a different notation. As with u-substitution, this new notation takes some getting used to but is usually preferred by students once they've done a few examples with it.

Example 3.3.
Compute $\int x^2 \cos^{-1}(x)\,dx$.

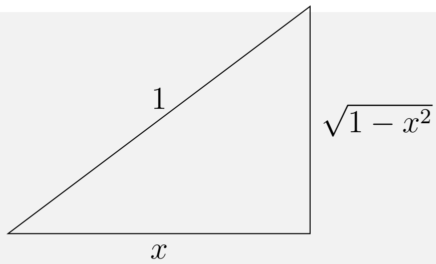
If we take $u = \cos^{-1}(x)$ and $dv = x^2\,dx$, then we must have $v = \frac{x^3}{3}$ and

$$du = \frac{d}{dx}\cos^{-1}(x)\,dx = \frac{1}{\sin(\cos^{-1}(x))}dx.$$

Here we have used the formula for the derivative of an inverse function. Just to remind you of how this works, if $f^{-1}(x)$ is the inverse of $f(x)$, then we must have $f(f^{-1}(x)) = x$. If we differentiate both sides of this equality we must still have an equality, and we can differentiate $f(f^{-1}(x))$ using the chain rule. The chain rule will give us a factor of $\frac{d}{dx}f^{-1}(x)$, and from there we can solve for $\frac{d}{dx}f^{-1}(x)$:

$$\begin{aligned}
& f(f^{-1}(x)) = x \\
\implies & \frac{d}{dx}f(f^{-1}(x)) = \frac{d}{dx}x \\
\implies & f'(f^{-1}(x)) \cdot \frac{d}{dx}f^{-1}(x) = 1 \\
\implies & \frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.
\end{aligned}$$

We can actually use some trig to rewrite the expression $1/\sin(\cos^{-1}(x))$ that appeared above. If we imagine a right triangle with hypotenuse 1 and a non-right angle θ with x as the adjacent side, then the Pythagorean theorem tells us the opposite side must have length $\sqrt{1-x^2}$:



Let's notice that since cosine is adjacent over hypotenuse, for this triangle we have $\cos(\theta) = x$, or $\cos^{-1}(x) = \theta$. Thus $\sin(\cos^{-1}(x)) = \sin(\theta)$ and recalling that that sine is opposite over hypotenuse, we have

$$\sin(\cos^{-1}(x)) = \sin(\theta) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$
$$\implies \frac{1}{\sin(\cos^{-1}(x))} = \frac{1}{\sqrt{1-x^2}}.$$

All of this means that our dv that appears in our integration by parts formula can be written as $du = \frac{dx}{1-x^2}$.

Now we use the integration by parts formula $\int u\,dv = uv - \int v\,du$ to write

$$\int x^3 \cos^{-1}(x)\,dx = \cos^{-1}(x) \cdot \frac{x^3}{3} - \int \frac{\left(\frac{x^3}{3}\right)}{\sqrt{1-x^2}}\,dx.$$

Now we must evaluate this late integral. First we rewrite that integral as

$$\int \frac{\left(\frac{x^3}{3}\right)}{\sqrt{1-x^2}}\,dx = \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx.$$

We can compute this integral using a substitution. As we already used the variable u above, we will use w in our substitution. Letting $w = 1 - x^2$, $dw = -2x\,dx$. Before we plug this into our integral let's rewrite it once more so that the $-2x\,dx$ appears:

$$\frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \frac{1}{3} \cdot \frac{1}{-2}\int \frac{x^2(-2x)}{\sqrt{1-x^2}}\,dx$$

With our w-substitution above this becomes

$$\frac{-1}{6}\int \frac{1-w}{\sqrt{w}}\,dw = \frac{-1}{6}\int \left(w^{-1/2} - w^{1/2}\right)\,dw.$$

(Here we used the fact that if $w = 1 - x^2$, then $x^2 = 1 - w$.) This integral is easy to compute,

$$\frac{-1}{6}\left(w^{-1/2} - w^{1/2}\right)\,dw = \frac{-1}{6}\left(2w^{1/2} - \frac{2}{3}w^{3/2}\right) + C.$$

Rewriting this in terms of x we have

$$\frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \frac{-1}{6}\left(2\sqrt{1-x^2} - \frac{2}{3}\left(1-x^2\right)^{3/2}\right) + C.$$

Finally combining this with the above we have

$$\begin{aligned}
&\int x^2\cos^{-1}(x)\,dx \\
=&\frac{x^3\cos^{-1}(x)}{3} - \int \frac{\left(\frac{x^3}{3}\right)}{\sqrt{1-x^2}}\,dx \\
=&\frac{x^3\cos^{-1}(x)}{3} + \frac{1}{6}\left(2\sqrt{1-x^2}\right) - \frac{2}{3}\left(1-x^2\right)^{3/2} + C \\
=&\frac{x^3\cos^{-1}(x) + \sqrt{1-x^2} - 2(1-x^2)^{3/2}}{3} + C
\end{aligned}$$

In doing these problems, we have to make a choice of what u and dv are, and that choice influences how easy or difficult it will be to solve the problem. For example, if we were to repeat the example above but making the choice $u = x^2$, $dv = \cos^{-1}(x)$, then we would be required to compute $v = \int \cos^{-1}(x)\,dx$, which is not so obvious.

Notice that when we perform integration by parts we will need to differentiate u, which is usually easy, and integrate dv, which may be hard. So, we should try to choose our u and dv to be things where we are more likely to be able to integrate the chosen dv. To help guide us in this decision there's a convenient acronym, ILATE:

I	inverse trig
L	logarithm
A	algebraic (e.g., polynomial)
T	trig
E	exponential

The idea here is that we will choose our u using the letters of ILATE left-to-right. If there's an inverse trig involved in our integral, we'll take our u to be that. If there's no inverse trig, then we'll take our u to be any logarithms that appear, and so on. Once we make a choice of u, then dv is everything that remains.

Remark.
Of course, ILATE is not magical. The strategy here is just that we can pretty much differentiate any function we're given, but integration may be much more difficult. So, we'll take our u to be things that are difficult to integrate (like inverse trig functions and logarithms), so that dv is hopefully easier to integrate. This is what happened in our example above where we took u to be $\cos^{-1}(x)$ – an inverse trig function – and then dv was simply $x^2\,dx$, and x^2 is easy to integrate.

Example 3.4.
Compute $\int 2^x\left(x^2+3x-4\right)\,dx$.

We will use integration by parts, using ILATE to guide our choice of u. There are no inverse trig functions and no logarithms in our integrand, but there is something "algebraic," the polynomial $x^2 + 3x - 4$. So we will take $u = x^2 + 3x - 4$, and then $dv = 2^x\,dx$. Notice that du is easily computed to be $du = (2x + 3)\,dx$, and v is given by integrating 2^x:

$$v = \int dv = \int 2^x\,dx = \frac{2^x}{\ln(2)}.$$

Now applying the integration by parts formula we can rewrite our original integral as

$$\int 2^x \left(x^2+3x-4\right)\, dx = \left(x^2+3x-4\right)2^x - \int \frac{2^x}{\ln(2)}(2x+3)\, dx.$$

Now we need to compute this integral that appears on the right. To do this we will rewrite it as

$$\int \frac{2^x}{\ln(2)}(2x+3)\, dx = \frac{1}{\ln(2)}\left(2\int 2^x x\, dx + 3\int 2^x\, dx\right).$$

The right-most integral we have already computed, $\int 2^x\, dx = \frac{2^x}{\ln(2)} + C$. For the other integral, $\int 2^x x\, dx$, however, we will have to do *another* integration by parts.

In order to not confuse the "new" u and dv in our second integration by parts, we will use u_2 and dv_2. Notice that, using ILATE again, we will take $u_2 = x$ and $dv_2 = 2^x\, dx$. Thus $du_2 = dx$ and $v_2 = \frac{2^x}{\ln(2)}$. Now we apply the integration by parts formula to compute

$$\int 2^x\, xdx = \frac{x2^x}{\ln(2)} - \frac{1}{\ln(2)}\int 2^x\, dx = \frac{x2^x}{\ln(2)} - \frac{2^x}{(\ln(2))^2} + C.$$

Now we can put all of this together to solve our original problem:

$$\begin{aligned}
&\int 2^x\left(x^2+3x-4\right)\, dx\\
&= \left(x^2+3x-4\right)2^x - \int \frac{2^x}{\ln(2)}\,(2x+3)\, dx\\
&= \left(x^2+3x-4\right)2^x - \frac{3}{\ln(2)^2}2^x + \frac{x}{\ln(2)^2}2^{x+1} - \frac{1}{\ln(2)^3} + C.
\end{aligned}$$

It is worth pointing out here that it is not uncommon to have to perform integration by parts multiple times in solving one problem! Sometimes, though, you may wind up performing an infinite loop of integration by parts with integrals that seem to be getting harder and harder. When this happens, you likely have a mistake somewhere in your work, or should pick a different u and dv. (ILATE is a useful guideline, but it is not always the definitive way to choose u and dv.)

There are a few cases, though, when this "infinite loop" of integration by parts can be solved.

Example 3.5.
Compute $\int e^x \sin(x)\,dx$.

Using ILATE we will take $u = \sin(x)$ and $dv = e^x\,dx$. Notice this means $du = \cos(x)\,dx$ and $v = e^x$. Our integral can now be written as

$$\int e^x \sin(x)\,dx = e^x \sin(x) - \int e^x \cos(x)\,dx.$$

We will perform integration by parts again to compute $\int e^x \cos(x)\,dx$. Letting $u_2 = \cos(x)$ and $dv_2 = e^x\,dx$ gives us $du_2 = -\sin(x)\,dx$ and $v_2 = e^x$. Thus integration by parts tells us

$$\int e^x \cos(x)\,dx = e^x \cos(x) + \int e^x \sin(x)\,dx.$$

Plugging this into the $\int e^x \cos(x)\,dx$ that appeared before we have

$$\begin{aligned}\int e^x \sin(x)\,dx =& e^x \sin(x) - \int e^x \cos(x)\,dx \\ =& e^x \sin(x) - \left(e^x \cos(x) + \int e^x \sin(x)\,dx\right) \\ =& e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)\,dx.\end{aligned}$$

Notice that our original integral reappeared on the right-hand side, and so we can now try to solve for our integral!

$$\begin{aligned}&\int e^x \sin(x)\,dx = e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)\,dx \\ \Longrightarrow\ & 2\int e^x \sin(x)\,dx = e^x \sin(x) - e^x \cos(x) \\ \Longrightarrow\ & \int e^x \sin(x) = \frac{1}{2}\left(e^x \sin(x) - e^x \cos(x)\right) + C.\end{aligned}$$

(In the last step we added a "$+C$" to account for the fact that a general antiderivative is only determined up to a constant.)

There are some times when the choice of dv is somewhat hidden, as in the next example.

Example 3.6.
Compute $\int \ln(x)\,dx$.

If we try to perform integration by parts using ILATE to guide our choice of u, notice we have $u = \ln(x)$. But what is dv? In general our dv is everything that's left over, so we must have $dv = dx$. Alternatively think of $\int \ln(x)\,dx$ as $\int \ln(x) \cdot 1\,dx$. Then $dv = 1 \cdot dx$. Notice we then have $du = \frac{1}{x}\,dx$ and $v = \int dx = \int 1\,dx = x$. Now we have

$$\begin{aligned} \int \ln(x)\,dx &= x\ln(x) - \int x \cdot \frac{1}{x}\,dx \\ &= x\ln(x) - \int 1\,dx \\ &= x\ln(x) - x + C \end{aligned}$$

Exercise 3.1.
Verify the antiderivative computed in Example 3.6 is correct.

All of the examples we have seen thus far have been indefinite integrals, and so you should naturally ask what will happen when we compute a definite integral. Let's examine this by writing our integration by parts formula as follows. Let's rewrite the $\int g(x)f'(x)\,dx$ that appears on the right-hand side of the integral as $H(x) + C$; whatever the antiderivative of $g(x)f'(x)$ happens to be, let's just momentarily call it $H(x) + C$. Then, by the integration by parts formula, the antiderivative of $f(x)g'(x)$ is

$$\begin{aligned} \int f(x)g'(x)\,dx &= f(x)g(x) - \int g(x)f'(x)\,dx \\ &= f(x)g(x) - H(x) + C. \end{aligned}$$

The fundamental theorem of calculus then tells us that we can compute the definite integral $\int_a^b f(x)g'(x)\,dx$ as follows:

$$\begin{aligned}
\int_a^b f(x)g'(x)\,dx &= (f(x)g(x) - H(x) + C)\Big|_a^b \\
&= f(b)g(b) - H(b) + C - (f(a)g(a) - H(a) + C) \\
&= f(b)g(b) - H(b) + C - f(a)f(a) + H(a) - C \\
&= f(b)g(b) - f(a)g(a) - H(b) + H(a) \\
&= f(b)g(b) - f(a)g(a) - (H(b) - H(a)).
\end{aligned}$$

Notice, however, that since $H(x) + C$ is the antiderivative of $g(x)f'(x)\,dx$ we must have

$$H(b) - H(a) = \int_a^b f(x)g'(x)\,dx$$

by the fundamental theorem of calculus. Plugging this in for the above and rewriting $f(b)g(b) - f(a)g(a)$ as $f(x)g(x)\Big|_a^b$ we have

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx.$$

In our u, dv notation we have

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

Example 3.7.

Compute $\displaystyle\int_0^{\pi/2} x\sin(2x)\,dx$.

Letting $u = x$ and $dv = \sin(2x)\,dx$ we have $du = dx$ and $v =$

$\frac{-1}{2}\cos(2x)$. Thus

$$\begin{aligned}
\int_0^{\pi/2} x\sin(2x)\,dx &= \left.\frac{-x\cos(2x)}{2}\right|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2}\cos(2x)\,dx \\
&= \frac{-\pi\cos(\pi)}{4} + \frac{0\cos(0)}{2} + \frac{1}{2}\left(\left.\frac{\sin(2x)}{2}\right|_0^{\pi/2}\right) \\
&= \frac{\pi}{4} + \frac{1}{4}\left(\sin(\pi) - \sin(0)\right) \\
&= \frac{\pi}{4}.
\end{aligned}$$

3.2 Trigonometric integrals

When integrating various combinations of trigonometric functions, it can be helpful to take advantage of various trigonometric identities, such as the Pythagorean identity, $\sin^2(x) + \cos^2(x) = 1$. For instance, to compute the integral $\int \cos^3(x)\,dx$ we can write

$$\int \cos^3(x)\,dx = \int \cos^2(x)\cos(x)\,dx = \int \left(1 - \sin^2(x)\right)\cos(x)\,dx.$$

We can now perform the substitution $u = \sin(x)$, $du = \cos(x)\,dx$ and our integral becomes

$$\int \left(1 - u^2\right)\,du$$

which we can easily compute,

$$\int \left(1 - u^2\right)\,du = u - \frac{u^3}{3} + C.$$

Rewriting this in terms of x we have

$$\int \cos^3(x) = \sin(x) - \frac{1}{3}\sin^3(x) + C.$$

Exercise 3.2.
Verify that $\sin(x) - \frac{1}{3}\sin^3(x) + C$ is the general antiderivative of $\cos^3(x)$.

$$\begin{aligned}
\frac{d}{dx}\left(\sin(x) - \frac{1}{3}\sin^3(x) + C\right) &= \cos(x) - \frac{1}{3}\cdot 3\sin^2(x)\cos(x) + 0 \\
&= \cos(x) - \sin^2(x)\cos(x) \\
&= \cos(x)\left(1 - \sin^2(x)\right) \\
&= \cos(x)\cdot\cos^2(x) \\
&= \cos^3(x)
\end{aligned}$$

In general, we want to use some trig to rewrite the integrand as something we can easily compute. If we can write our integral as some function of $\sin(x)$ together with a single factor of $\cos(x)$ as above, then we can use a u-substitution. In particular, if we have an odd power of $\sin(x)$ (or $\cos(x)$), then we can pull of one factor and use the Pythagorean identity to rewrite the remaining even power of $\sin(x)$ (or $\cos(x)$) using the Pythagorean identity.

Example 3.8.
Compute $\int \sin^7(x)\cos^2(x)\,dx$.

We begin by trying to rewrite the integrand using the strategy mentioned just above:

$$\begin{aligned}
\int \sin^7(x)\cos^2(x)\,dx &= \int \sin(x)\cdot\sin^6(x)\cos^2(x)\,dx \\
&= \int \sin(x)\left(\sin^2(x)\right)^3\cos^2(x)\,dx \\
&= \int \sin(x)\left(1 - \cos^2(x)\right)^3\cos^2(x)\,dx.
\end{aligned}$$

We are now set up to perform a u-substitution. Letting $u = \cos(x)$, then $du = -\sin(x)\,dx$ and we can write our integral as

$$\begin{aligned}
-\int \left(1-u^2\right)^3 u^2\,du &= -\int \left(1-3u^2+3u^4-u^6\right)\,u^2\,du \\
&= -\int \left(u^2-3u^4+3u^6-u^8\right)\,du \\
&= -\left(\frac{u^3}{3}-\frac{3u^5}{5}+\frac{3u^7}{7}-\frac{u^9}{9}\right)+C
\end{aligned}$$

We can now rewrite u as $\cos(x)$ to obtain

$$\int \sin^7(x)\cos^2(x)\,dx = -\left(\frac{\cos(x)^3}{3}-\frac{3\cos(x)^5}{5}+\frac{3\cos(x)^7}{7}-\frac{\cos(x)^9}{9}\right)+C$$

Similarly, because we have the identity $1+\tan^2(x) = \sec^2(x)$, we can integrate powers of $\sec(x)$ and $\tan(x)$ using the same strategy. Here we want to write our integrand as a function of $\tan(x)$ and have a single $\sec^2(x)$ left over, or as a function of $\sec(x)$ with a single factor of $\sec(x)\tan(x)$ left over. Once we do that, we are set up to do a u-substitution.

Example 3.9.
Compute $\int \sec^3(x)\tan^3(x)\,dx$.

First note we can rewrite the integral as

$$\int \sec^3(x)\tan^3(x)\,dx = \int \sec^2(x)\tan^2(x)\sec(x)\tan 9x)\,dx$$

Letting $u = \sec(x)$, $du = \sec(x)\tan(x)\,dx$ our integral becomes

$$\int u^2\left(u^2-1\right)\,du = \int \left(u^4-u^2\right)\,du = \frac{u^5}{5}-\frac{u^3}{3}+C$$

Hence,

$$\int \sec^3(x)\tan^3(x)\,dx = \frac{\sec^5(x)}{5}-\frac{\sec^3(x)}{3}+C.$$

In some cases we may need to use more involved trig identities, as in the next example.

Example 3.10.
Compute $\int \sin^4(x)\cos^2(x)\,dx$.

Here just trying to pull off a single $\sin(x)$ or $\cos(x)$ and then apply the Pythagorean identity as we had before won't get us very far as there will be an "extra" factor we won't be able to swallow up with du, or we will have only $\sin(x)$'s or $\cos(x)$. That is, pulling off a single $\sin(x)$ won't help because we will have the following:

$$\begin{aligned}\int \sin^4(x)\cos^2(x)\,dx &= \int \sin(x)\sin^3(x)\cos^2(x)\,dx\\ &= \int \sin^2(x)\sin^2(x)\cos^3(x)\,dx\\ &= \int \sin^2(x)\left(1-\cos^2(x)\right)\cos^2(x)\,dx\end{aligned}$$

The substitution $u = \cos(x)$, $du = -\sin(x)\,dx$ doesn't help us here because of the extra $\sin(x)$ that will be left over.

If we tried to pull off a single $\cos(x)$ we'd run into a different problem:

$$\int \sin^4(x)\cos^2(x)\,dx = \int \sin^4(x)\left(1-\sin^2(x)\right)\,dx$$

Now we're stuck: since there are only sines and no cosines we can't perform a u-substitution.

In a situation like this it's useful to recall to recall the half-angle identities,

$$\cos(2x) = 1 - 2\sin^2(x) = -1 + 2\cos^2(x).$$

We can do a little bit of algebra to these equalities to obtian the following identities which are helpful in our current situation:

$$\begin{aligned}\sin^2(x) &= \frac{1-\cos(2x)}{2}\\ \cos^2(x) &= \frac{1+\cos(2x)}{2}\end{aligned}$$

Using this, our strategy will be to rewrite our integrand to contain terms that are all powers of cosine. Once we're in that situation we can try to deal with integrating each term individually, possibly using the half-angle identities or the Pythagorean identity for each integrating each term.

For example, we can rewrite the $\sin^4(x)$ in our original problem as

$$\begin{aligned}\int \sin^4(x)\cos^2(x)\,dx &= \int \left(\sin^2(x)\right)^2 \cos^2(x)\,dx \\ &= \int \left(\frac{1-\cos(2x)}{2}\right)^2 \cos^2(x)\,dx \\ &= \frac{1}{4}\int \left(1-2\cos(2x)+\cos^2(2x)\right)\cos^2(x)\,dx \\ &= \frac{1}{4}\int \left(\cos^2(x)-2\cos(2x)\cos^2(x)+\cos^2(2x)\cos^2(x)\right)\,dx.\end{aligned}$$

Now we simply deal with each part of the integral by itself.

Using the half-angle identity we may write

$$\begin{aligned}\int \cos^2(x)\,dx &= \frac{1}{2}\int \left(1+\cos(2x)\right)\,dx \\ &= \frac{1}{2}\left(x+\frac{\sin(2x)}{2}\right)+C \\ &= \frac{x}{2}+\frac{\sin(2x)}{4}+C\end{aligned}$$

For the second term we will use the identity $\cos^2(x) = \frac{1}{2}\left(1+\cos(2x)\right)$ to rewrite the integral as follows:

$$\begin{aligned}\int 2\cos(2x)\cos^2(x)\,dx &= 2\int \cos(2x)\cdot\frac{1+\cos(2x)}{2}\,dx \\ &= \int \left(\cos(2x)+\cos^2(2x)\right)\,dx \\ &= \int \left(\cos(2x)+\frac{1+\cos(4x)}{2}\right)\,dx \\ &= \int \left(\frac{1}{2}+\cos(2x)+\frac{\cos(4x)}{2}\right)\,dx \\ &= \frac{x}{2}-\frac{\sin(2x)}{2}-\frac{\sin(4x)}{8}+C\end{aligned}$$

For the last term we compute

$$\begin{aligned}\int \cos^2(2x)\cos^2(x)\,dx &= \int \cos^2(2x)\cdot\frac{1+\cos(2x)}{2}\,dx\\ &= \frac{1}{2}\int \left(\cos^2(2x)+\cos^3(2x)\right)\,dx\\ &= \frac{1}{2}\left(\int \frac{1+\cos 2x}{2}\,dx + \int \cos^2(2x)\cos(2x)\,dx\right)\\ &= \frac{1}{2}\left(\frac{x}{2}+\frac{\sin(2x)}{4}+\int \left(1-\sin^2(2x)\right)\cos(2x)\,dx\right)\end{aligned}$$

For the remaining integral we perform the substitution $u = \sin(2x)$, $du = 2\cos(2x)\,dx$. We then write the last integral as

$$\frac{1}{2}\int (1-u^2)\,du = \frac{u}{2}-\frac{u^3}{6}+C$$

in terms of x this gives us

$$\int \left(1-\sin^2(2x)\right)\cos(2x)\,dx = \frac{\sin(2x)}{2}-\frac{\sin^3(2x)}{6}+C.$$

Finally, putting everything back together we have

$$\begin{aligned}&\int \sin^4(x)\cos^2(x)\,dx\\ =&\frac{x}{8}+\frac{\sin(2x)}{16}-\\ &\frac{x}{8}+\frac{\sin(2x)}{8}+\frac{\sin(4x)}{32}+\\ &\frac{x}{16}+\frac{\sin(2x)}{16}+\frac{\sin(2x)}{8}-\frac{\sin^3(2x)}{24}+C\\ =&\frac{x}{16}+\frac{3\sin(2x)}{8}+\frac{\sin(2x)}{16}+\frac{\sin(4x)}{16}-\frac{\sin^3(2x)}{24}+C\end{aligned}$$

Remark.
It's worth pointing out that there are multiple different routes for

solving many of these problems with trig identities, and several different ways to write the final answer by using trig substitutions to simplify the answer. Thus two answers to the same problem could be correct but look very different because they have been simplified with a different sequence of trig identities. The answers really are the same in a case like this, just written in different ways.

There are several other trig identities which can be useful when rewriting integrals, but for now we'll only introduce one more: for any integers m and n we have

$$\sin(mx)\cos(nx) = \frac{1}{2}\left(\sin\left((m-n)x\right) + \sin\left((m+n)x\right)\right)$$

Example 3.11.
Compute $\int \sin(3x)\cos(2x)\,dx$.

Using the above identity where $m = 3$ and $n = 2$ we simply have

$$\begin{aligned}\int \sin(3x)\cos(2x)\,dx &= \frac{1}{2}\int \left(\sin(x) + \sin(5x)\right)\,dx \\ &= \frac{1}{2}\left(-\cos(x) - \frac{\cos(5x)}{5}\right) + C\end{aligned}$$

3.3 Trigonometric substitution

We often perform u-substitution to take a difficult integral and rewrite it as something simpler. For example, given

$$\int f'(g(x))\,g'(x)\,dx$$

we introduce a variable $u = g(x)$, and so $du = g'(x)\,dx$, and then rewrite the integral as

$$\int f'(u)\,du.$$

That is, in u-substitution we often use our u to take the place of some complicated function. There are times, however, when we may want to reverse the process, and instead replace our variable x by a possibly complicated-looking function of x. This may sound strange, but the underlying idea is that the resultant integral may be more amenable to some of our other integration techniques. For instance, if we wanted to compute $\int f(x)\,dx$ where $f(x)$ was a function that didn't have an "obvious" antiderivative, we could let $x = g(\theta)$ and $dx = g'(\theta)\,d\theta$ so that our integral becomes

$$\int f(g(\theta))g'(\theta)\,d\theta.$$

At first glance this may look like we're making the problem more difficult, but sometimes making an appropriate choice of $g(\theta)$ will allow us to use some other tricks up our sleeves to compute the integral.

The most common instance of this occurs when the original integral involves a square root of a sum or difference of squares. Often when this occurs we can take $x = g(\theta)$ to be some trig function and then take advantage of various trig identities to help us solve the integral. In that case the procedure described above is often referred to as a ***trig substitution***.

Example 3.12.
Compute $\int \sqrt{1-x^2}\,dx$.

Notice that if we were to write $x = \sin(\theta)$ and $dx = \cos(\theta)\,d\theta$, then the integral becomes

$$\int \sqrt{1-\sin^2(\theta)}\,\cos(\theta)\,d\theta.$$

Now we can use the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ to write $\cos(\theta) = \sqrt{1-\sin^2(\theta)}$ and write our integral as

$$\int \cos^2(\theta)\,d\theta.$$

Now we take advantage of the identity

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

and we have

$$\begin{aligned}\int \cos^2(\theta)\,d\theta &= \frac{1}{2}\int (1+\cos(2\theta))\;d\theta\\ &= \frac{1}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C\\ &= \frac{2\theta + \sin(2\theta)}{4} + C.\end{aligned}$$

This is not the final answer to our orignal problem, however: we were asked to compute an antiderivative of a function of x, so we need to give a final answer in terms of x. Thus we need to convert our θ's above back into x's. To do this let's notice that if $x = \sin(\theta)$, then by taking the arcsin of each side we have

$$\sin^{-1}(x) = \sin^{-1}(\sin(\theta)) = \theta$$

and so we can replace our θ's in the above with x's. Before we do that, let's recall the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. Using this we can write our antiderivative above as

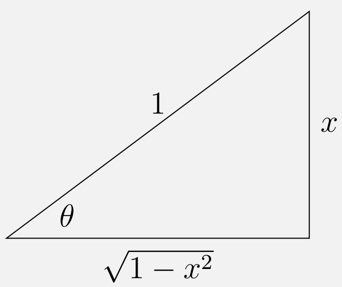
$$\frac{2\theta + 2\sin(\theta)\cos(\theta)}{4} + C$$

Now we convert θ back to $\sin^{-1}(x)$ to obtain

$$\int \sqrt{1-x^2}\,dx = \frac{2\sin^{-1}(x) + 2\sin(\sin^{-1}(x))\cos(\sin^{-1}(x))}{4} + C$$

With a little bit of thought we can simplify this nicely.

First note $\sin(\sin^{-1}(x)) = x$. Also notice that if we consider a right triangle with hypotenuse 1 and non-right angle θ with opposite side x, this triangle will have $\sin(\theta) = x$ (as in our substitution). The Pythagorean theorem will then tell us that $\cos(\theta) = \sqrt{1-x^2}$.



Keeping in mind $\theta = \sin^{-1}(x)$, we then see that the $\cos(\sin^{-1}(x))$ appearing in our integral above can simply be written as $\sqrt{1-x^2}$. Hence

$$\int \sqrt{1-x^2}\,dx = \frac{2\sin^{-1}(x) + 2x\sqrt{1-x^2}}{4} + C.$$

Exercise 3.3.
Verify that

$$\frac{2\sin^{-1}(x) + 2x\sqrt{1-x^2}}{4} + C$$

is the antiderivative of $\sqrt{1-x^2}$ by differentiating.

If we had a multiple in each term appearing in our integral in the previous example, then we can take care of that using our substitution.

Example 3.13.
Compute $\displaystyle\int \sqrt{9-x^2}\,dx$.

Here we want to try to turn $9-x^2$ into $1-\sin^2(\theta)$, but the 9 seems to present a problem. To fix this, we'll actually try to put an extra 9 in the integral that we can factor out. That is, instead of turning $9-x^2$ into $1-\sin^2(\theta)$, we'll try to turn it into

$$9 - 9\sin^2(\theta) = 9\left(1-\sin^2(\theta)\right).$$

To get this extra 9 that we need so that we can factor 9 out, we'll perform the substitution $x = 3\sin(\theta)$ (so then $x^2 = 9\sin^2(\theta)$), and

$dx = 3\cos(\theta)d\theta$. The integral then becomes

$$\begin{aligned}
&\int \sqrt{9-(3\sin(\theta))^2}3\cos(\theta)\,d\theta \\
&= \int \sqrt{9-9\sin^2(\theta)}3\cos(\theta)\,d\theta \\
&= \int \sqrt{9\left(1-\sin^2(\theta)\right)}3\cos(\theta)\,d\theta \\
&= \int \sqrt{9}\cdot\sqrt{1-\sin^2(\theta)}3\cos(\theta)\,d\theta \\
&= 9\int \sqrt{\cos^2(\theta)}\cos(\theta)\,d\theta \\
&= 9\int \cos^2(\theta)\,d\theta \\
&= 9\int \frac{1+\cos(2\theta)}{2}\,d\theta \\
&= 9\left(\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right)+C \\
&= \frac{9\theta}{2}+\frac{9\sin(2\theta)}{4}+C
\end{aligned}$$

Now we need to convert this back in terms of x. Before we do that it will be convenient to notice that

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

(This follows from our earlier identity $\sin(mx)\cos(nx) = \frac{1}{2}\left(\sin((m-n)x)+\sin((m+n)x)\right)$ by taking $m = n = 1$.)

Our antiderivatve in terms of θ is thus

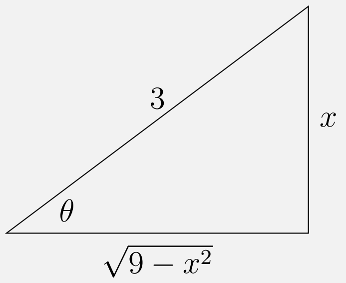
$$\frac{9\theta}{2}+\frac{9\sin(\theta)\cos(\theta)}{2}+C.$$

The point of the most recent manipulation is that it will be easier for us to convert $\sin(\theta)$ and $\cos(\theta)$ into a function of x than $\sin(2\theta)$, as we will see in just a moment.

Notice that as $x = 3\sin(\theta)$, we must have $\sin(\theta) = \frac{x}{3}$, and so $\theta = \sin^{-1}(x/3)$. This allows us to write our antiderivative above as

$$\int \sqrt{9-x^2}\,dx = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)+\frac{9}{2}\cdot\frac{x}{3}\cdot\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)+C.$$

To rewrite the $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ that appears we again think about right triangles. Consider the right triangle whose hypotenuse is 3 and whose side opposite the angle θ is x, as pictured below.



Notice we do in fact have $\sin(\theta) = \frac{x}{3}$. The $\cos(\theta)$ term can then be written as "adjacent over hypotenuse" to obtain

$$\cos(\theta) = \cos\left(\sin\left(\frac{x}{3}\right)\right) = \frac{\sqrt{9-x^2}}{3}.$$

Plugging this into the above we have

$$\begin{aligned}\int \sqrt{9-x^2}\,dx &= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) + C\\ &= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x}{2}\sqrt{9-x^2} + C.\end{aligned}$$

If our last example had been modified to $\int \sqrt{9-4x^2}\,dx$, then we could first rewrite this as $\int \sqrt{9-(2x)^2}\,dx$, and performing the substitution $u = 2x$, $du = 2\,dx$ rewrite the integral as

$$\frac{1}{2}\int \sqrt{9-u^2}\,du$$

then proceed as in the last example.

Exercise 3.4.

Compute $\int \sqrt{9-4x^2}\,dx$.

In the examples we have seen thus far, we took advantage of the Pythagoren identity $\sin^2(\theta)+\cos^2(\theta)=1$ to rewrite integrands involving $\sqrt{a^2-x^2}$ where a was a constant (e.g., 1 and 3 in our earlier examples, respectively). Recall, though, there are other useful trig identities such as $1+\tan^2(\theta) = \sec^2(\theta)$. This can be helpful when our integrand has involves $\sqrt{a^2+x^2}$ as seen in the next example.

Example 3.14.
Compute $\displaystyle\int \frac{dx}{x^2\sqrt{25+x^2}}$.

We want to rewrite $25+x^2$ as something like $1+\tan^2(\theta)$ so that we can then rewrite that as $\sec^2(\theta)$. The 25 that appears slightly complicates things, so we will first try to write our integral as $25+25\tan^2(\theta)$. If we can do that then we can factor out 25 to obtain $25\left(1+\tan^2(\theta)\right)$.

Notice that if we want $25+x^2 = 25+25\tan^2(\theta)$, then we must have $x^2 = 25\tan^2(\theta)$ and so $x=\sqrt{25\tan^2(\theta)} = 5\tan(\theta)$. Thus we will use the trig substitution $x=5\tan(\theta)$, $dx = 5\sec^2(\theta)\,d\theta$ to write our integral as

$$\begin{aligned}
&\int \frac{5\sec^2(\theta)\,d\theta}{25\tan^2(\theta)\sqrt{25+25\tan^2(\theta)}}\\
=&\int \frac{5\sec^2(\theta)\,d\theta}{25\tan^2(\theta)\sqrt{25\left(1+\tan^2(\theta)\right)}}\\
=&\int \frac{5\sec^2(\theta)\,d\theta}{25\tan^2(\theta)\cdot 5\sec(\theta)}\,d\theta\\
=&\frac{1}{25}\int \frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta
\end{aligned}$$

In order to proceed with the integral we will write $\sec(\theta)$ and $\tan(\theta)$

in terms of sines and cosines:

$$\begin{aligned}
&\frac{1}{25}\int \frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta \\
=&\frac{1}{25}\int \frac{1/\cos(\theta)}{\left(\sin(\theta)/\cos(\theta)\right)^2}\,d\theta \\
=&\frac{1}{25}\int \frac{1}{\cos(\theta)}\cdot\frac{\cos^2(\theta)}{\sin^2(\theta)}\,d\theta \\
=&\frac{1}{25}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta
\end{aligned}$$

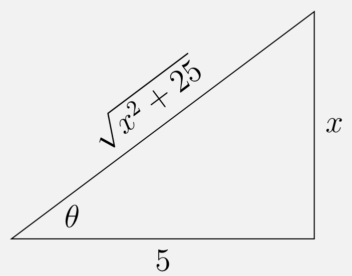
Now we can use the substitution $u = \sin(\theta)$, $du = \cos(\theta)\,d\theta$ to write the integral as

$$\begin{aligned}
\frac{1}{25}\int \frac{1}{u^2}\,du &= \frac{1}{25}\int u^{-2}\,du \\
&= \frac{-1}{25}u^{-1} + C \\
&= \frac{-1}{25u} + C.
\end{aligned}$$

We can easily rewrite this in terms of θ as

$$\frac{-1}{25\sin(\theta)} + C.$$

To go from θ's to x's, we consider the right triangle where the side opposite the angle θ has length x and the adjacent side has length 5.



Notice $\tan(\theta) = \frac{x}{5}$, and so $x = 5\tan(\theta)$, as in our substitution. We want to replace the $\sin(\theta)$ that appeared above and so we simply consider that $\sin(\theta)$ is "opposite over hypotenuse" to write $\sin(\theta) =$

${}^{x}\!/\!\sqrt{x^2+25}$. We then have

$$\int \frac{dx}{x^2\sqrt{25+x^2}} = \frac{-1}{25\left(\frac{x}{\sqrt{x^2+25}}\right)} + C = \frac{-\sqrt{x^2+25}}{25x} + C.$$

Noticing that $1+\tan^2(\theta) = \sec^2(\theta)$ implies $\sec^2(\theta)-1 = \tan^2(\theta)$, we can use similar trigonometric substitutions to deal with integrals involving $\sqrt{x^2-a^2}$.

Example 3.15.

Compute $\displaystyle\int \frac{\sqrt{x^2-4}}{x^3}\,dx$.

Our strategy here will be to convert x^2-4 into something like $\sec^2(\theta)-1$ and then write that as $\tan^2(\theta)$. We can deal with the 4 that appears by first writing x^2-4 as $4\sec^2(\theta)-4$ and then factoring the 4 out. If we want

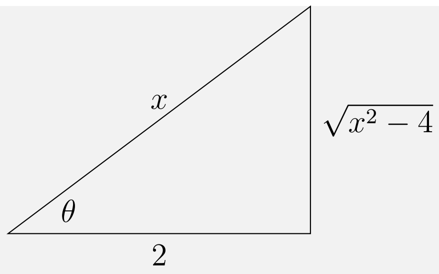
$$x^2-4 = 4\sec^2(\theta)-4$$

then we will require $x^2 = 4\sec^2(\theta)$ and so $x = 2\sec(\theta)$, which means

$dx = 2\sec(\theta)\tan(\theta)\,d\theta$. Using this substitution our integral becomes

$$\begin{aligned}
&\int \frac{\sqrt{4\sec^2(\theta)-4}}{8\sec^3(\theta)} \cdot 2\sec(\theta)\tan(\theta)\,d\theta \\
=&\int \frac{2\sqrt{\sec^2(\theta)-1}}{8\sec^3(\theta)} \cdot 2\sec(\theta)\tan(\theta)\,d\theta \\
=&\frac{1}{2}\int \frac{\sqrt{\sec^2(\theta)-1}}{\sec^3(\theta)} \cdot \sec(\theta)\tan(\theta)\,d\theta \\
=&\frac{1}{2}\int \frac{\sqrt{\sec^2(\theta)-1}}{\sec^2(\theta)} \cdot \tan(\theta)\,d\theta \\
=&\frac{1}{2}\int \frac{\sqrt{\tan^2(\theta)}}{\sec^2(\theta)} \cdot \tan(\theta)\,d\theta \\
=&\frac{1}{2}\int \frac{\tan(\theta)}{\sec^2(\theta)} \cdot \tan(\theta)\,d\theta \\
=&\frac{1}{2}\int \frac{\tan^2(\theta)}{\sec^2(\theta)}\,d\theta \\
=&\frac{1}{2}\int \tan^2(\theta)\frac{1}{\sec^2(\theta)}\,d\theta \\
=&\frac{1}{2}\int \frac{\sin^2(\theta)}{\cos^2(\theta)}\cos^2(\theta)\,d\theta \\
=&\frac{1}{2}\int \sin^2(\theta)\,d\theta \\
=&\frac{1}{2}\int \frac{1-\cos(2\theta)}{2}\,d\theta \\
=&\frac{1}{4}\int (1-\cos(2\theta))\,d\theta \\
=&\frac{1}{4}\left(\theta - \frac{\sin(2\theta)}{2}\right) + C \\
=&\frac{1}{4}\left(\theta - \sin(\theta)\cos(\theta)\right) + C
\end{aligned}$$

Since $x = 2\sec(\theta)$, we have $\sec(\theta) = \frac{x}{2}$. As secant is "hypotenuse over adjacent," we want to consider the right triangle where the hypotenuse has length x and the side adjacent to θ has length 2.



Notice this tells us $\sin(\theta) = \sqrt{x^2-4}/x$ and $\cos(\theta) = 2/x$, and so we have

$$\int \frac{\sqrt{x^2-4}}{x^3}\,dx = \frac{1}{4}\left(\sec^{-1}\left(\frac{x}{2}\right) - \frac{\sqrt{x^2-4}}{x}\cdot\frac{2}{x}\right) + C$$

To summarize what we've described, whenever we see an integral involving $\sqrt{a^2-x^2}$, we want to use the substitution $x = a\sin(\theta)$, $dx = a\cos(\theta)\,d\theta$ and take advantage of the trig identity $1-\sin^2(\theta) = \cos^2(\theta)$.

When we see an integral involving $\sqrt{a^2+x^2}$, we'll use the substitution $x = a\tan(\theta)$, $dx = a\sec^2(\theta)$ and use the identity $1+\tan^2(\theta) = \sec^2(\theta)$.

If $\sqrt{x^2-a^2}$ appears, then we'll use the substitution $x = a\sec(\theta)$, $dx = a\sec(\theta)\tan(\theta)\,d\theta$ and $\sec^2(\theta)-1 = \tan^2(\theta)$.

This is not an exhaustive list of all possible trig substitutions or useful trig identities, but this is a summary of the most commonly used ones, especially for the purposes of integration.

3.4 Partial fractions

Recall that when two fractions are added together, we must get a common denominator. For example,

$$\begin{aligned}\frac{3}{x+2} + \frac{7}{x-5} &= \frac{3}{x+2}\cdot\frac{x-5}{x-5} + \frac{7}{x-5}\cdot\frac{x+2}{x+2}\\ &= \frac{3x-15+7x+14}{(x+2)(x-5)}\\ &= \frac{10x-1}{x^2-3x-10}.\end{aligned}$$

If we wanted to integrate $\frac{10x-1}{x^2-3x-10}$, being aware that it may be written as $\frac{3}{x+2}+\frac{7}{x-5}$ is helpful:

$$\begin{aligned}\int \frac{10x-1}{x^2-3x-10}\,dx &= \int \left(\frac{3}{x+2}+\frac{7}{x-5}\right) dx\\ &= 3\int \frac{dx}{x+2}+7\int \frac{dx}{x-5}\\ &= 3\ln|x+2|+7\ln|x-5|+C.\end{aligned}$$

Thus, knowing that a fraction can be decomposted as a sum of simpler fractions is often helpful for integration as it allows us to express a difficult to integrate function in terms of an easier to integrate function.

In the previous example we cheated a little bit because we started with the simpler functions. So, the obvious question we need to address is how do we go backwards? That is, given a "complicated" fraction, how do we break it up into simpler fractions?

Let's begin by considering a concrete example. Suppose we wanted to compute

$$\int \frac{5x+23}{x^2+2x-3}\,dx$$

by writing the integrand as the sum of two simpler fractions. What should those fractions be?

We know the fractions will have common denominator x^2+2x-3, and we get the common denominator by multiplying the denominators of those simpler fractions together. So, what polynomials can we multiply together to get x^2+2x-3? Put another way, how does x^2+2x-3 factor?

Thinking about factoring the polynomial for a moment we may realize that $x^2+2x-3=(x+3)(x-1)$. So, our decomposition into simpler fractions should have the form

$$\frac{???}{x+3}+\frac{???}{x-1}$$

where the numerators still need to be determines.

How will we go about finding these numerators? As the numerator of our initial fraction is a polynomial, it seems reasonable that our simpler fractions should have polynomial numerators as well. Since our initial numerator was a polynomial of degree one, and our numerators of the simpler summands will get multiplied by the denominator of the other fraction, which has degree one, these polynomials should simply be constants. I.e., we want to find the constants A and B so that

$$\frac{A}{x+3} + \frac{B}{x-1} = \frac{5x+23}{x^2+2x-3}.$$

Now we have an algebra problem, finding the correct A and B. To solve the problem let's simply determine what would happen if we added the fractions on the left, leaving A's and B's in our expressions:

$$\begin{aligned}\frac{A}{x+3} + \frac{B}{x-1} &= \frac{A}{x+3}\cdot\frac{x-1}{x-1} + \frac{B}{x-1}\cdot\frac{x+3}{x+3}\\ &= \frac{Ax - A + Bx + 3B}{x^2+2x-3}\\ &= \frac{(A+B)x + (-A+3B)}{x^2+2x-3}.\end{aligned}$$

If this is to equal our initial fraction, then we will require that the denominators match up. That is, we need $(A+B)x + (-A+3B) = 5x+23$. This gives us a system of equations,

$$\begin{aligned} A + B &= 5\\ -A + 3B &= 23.\end{aligned}$$

To solve this, we will add the equations together to obtain

$$(A+B) + (-A+3B) = 5 + 23$$

To see why this is equality holds, simply notice that $A+B$ is assumed to be 5 and $-A+3B$ is assumed to be 23. We can of course rewrite this equation as simply $4B = 28$ which means $B = {}^{28}/_{4} = 7$. Once we know $B = 7$, we can then plug this back into our first equation to determine $A + 7 = 5$ and so $A = 5 - 7 = -2$. That is, we have determined

$$\frac{5x+23}{x^2+2x-3} = \frac{-2}{x+3} + \frac{7}{x-1}$$

Now we can easily compute the integral:

$$\begin{aligned}\int \frac{5x+23}{x^2+2x-3}\,dx &= \int\left(\frac{-2}{x+3} + \frac{7}{x-1}\right)dx\\ &= -2\int\frac{dx}{x+3} + 7\int\frac{dx}{x-1}\\ &= -2\ln|x+3| + 7\ln|x-1| + C.\end{aligned}$$

In the example above the degree of the numerator was less than the degree of the denominator, and this is important for the system of equations we came up with to have a unique solution. However, we can always ensure this happens because of the following fact from algebra.

Theorem 3.1 (The Division Algorithm)**.**
Given any two polynomials $f(x)$ and $g(x)$, there exists a unique pair of polynomials $q(x)$ and $r(x)$ with $\deg(r(x)) < \deg(g(x))$ such that $f(x) = q(x)g(x) + r(x)$.

For example, if $f(x) = x^4 + 2x^3 - x^2 + 5x + 1$ and $g(x) = x^2 + 2x + 1$, then the above says there must be polynomials $q(x)$ and $r(x)$ such that $\deg(r(x)) < 2$ and $f(x) = q(x)g(x) + r(x)$. The theorem only tells us these polynomials doesn't exist, but it doesn't tell us what they are. So, how can we go about finding $q(x)$ and $r(x)$? Let's first notice that $q(x)$ *must* have degree 2 since $g(x)$ has degree 2 and $q(x)g(x) + r(x)$ should have degree 4 (and $r(x)$ can't bump the degree up any higher than what $q(x)$ already provides). That means $q(x)$ must have the form $q(x) = Ax^2 + Bx + C$. Now if we multiply this by our $g(x)$ then we have

$$\begin{aligned}
&\left(Ax^2 + Bx + C\right)\left(x^2 + 2x + 1\right) \\
=&Ax^4 + 2Ax^3 + Ax^2 + Bx^3 + 2Bx^2 + Bx + Cx^2 + 2Cx + C \\
=&Ax^4 + (2A + B)x^3 + (A + 2B + C)x^2 + (B + 2C)x + C
\end{aligned}$$

Since $r(x)$ has degree less than 2, the x^4, x^3, and x^2 terms that appear in $q(x)g(x) + r(x)$ must come from the $q(x)g(x)$ part, and so we must have

$$\begin{aligned}
A &= 1 \\
2A + B &= 2 \\
A + 2B + C &= -1
\end{aligned}$$

We can easily solve the second two equations to determine $B = 0$ and $C = -2$. Thus we must have $q(x) = x^2 - 2$, and we can compute

$$q(x)g(x) = x^4 + 2x^3 - x^2 - 4x - 2.$$

This isn't quite $f(x)$, and we need $f(x)$ to make up the difference:

$$\begin{aligned}
f(x) &= q(x)g(x) + r(x) \\
\implies r(x) &= f(x) - q(x)g(x)
\end{aligned}$$

In our case this allows us to compute

$$\begin{aligned} r(x) &= x^4 + 2x^3 - x^2 + 5x + 1 - \left(x^4 + 2x^3 - x^2 - 4x - 2\right) \\ &= 9x + 3. \end{aligned}$$

This process of finding $q(x)$ and $r(x)$ is streamlined by the polynomial division algorithm you learned in high-school,

$$\begin{array}{r|l} & \quad x^2 \qquad\quad -2 \\ \hline x^2 + 2x + 1 & x^4 + 2x^3 \ - x^2 + 5x + 1 \\ & -x^4 - 2x^3 \ - x^2 \\ \hline & \qquad\quad -2x^2 + 5x + 1 \\ & \qquad\qquad 2x^2 + 4x + 2 \\ \hline & \qquad\qquad\qquad 9x + 3 \end{array}$$

Notice the quotient $x^2 - 2$ at the top is exactly our $q(x)$ and the remainder $9x + 3$ at the bottom is exactly our $r(x)$.

The reason this whole division algorithm and $f(x) = q(x)g(x) + r(x)$ stuff is helpful for us right now is that if divide both sides of $f(x) = q(x)g(x) + r(x)$ by $g(x)$ we're left with

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}.$$

For example, our earlier calculation tells us

$$\frac{x^4 + 2x^3 - x^2 + 5x + 1}{x^2 + 2x + 1} = x^2 - 2 + \frac{9x + 3}{x^2 + 2x + 1}.$$

This can be helpful for integration as it allows us write the following:

$$\begin{aligned} &\int \frac{x^4 + 2x^3 - x^2 + 5x + 1}{x^2 + 2x + 1}\, dx \\ =&\int \left(x^2 - 2 + \frac{9x + 3}{x^2 + 2x + 1}\right) dx \\ =&\frac{x^3}{3} - 2x + \int \frac{9x + 3}{x^2 + 2x + 1}\, dx. \end{aligned}$$

Now we can try to write the fraction in our last integral as a sum of simpler fractions. Noticing that $x^2 + 2x + 1$ factors as $(x + 1)(x + 2)$, we may try to write our fraction as

$$\begin{aligned}\frac{9x+3}{x^2+2x+1} &= \frac{A}{x+1}+\frac{B}{x+2}\\ &= \frac{A(x+2)+B(x+1)}{x^2+2x+1}\\ &= \frac{(A+B)x+2A+B}{x^2+2x+1}.\end{aligned}$$

To determine A and B we must solve the following system of equations,

$$\begin{aligned}A+B&=9\\ 2A+B&=3.\end{aligned}$$

Subtracting the first equation from the secon dequation gives us

$$\underbrace{(2A+B)}_{3}-\underbrace{(A+B)}_{9}=-6$$

This leaves us with $A=-6$, and so $B=15$ and we have

$$\frac{9x+3}{x^2+2x+1}=\frac{-6}{x+1}+\frac{15}{x+2}.$$

Now we can easily compute our earlier integral,

$$\begin{aligned}\int \frac{9x+3}{x^2+2x+1}\,dx &= \int\left(\frac{-6}{x+1}+\frac{15}{x+2}\right)dx\\ &= -6\ln|x+1|+15\ln|x+2|+C.\end{aligned}$$

Plugging this into our integral from before, we now have

$$\int \frac{x^4+2x^3-x^2+5x+1}{x^2+2x+1}\,dx=\frac{x^3}{3}-2x-6\ln|x+1|+15\ln|x+2|+C.$$

Just to summarize what we have thus far: we want to write complicated fractions of polynomials as sums of simpler fractions, and in order to do this we need the polynomial in the numerator has smaller degree than the polynomial in the denominator. If the denominator has smaller degree than the numerator,then we first need to apply the division algorithm to write our fraction in the form $f(x) = q(x)g(x) + r(x)$ where

$f(x)$ is the numerator and $g(x)$ is the denominator. To do this we need to divide $g(x)$ into $f(x)$ and can either do this "by hand" (writing out a corresponding system of equations to give the coefficients of $q(x)$ and then subtracting to determine $r(x)$), or by using the polynomial long division you learned in high-school.

In the examples we have seen thus far we have had denominators that factored into two distinct factors, but this need always happen.

In general, a polynomial of degree n can have at most n roots and so at most n factors (this is called the *fundamental theorem of algebra*). These roots don't have to be distinct, however, and this can result in a repeated factor. For example, $x^3 - 5x^2 + 3x + 9$ factors as $(x+1)(x-3)^2$. Polynomials which can be written as a product of linear factors like this – even if some of the factors are repated – is said to *factor completely*. If a polynomial does not factor completely, then it has irreducible quadratic factors. Conveniently, though, this is the worst possible scenario.

Theorem 3.2.
Every polynomial can be written as a product of linear factors and irreducible quadratic factors. Factors may repeat in both cases.

We will have to modify the procedure of "breaking a fraction into simpler fractions" for each possible situation involving linear factors, quadratic factors, and unique factors or repeated factors. That means there are four cases we need to know how to deal with:

1. Unique linear factors.
2. Repeated linear factors.
3. Unique irreducible quadratic factors.
4. Repeated irreducible quadratic factors.

We will deal with the various cases through the examples that are to follow. For simplicity we will restrict our examples to the situations where the numerator has smaller degree than the denominator to avoid having to apply the division algorithm, but in principle this may be a necessary first step regardless of how the denominator factors.

Example 3.16.
Compute the general antiderivative of

$$\frac{6x^2 + 23x + 10}{x^3 + 3x^2 - 4x - 12}.$$

Our first goal is to write our fraction as a sum of simpler, easier-to-integrate fractions. To do this we need to factor the denominator. In this particular example we can factor by grouping:

$$\begin{aligned} x^3 + 3x^2 - 4x - 12 &= x^2(x+3) - 4(x+3) \\ &= (x^2 - 4)(x+3) \\ &= (x+2)(x-2)(x+3) \end{aligned}$$

Now that we know how the denominator factors, we will write our fraction as a sum of three fractions (one for each factor that appeared) with constants for the numerators:

$$\frac{6x^2 + 23x + 10}{x^3 + 3x^2 - 4x - 12} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{x+3}.$$

Now, to determine what A, B, and C actually are, we want to add the fractions on the left and compare the result with the original fraction we started with.

$$\frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{x+3}$$

$$= \frac{A}{x+2} \cdot \frac{(x-2)(x+3)}{(x-2)(x+3)} + \frac{B}{x-2} \cdot \frac{(x+2)(x+3)}{(x+2)(x+3)} + \frac{C}{x+3} \cdot \frac{(x+2)(x-2)}{(x+2)(x-2)}$$

$$= \frac{A(x^2 + x - 6) + B(x^2 + 5x + 6) + C(x^2 - 4)}{x^3 + 3x^2 - 4x - 12}$$

$$= \frac{(A + B + C)x^2 + (A + 5B)x + (-6A + 6B - 4C)}{x^3 + 3x^2 - 4x - 12}$$

Now we set up our system of equations,

$$\begin{aligned} A + B + C &= 6 \\ A + 5B &= 23 \\ -6A + 6B - 4C &= 10 \end{aligned}$$

To solve this system we'll try to eliminate A from the second and third equation. Subtracting the first equation from the second the system becomes

$$\begin{aligned} A + B + C &= 6 \\ 4B - C &= 17 \\ -6A + 6B - 4C &= 10 \end{aligned}$$

Now adding six times the first equation to the third equation we have

$$\begin{aligned} A + B + C &= 6 \\ 4B - C &= 17 \\ 12B + 2C &= 46 \end{aligned}$$

We can go one step further and eliminate B from the third equation by subtracting three times the second equation from the third equation, giving us

$$\begin{aligned} A + B + C &= 6 \\ 4B - C &= 17 \\ 5C &= -5 \end{aligned}$$

Once we've turned our system into something like this where we have a "triangular" system, we can easily compute the solution to the system by solving for one variable at a time. In this example the third equation instantly tells us $C = -1$. Once we know $C = -1$ we can plug this into the second equation to obtain $4B + 1 = 7$, or $4B = 16$, and so $B = 4$. Now we plug $C = -1$ and $B = 4$ into the first equation to obtain $A + 4 - 1 = 6$, or $A + 3 = 6$, and thus $A = 3$.

Now that we've solved the system we know that our fraction can be rewritten as

$$\frac{6x^2+23x+10}{x^3+3x^2-4x-12} = \frac{3}{x+2} + \frac{4}{x-2} + \frac{-1}{x+3}.$$

This makes our integral much easier to compute:

$$\int \frac{6x^2+23x+10}{x^3+3x^2-4x-12}\,dx = \int \left(\frac{3}{x+2} + \frac{4}{x-2} + \frac{-1}{x+3}\right)dx$$

$$= 3\ln|x+2| + 4\ln|x-2| - \ln|x+3| + C$$

In the last example, and all of our previous examples, the denominator has had distinct roots, so now we turn our attention to the case when a root is repeated. That is, when the polynomial has factors of the form $(x-a)^n$.

When a factor of the denominator repeats, we need to add more terms to our expression with simpler fractions. In particular, if $(x-a)^n$ occurs as a factor of the denominator, then our sum of simpler fractions must include the following terms,

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \cdots + \frac{A_n}{(x-a)^n}.$$

The reason for this has to do with ensuring our system of equations we write down will have a solution. If we did not include all of these terms, we may not be able to solve the system. (The exact reasons for this aren't terribly advanced, but are a bit beyond the scope of our class. If you go on to take linear algebra, you'll learn more about the conditions required to guarantee a system of linear equations has a solution.)

Example 3.17.
Compute $\displaystyle\int \frac{5x^2+36x+62}{x^3+9x^2+27x+27}\,dx.$

Factoring the denominator gives us $(x+3)^3$, and so we will write our fraction as

$$\frac{5x^2+36x+62}{x^3+9x^2+27x+27} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3}.$$

Adding these simpler fractions together gives us

$$\frac{A(x+3)^2+B(x+3)+C}{(x+3)^2} = \frac{A(x^2+6x+9)+Bx+3B+C}{(x+3)^3} = \frac{Ax^2+(6A+B)x+(9A+3}{(x+3)^3}$$

Comparing this to the original fraction we were given, we obtain the system of equations

$$\begin{aligned} A &= 5 \\ 6A+B &= 36 \\ 9A+3B+C &= 62. \end{aligned}$$

Solving this system tells us $A = 5$, $b = 6$, and $C = -1$, thus

$$\int \frac{5x^2+36x+62}{x^3+9x^2+27x+27}\,dx = \int \left(\frac{5}{x+3} + \frac{6}{(x+3)^2} - \frac{1}{(x+3)^3}\right)\,dx$$

In order to integrate this we will perform the substitution $u = x+3$, $du = dx$, and thus rewrite the integral as

$$\int \left(\frac{5}{u} + 6u^{-2} - u^{-3}\right)\,du = 5\ln|u| - 6u^{-1} + \frac{u^{-2}}{2} + C$$

Rewriting this in terms of x we have

$$\int \frac{5x^2+36x+62}{x^3+9x^2+27x+27}\,dx = 5\ln|x+3| - \frac{6}{x+3} + \frac{1}{2(x+3)^2} + C.$$

All of the examples we have seen so far have been concerned with denominators that factor completely into linear factors. Sometimes this does not happen, though, and we will have irreducible quadratic factors. In order to guarantee that the system of equations giving us the numerators of our simpler fractions has a solution, our numerators will need to be linear polynomials, and so have the form $Ax + B$.

Example 3.18.
Compute $\int \frac{x^3+4x^2-6x+7}{(x^2+1)(x-1)^2}\,dx.$

We will try to write our fraction as

$$\frac{x^3+4x^2-6x+7}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}.$$

Notice that we only have a linear numerator for the factor with the quadratic denominator. The other terms, which come from a repeated linear factor, simply have constants as their numerators.

Adding together the terms on the right-hand side gives us

$$\frac{(Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)}{(x^2+1)(x-1)^2}.$$

After a bit of slightly tedious arithmetic this will become

$$\frac{(A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x + (B-C+D)}{(x^2+1)(x-1)^2}$$

This gives us the following system of equations,

$$\begin{aligned} A + C &= 1 \\ -2A + B - C + D &= 4 \\ A - 2B + C &= -6 \\ B - C + D &= 7 \end{aligned}$$

Subtracting the third equation from the first tells us $2B = 7$, so $B = {}^7\!/\!_2$. Subtracting the second equation from the fourth then gives us $2A = 3$, and so $A = {}^3\!/\!_2$. The first equation then becomes ${}^3\!/\!_2 + C = 1$, and so $C = {}^{-1}\!/\!_2$. Finally, we may write the fourth equation as ${}^7\!/\!_2 - (-{}^1\!/\!_2) + D = 7$ and so $D = 3$.

That is, our fraction may be written as

$$\frac{x^3+4x^2-6x+7}{(x^2+1)(x-1)^2} = \frac{({}^3\!/\!_2)x + {}^7\!/\!_2}{x^2+1} + \frac{{}^{-1}\!/\!_2}{x-1} + \frac{3}{(x-1)^2}$$

which we can further rewrite, by bringing the denominators from the terms in the numerators down, as

$$\frac{3x+7}{2(x^2+1)} - \frac{1}{2(x-1)} + \frac{3}{(x-1)^2}.$$

We can more easily integrate these individual terms. For example,

$$\int \frac{3x+7}{2(x^2+1)}\,dx = \frac{3}{2}\int \frac{x}{x^2+1}\,dx + \frac{7}{2}\int \frac{dx}{x^2+1}.$$

The first term on the right can be computed using the substitution $u = x^2+1$, $du = 2x\,dx$ to obtain

$$\frac{3}{4}\int \frac{du}{u} = \frac{3}{4}\ln|u| + C,$$

and so

$$\frac{3}{2}\int \frac{x}{x^2+1}\,dx = \frac{3}{4}\ln|x^2+1| + C.$$

For the second integral above we will perform the trig substitution $x = \tan(\theta)$, $dx = \sec^2(\theta)\,d\theta$. Then our integral becomes

$$\frac{7}{2}\int \frac{\sec^2(\theta)}{\tan^2(\theta)+1}\,d\theta = \frac{7}{2}\frac{\sec^2(\theta)}{\sec^2(\theta)}\,d\theta = \frac{7}{2}\int d\theta = \frac{7}{2}\theta + C.$$

As $x = \tan(\theta)$, $\theta = \tan^{-1}(x)$ and so we have

$$\frac{7}{2}\int \frac{dx}{x^2+1} = \frac{7}{2}\tan^{-1}(x) + C$$

and so

$$\int \frac{3x+7}{2(x^2+1)}\,dx = \frac{3}{4}\ln|x^2+1| + \frac{7}{2}\tan^{-1}(x) + C$$

The second term, $\int \frac{dx}{2(x-1)}$, of our earlier integral is simply $\frac{1}{2}\ln|x-1|+C$. The third term, $\int \frac{3}{(x-1)^2}\,dx$ is easily seen to be $\frac{-3}{x-1}+C$ through a simple u-substitution.

Putting all of the pieces back together we finally have

$$\int \frac{x^3+4x^2-6x+7}{(x^2+1)(x-1)^2}\,dx = \int \frac{3x+7}{2(x^2+1)}\,dx - \int \frac{1}{2(x-1)}\,dx + \int \frac{3}{(x-1)^2}\,dx$$

$$= \frac{3}{4}\ln\left|x^2+1\right| + \frac{7}{2}\tan^{-1}(x) - \frac{1}{2}\ln|x-1| - \frac{3}{x-1} + C.$$

The last case we need to consider is when the denominator has repeated irreducible factors. That is, when the denominator has factors of the form $(\alpha x^2+\beta x+\gamma)^n$ where the quadratic $\alpha x^2+\beta x+\gamma$ will not factor any further. In this situation we will do something analogous to what we did for repeated linear factors: our simpler fraction will contain the terms

$$\frac{A_1x+B_1}{\alpha x^2+\beta x+\gamma} + \frac{A_2x+B_2}{\left(\alpha x^2+\beta x+\gamma\right)^2} + \frac{A_3x+B_3}{\left(\alpha x^2+\beta x+\gamma\right)^3} + \ldots \frac{A_nx+B_n}{\left(\alpha x^2+\beta x+\gamma\right)^n} +$$

Example 3.19.
Compute $\displaystyle\int \frac{3x^3+5x^2+6x+2}{(x^2+2x+2)^2}\,dx$.

We will equate our original fraction with

$$\frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{\left(x^2+2x+2\right)^2}$$

Adding these fractions together yields

$$\frac{Ax^3+(2A+B)x^2+(2A+2B+C)x+2B+D}{(x^2+2x+2)^2}.$$

Setting the numerator of this fraction equal to the numerator of the fraction in our integrand and equating coefficients gives us the system of equations

$$\begin{aligned} A &= 3 \\ 2A + B &= 5 \\ 2A + 2B + C &= 6 \\ 2B + D &= 2 \end{aligned}$$

Solving for the variables one at a time tells us $A = 3$, $B = -1$, $C = 2$, and $D = 4$ and thus we have

$$\int \frac{3x^3 + 5x^2 + 6x + 2}{(x^2 + 2x + 2)^2}\,dx = \int \frac{3x - 1}{x^2 + 2x + 2}\,dx + \int \frac{2x + 4}{(x^2 + 2x + 2)^2}\,dx$$

In order to evluate these two integrals, let's notice that we may complete the square to write our denominator as

$$x^2 + 2x + 2 = x^2 + 2x + 1 - 1 + 2 = (x + 1)^2 + 1.$$

Performing the substitution $u = x + 1$, $du = dx$, the integrals then become

$$\int \frac{3u - 4}{u^2 + 1}\,du + \int \frac{2u + 2}{(u^2 + 1)^2}\,du$$

We now perform the trig substitution $u = \tan(\theta)$, $du = \sec^2(\theta)\,d\theta$ and the integrals become

$$\int \frac{(3\tan(\theta) - 4)\sec^2(\theta)}{\tan^2(\theta) + 1}\,d\theta + \int \frac{2\tan(\theta) + 2}{(\tan^2(\theta) + 1)^2}\sec^2(\theta)\,d\theta$$

As $\sec^2(\theta) = \tan^2(\theta) + 1$ the integrals then become

$$\int (3\tan(\theta) - 4)\,d\theta + \int \frac{2\tan(\theta) + 2}{\sec^2(\theta)}\,d\theta$$

To evalute these, let's keep in mind that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and $\sec(\theta) = \frac{1}{\cos(\theta)}$, so $\frac{1}{\sec(\theta)} = \cos(\theta)$. Our integrals then become

$$\int 3\frac{\sin(\theta)}{\cos(\theta)}\,d\theta - \int 4\,d\theta + \int 2\sin(\theta)\cos(\theta)\,d\theta + \int 2\cos^2(\theta)\,d\theta.$$

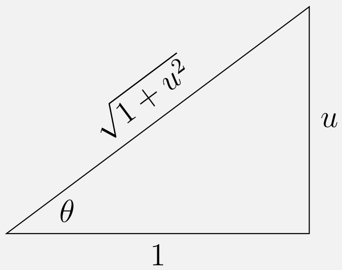
Each of these integral is straight-forward to compute:

$$\int 3\frac{\sin(\theta)}{\cos(\theta)}\,d\theta = -3\ln|\cos(\theta)| + C$$
$$\int 4\,d\theta = 4\theta + C$$
$$\int 2\sin(\theta)\cos(\theta)\,d\theta = -\cos^2(\theta) + C$$
$$\int 2\cos^2(\theta)\,d\theta = \int (1-\cos(2\theta))\ d\theta = \theta - \frac{\sin(2\theta)}{2} + C.$$

Our integral in terms of θ is thus

$$-3\ln|cos(\theta)| - 4\theta - \cos^2(\theta) + \theta - \frac{\sin(2\theta)}{2} + C.$$

Ultimately we need to write this in terms of x, but first we have to go back to writing it in terms of u. Since $u = \tan(\theta)$, we have $\theta = \tan^{-1}(u)$. We can rewrite $\sin(\theta)$ and $\cos(\theta)$ by considering the right triangle with angle θ which has adjacent side of length 1 and opposite side of length u,



From this we see

$$\cos(\theta) = \frac{1}{\sqrt{1+u^2}} \qquad \sin(\theta) = \frac{u}{\sqrt{1+u^2}}.$$

Recalling that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, our integral in terms of u may be written as

$$-3\ln\left|\frac{1}{\sqrt{1+u^2}}\right| - 3\tan^{-1}(u) - \frac{1}{1+u^2} - \frac{u}{1+u^2} + C$$

$$= \frac{3}{2}\ln\left|1+u^2\right| - 3\tan -1(u) - \frac{1+u}{1+u^2} + C$$

Finally, in terms of x, our integral becomes

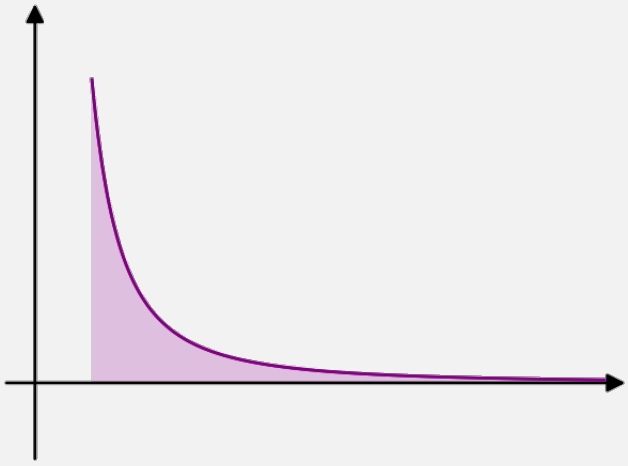
$$\frac{3}{2}\ln\left|1+(1+x)^2\right| - 3\tan^{-1}(x+1) - \frac{x+2}{1+(x+1)^2} + C$$

3.5 Improper integrals

When we've dealt with integration before we have implicitly made two key assumptions: the integral was over a closed, bounded interval $[a, b]$, and the function we were integrating was continuous on the interval we were integrating over. We will now dispense of these assumptions to study so-called "improper integrals."

Let's begin with an example to motivate what's to come.

Example 3.20.
Find the area under the curve $y = {}^1/_{x^2}$ over the interval $[1, \infty)$.



Notice that we can not simply use our usual techniques of "find the antiderivative and evaluate at the endpoints" as one of these endpoints is infinite; we can't really plug ∞ into a function. So, we do the next best thing and approximate the quantity we care about with something easier. In particular, we will approximate the integral over this infinite region with integrals over simpler, finite regions that become larger and larger and larger. That is, we want to say something along the lines of

$$\int_1^\infty \frac{dx}{x^2} \approx \int_1^b \frac{dx}{x^2} \text{ for very large values of } b.$$

Of course, there are some subtle issues here that we'll have to worry about, but we'd like something like this to be true.

Let's notice that

$$\int_1^b \frac{dx}{x^2} = \int_1^b x^{-2}\,dx = -x^{-1}\Big|_1^b = 1 - \frac{1}{b}.$$

Thus

$$\int_1^{10} \frac{dx}{x^2} = 1 - \frac{1}{10} = 0.9$$
$$\int_1^{100} \frac{dx}{x^2} = 1 - \frac{1}{100} = 0.99$$
$$\int_1^{10^6} \frac{dx}{x^2} = 1 - \frac{1}{10^6} = 0.999999$$

As b gets larger and larger, the value of $\int_1^b dx/x^2$ seems to get closer and closer to 1. To make this more precise, we should use limits and define $\int_1^\infty dx/x^2$ as

$$\int_1^\infty \frac{dx}{x^2} := \lim_{b\to\infty} \int_1^b \frac{dx}{x^2}.$$

Of course, in this case we would simply have

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} \left(1 - \frac{1}{b}\right) = 1.$$

In general, we will define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

provided this limit exists. Of course, it's possible the limit will not exist. For example,

$$\int_0^\infty \cos(x)\,dx = \lim_{b\to\infty} \sin(b) \text{ DNE}.$$

Or, the limit could blow up to infinity:

$$\int_0^\infty x\,dx = \lim_{b\to\infty} \int_0^b x\,dx = \lim_{b\to\infty} \frac{b^2}{2} = \infty.$$

We say that the integral $\int_a^\infty f(x)\,dx$ ***converges*** if the corresponding limit exists and is finite, and we say $\int_a^\infty f(x)\,dx$ ***diverges*** otherwise.

In the examples we've seen thus far, $\int_1^\infty dx/x^2$ converges while $\int_0^\infty \cos(x)\,dx$ and $\int_0^\infty x\,dx$ both diverge.

Of course, an integral over a region $(-\infty, a]$ is defined similarly:

$$\int_{-\infty}^a f(x)\,dx := \lim_{b\to-\infty} \int_b^a f(x)\,dx.$$

Example 3.21.
Compute $\int_{-\infty}^{0} xe^x\,dx$.

By definition we must compute the limit

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{b\to-\infty}\int_{b}^{0} xe^x\,dx.$$

Let's first evaluate the integral on the right-hand side using integration by parts with $u = x$ and $dv = e^x\,dx$, so $du = dx$ and $v = e^x$. We then have

$$\begin{aligned}\int_{b}^{0} xe^x\,dx &= xe^x\Big|_b^0 - \int_b^0 e^x\,dx \\ &= \left(0\cdot e^0 - b\cdot e^b\right) - e^x\Big|_b^0 \\ &= -be^b - \left(e^0 - e^b\right) \\ &= -be^b - 1 + e^b \\ &= e^b(1-b) - 1.\end{aligned}$$

Thus our integral is given by

$$\begin{aligned}\int_{-\infty}^{0} xe^x\,dx &= \lim_{b\to-\infty}\int_b^0 xe^x\,dx \\ &= \lim_{b\to-\infty}\left(e^b(1-b) - 1\right) \\ &= \left(\lim_{b\to-\infty} e^b(1-b)\right) - 1 \\ &= \left(\lim_{b\to-\infty}\frac{1-b}{e^{-b}}\right) - 1 \\ &\stackrel{\mathscr{L}}{=} \left(\lim_{b\to-\infty}\frac{-1}{-e^{-b}}\right) - 1 \\ &= \left(\lim_{b\to-\infty}\frac{1}{e^{-b}}\right) - 1 \\ &= 0 - 1 \\ &= -1\end{aligned}$$

If we wished to integral a function $f(x)$ over the entire real line, what we could do is split the integral into two pieces, such as

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx \\ &= \lim_{b\to-\infty}\int_{b}^{a} f(x)\,dx + \lim_{c\to\infty}\int_{a}^{c} f(x)\,dx\end{aligned}$$

Example 3.22.
Compute $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

First we break the integral up as

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

Now let's recall that the antiderivative of $\frac{1}{1+x^2}$ is computed by performing the trig substitution $x = \tan(\theta)$, $dx = \sec^2(\theta)\,d\theta$. Thus $\int \frac{dx}{1+x^2}$ becomes

$$\int \frac{\sec^2(\theta)}{1+\tan^2(\theta)}\,d\theta = \int \frac{\sec^2(\theta)}{\sec^2(\theta)}\,d\theta = \int 1\,d\theta = \theta + C$$

Since $x = \tan(\theta)$, we must have $\theta = \tan^{-1}(x)$ and so $\int \frac{dx}{1+x^2}$ is simply $\tan^{-1}(x) + C$.

Now we can compute our integral as

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to-\infty}\int_{b}^{0} \frac{dx}{1+x^2} + \lim_{c\to\infty}\int_{0}^{c} \frac{dx}{1+x^2} \\ &= \lim_{b\to-\infty} \tan^{-1}(x)\Big|_{b}^{0} + \lim_{c\to\infty} \tan^{-1}(x)\Big|_{0}^{c} \\ &= \lim_{b\to-\infty} \big(\tan^{-1}(0) - \tan^{-1}(b)\big) + \lim_{c\to\infty} \big(\tan^{-1}(c) - \tan^{-1}(0)\big) \\ &= \lim_{b\to-\infty} -\tan^{-1}(b) + \lim_{c\to\infty} \tan^{-1}(c)\end{aligned}$$

Now to evaluate these limits, we need to think about what values of θ can be plugged into $\tan(\theta)$ to give us larger and larger values (as $c \to \infty$) as well as more and more negative values (as $b \to -\infty$).

To do this, we note that as $\theta \to {}^{\pi}/_{2}$, $\tan(\theta)$ goes to infinity, and as $\theta \to -{}^{\pi}/_{2}$, $\tan(\theta)$ goes to negative infinity. This tells us

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to-\infty} -\tan^{-1}(b) + \lim_{c\to\infty} \tan^{-1}(c) \\ &= -\left(\frac{-\pi}{2}\right) + \frac{\pi}{2} \\ &= \pi\end{aligned}$$

Above we observed that $\int_1^{\infty} {}^{dx}/_{x^2}$ converged, but what about other powers of ${}^{1}/_{x}$?

Example 3.23.
Compute $\displaystyle\int_1^{\infty} \frac{dx}{x^3}$.

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x^3} &= \lim_{b\to\infty} \int_1^b x^{-3}\,dx \\ &= \lim_{b\to\infty} \left.\frac{-1}{2x^2}\right|_1^b \\ &= \lim_{b\to\infty} \left(\frac{-1}{2b^2} - \frac{-1}{2\cdot 1^2}\right) \\ &= \lim_{b\to\infty} \left(\frac{1}{2} - \frac{1}{2b^2}\right) \\ &= \frac{1}{2}\end{aligned}$$

Example 3.24.
Compute $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x}}$.

$$\begin{aligned}\int_1^\infty \frac{dx}{\sqrt{x}} &= \lim_{b\to\infty}\int_1^b x^{-1/2}\,dx\\ &= \lim_{b\to\infty} 2x^{1/2}\Big|_1^b\\ &= \lim_{b\to\infty}\left(2\sqrt{b}-2\right)\\ &= \infty\end{aligned}$$

In general, $\displaystyle\int_1^\infty \frac{dx}{x^p}$ may converge or diverge, depending on the value of p. Let's examine this by first supposing $p \neq 1$. If $p \neq$ then we may compute

$$\begin{aligned}\int_1^\infty \frac{dx}{x^p} &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx\\ &= \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg|_1^b\\ &= \lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1}-\frac{1}{-p+1}\right)\\ &= \lim_{b\to\infty}\left(\frac{1}{p-1}-\frac{b^{-p+1}}{p-1}\right)\\ &= \frac{1}{p-1}-\lim_{b\to\infty}\frac{b^{-(p-1)}}{p-1}\\ &= \frac{1}{p-1}-\frac{1}{p-1}\cdot\lim_{b\to\infty}\left(\frac{1}{b}\right)^{p-1}.\end{aligned}$$

Let's now notice that as $b \to \infty$, $\frac{1}{b} \to 0$. We are assuming $p \neq 1$, so there are two things that can happen: either $p > 1$ or $p < 1$. Let's examine each of these cases.

- If $p > 1$, then we have a fraction smaller than one (the $\frac{1}{b}$ above as b becomes larger) being raised to positive powers, and these go to zero as b goes to zero. Thus our integral converges to $\frac{1}{p-1}$.
- If $p < 1$, then our fraction $\frac{1}{b}$ becomes smaller than 1 as b gets large, but we are raising this number smaller than 1 to negative powers, as we're raising to $p-1$. Raising a small number to a negative

power gives us a larger fraction. For example, consider the case when $b = 4$ and $p = 1/2$. Then we'd have

$$\left(\frac{1}{b}\right)^{p-1} = \left(\frac{1}{4}\right)^{-1/2} = 4^{1/2} = 2.$$

As b gets larger and larger, this expression $\left(\frac{1}{b}\right)^{p-1}$ will also get larger and larger and will go off to infinity. Thus if $p < 1$, our integral diverges.

The other case we have to consider is when $p = 1$. In this case the integral is just

$$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty} \ln(b) = \infty$$

and again the integral diverges.

We have thus proven the following theorem:

Theorem 3.3.
The integral $\int_1^\infty \frac{1}{x^p}\,dx$ *converges if and only if* $p > 1$.

This is a helpful observation because we can often combine it with the following theorem.

Theorem 3.4.
Suppose $f(x) \geq g(x) \geq 0$ *for all* x *in* $[a, \infty)$.

- *If* $\int_a^\infty f(x)\,dx$ *converges, then* $\int_a^\infty g(x)\,dx$ *must converge as well.*
- *If* $\int_a^\infty g(x)\,dx$ *diverges, then* $\int_a^\infty f(x)\,dx$ *must diverge as well.*

In some problems we just want to know if an integral will converge or diverge, and combining our two theorems above can be useful if we can compare an integral we're interest in to $\int_1^\infty \frac{dx}{x^p}$ and use the theorem above to say if that integral converges or diverges.

Example 3.25.

Does the integral $\int_1^\infty \frac{2x+3}{x^2-1}\,dx$ converge or diverge?

Our strategy here will be to compare this integral to one of the integrals $\int_1^\infty \frac{dx}{x^p}$. We'll do this by making two simple observations about fractions:

1. If the numerator of a fraction is replaced with something smaller, the entire fraction is smaller. I.e., if $a > b$, then
$$\frac{a}{c} > \frac{b}{c}.$$
For example, $7 > 5$ and so $\frac{7}{3} > \frac{5}{3}$.

2. If the denominator of a fraction is replaced by something larger, the entire fraction is smaller. I.e., if $c < d$, then
$$\frac{a}{c} > \frac{a}{d}.$$
For example, $2 > 3$ and so $\frac{1}{2} > \frac{1}{3}$.

We will apply these rules repeatedly until we can say if $\frac{2x+3}{x^2-1}$ is larger or smaller than a fraction $\frac{1}{x^p}$.

$$\begin{aligned}
\frac{2x+3}{x^2-1} &> \frac{2x}{x^2-1} && \text{(Numerator decreases, fraction decreases.)}\\
&> \frac{x}{x^2-1} && \text{(Numerator decreases, fraction decreases.)}\\
&> \frac{x}{x^2} && \text{(Denominator increases, fraction decreases.)}\\
&= \frac{1}{x}
\end{aligned}$$

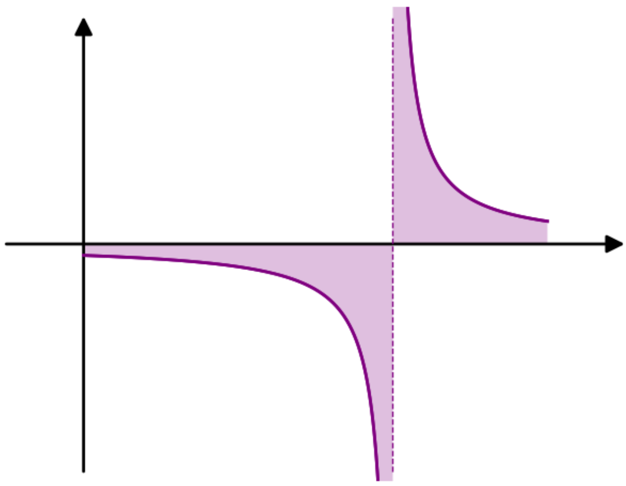
Thus, $\frac{2x+3}{x^2-1} > \frac{1}{x}$. Taking $f(x) = \frac{2x+3}{x^2-1}$ and $g(x) = \frac{1}{x}$ in Theorem 3.4, then since $\int_1^\infty \frac{dx}{x}$ diverges by Theorem 3.3, Theorem 3.4 tells us $\int_1^\infty \frac{2x+3}{x^2-1}$ must diverge as well.

Just to summarize what we've said so far: When an integral goes to $\pm\infty$ on one side, we write this as a limit of integrals as $b \to \pm\infty$, write the value of our integral as a function of b, and then take the limit.

Now let's examine another type of integral. Suppose that we are integrating on a closed, bounded interval $[a, b]$, but our function has a discontinuity. In particular, suppose our function had a vertical asymptote. For example, perhaps we wanted to evaluate the integral

$$\int_0^3 \frac{dx}{x-2}.$$

Notice the function has a vertical asymptote at $x = 2$.



We ***can not*** simply apply the fundamental theorem of calculus and evaluate at the endpoints $x = 0$ and $x = 3$ here. We haven't made a big deal about it because we didn't need to up until this point, but to apply the fundamental theorem of calculus we really need that the function we're integrating is continuous on the interval we're integrating over, but that's not the case here. To get around this, we'll break our integral up into two parts, where our function is continuous on each part. Since $\frac{1}{x-2}$ is continuous on $(0, 2)$ and on $(2, 3)$ (just removing the point $x = 2$ where we have a discontinuity), we will write ore integral as

$$\int_0^3 \frac{dx}{x-2} = \int_0^2 \frac{dx}{x-2} + \int_2^3 \frac{dx}{x-2}.$$

We could do this even if our function was continuous, but let's notice the issue with the discontinuity is really highlighted here. If you were try to apply the fundamental theorem of calculus on each of these pieces you'd have something like

$$\int_2^3 \frac{dx}{x-2} = \ln|x-2|\Big|_2^3 = \ln(1) - \ln(0)$$

and now you see that there's a problem: $\ln(0)$ is undefined! Since we can't actually integrate all the up to 2, because our supposed antiderivative is undefined there, we'll do the next best thing: take the limit as we get really, really close to 2.

In particular, to evaluate $\int_2^3 \frac{dx}{x-2}$, we'll take the limit of integrals of the form

$$\int_b^3 \frac{dx}{x-2}$$

as b gets closer and closer to 2. Notice, though, that we need to be careful that our b's never move "past" 2, or we'll run into this same issue again. So, we'll only take the right-hand limit as b approaches 2 and calculate

$$\begin{aligned}\int_2^3 \frac{dx}{x-2} &= \lim_{b\to 2^+} \frac{dx}{x-2} \\ &= \lim_{b\to 2^+} \ln|x-2|\Big|_b^3 \\ &= \ln(1) - \lim_{b\to 2^+} \ln(b-2)\end{aligned}$$

Notice that as b approaches 2 from the right, $b-2$ is always positive (which is good, since the natural log is not defined for negative numbers), and in particular $b-2$ shrinks down to zero. Thus $\ln(b-2)$ goes down to $-\infty$ and our limit above becomes

$$\int_2^3 \frac{dx}{x-2} = \ln(1) - \lim_{b\to 2^+} \ln(b-2) = 0 - (-\infty) = \infty.$$

And so our integral blows up to infinity; there is an infinite area between the graph $y = \frac{1}{x-2}$ and the interval $(2,3)$ on the x-axis!

Similarly, we can compute $\int_0^2 \frac{dx}{x-2}$ by taking the limit of integrals over $(0,b)$ as b moves closer and closer to 2 from the left. (Notice we're taking the left-hand limit here because our interval is to the left of the asymptote, whereas earlier we took the right-hand limit because the interval was to the right of the asymptote.)

$$\int_0^2 \frac{dx}{x-2} = \lim_{b \to 2^-} \int_0^b \frac{dx}{x-2}$$
$$= \lim_{b \to 2^-} \ln|x-2| \Big|_0^b$$
$$= \lim_{b \to 2^-} \ln|b-2| - \ln|0-2|.$$

Here we might point out that since b approaches 2 from the left, b is always less than 2, and so $b-2<0$. Thus $|b-2| = -(b-2) = 2-b$ and the integral becomes

$$\int_0^2 \frac{dx}{x-2} = \lim_{b \to 2^-} \ln|b-2| - \ln|0-2|. = \lim_{b \to 2^-} \ln(2-b) - \ln(2).$$

Let's notice that as b gets closer and closer 2 from the left, $2-b$ is a positive number (so $\ln(2-b)$ is defined), and $2-b$ approaches 0 as b approaches 2, thus the limit goes to $-\infty$ again:

$$\int_0^2 \frac{dx}{x-2} = -\infty.$$

Now, we might be tempted to put these two integrals back together to obtain $-\infty + \infty$, but here we have a problem: $-\infty + \infty$ is not defined. In particular, **negative infinity plus infinity is not zero**! That is, we can not assign a value to $\int_0^3 \frac{dx}{x-2}$! The best we can do is say that it diverges.

Remark.
Notice that if we did naively try to apply the fundamental theorem of calculus we would have computed $\ln(1)-\ln(2) \approx -0.6932$, which is very much the wrong answer!

In general, if a function $f(x)$ is continuous on an interval $[a,b)$ but is discontinuous at b, then we define

$$\int_a^b f(x)\,dx := \lim_{t \to b^-} \int_a^t f(x)\,dx \text{ provided the limit exists.}$$

If the limit exists (which implies the limit is a finite value), then we say the integral ***converges*** to that limit. If the limit does not exist (and this includes the cases when the limit is $\pm\infty$), we say the integral ***diverges***.

Similarly, if $f(x)$ is continuous on $(a, b]$ and discontinuous at a we define

$$\int_a^b f(x)\,dx := \lim_{t\to a^+} \int_t^b f(x)\,dx \text{ provided the limit exists.}$$

If $f(x)$ is defined on $[a, b]$ but has a discontinuity at some point c inside the interval, $a < c < b$, we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

and calculate those two integrals as limits,

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \\ &= \lim_{t\to c^-} \int_a^t f(x)\,dx + \int_{s\to c^+} \int_s^b f(x)\,dx \text{ provided both limits exist.}\end{aligned}$$

If both of the limits in the last expression exist (and so are finite), then we say the integral converges to the sum of those limits. If either of the limits in the last expression above do not exist, which includes the case that either is $\pm\infty$, then we say the integral diverges.

Remark.
If it happens that both limits are infinity, then there's not really any ambiguity or concern with writing $\infty + \infty = \infty$, and so we may be more precise and say the integral "diverges to infinity." Similarly if both limits were to go to $-\infty$, we can say $-\infty + -\infty = -\infty$ and say the integral "diverges to negative infinity." The issue we worry about is when one integral is ∞ and the other is $-\infty$. In those situations we can't assign a value to $\infty + -\infty$, and so the best we can do is say that the integral diverges.

To summarize, when a function we're integrating has a discontinuity, we need to approximate the integral by integrating over intervals that get closer and closer to that point of discontinuity, taking care that our intervals don't cross the discontinuity. This means we have to use one-sided limits, and we need to just pay close attention to which side we need to take the limit on.

Example 3.26.
Compute $\int_2^3 \frac{dx}{\sqrt{3-x}}$.

Let's notice this function has a discontinuity at $x = 3$, so we'll approximate the integral by integrating

$$\int_2^b \frac{dx}{\sqrt{3-x}}$$

for values of b that get closer and closer to 3. Since 2 is to the left of 3, our intervals $(2, b)$ will always end to the left of 3, and we'll want to take the left-hand limit as b approaches 3.

$$\int_2^3 \frac{dx}{3-x} = \lim_{b \to 3^-} \int_2^b \frac{dx}{3-x}$$

Now we'll perform a u-substitution with $u = 3 - x$ and $du = -\,dx$. The integral then becomes

$$\begin{aligned}
\lim_{b \to 3^-} -\int_1^{3-b} \frac{1}{\sqrt{u}}\, du &= \lim_{b \to 3^-} \int_{3-b}^1 u^{-1/2}\, du \\
&= \lim_{b \to 3^-} 2u^{1/2}\Big|_{3-b}^{1} \\
&= \lim_{b \to 3^-} 2\left(\sqrt{1} - \sqrt{3-b}\right)
\end{aligned}$$

Notice that as b approaches 3 from the left, we have that $3 - b$ approaches 0. (Also note $b < 3$ so $3 - b > 0$ and the square roots we're evaluating are defined.) Thus our limit is 2 and so

$$\int_2^3 \frac{dx}{\sqrt{3-x}} = 2.$$

3.6 Numerical integration

As we've seen, integration has a number of different uses. Usually the way we calculate an integral is by finding an antiderivative and evaluating it at the endpoints of the integral. Even for improper integrals this is basically our main tool, we just have to be a little bit careful and need to take limits sometimes. However, there are some integrals which simply cannot be computed in this way. One particularly important example is the following:

$$\frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx.$$

The expression above is extremely important in probability and statistics because it represents the probability that the standard normal random variable takes on a value between a and b. People that are interested in statistics *really* care about this because the main work horse of statistics, called the central limit theorem, allows us to basically treat very large random samples as if they were normal random variables.

The exact details of what a "random variable" is or what it means to be "normally distributed" are outside the scope of this class so we won't spend any time saying exactly what this means, we just want to point out that the integral above is something that has a lot of importance and application in the real world.

So, statisticians are interested in evaluating integrals like the one above, but there's a big problem. Even though the fundamental theorem of calculus promises us that the function appearing in the integral above does have an antiderivative (since it's continuous), we actually can not write the antiderivative as anything simpler than

$$F(x) = \int_a^x e^{-t^2/2}\,dt.$$

That is, we have several tricks up our sleeves for calculating antiderivatives at this point, but none of them help for this particular antiderivative. And it's not simply that we haven't learned the right trick yet, it's that there is no trick that helps with this antiderivative. Furthermore, it's not even that no one has been clever enough to come up with the right trick yet: there are in fact theoretical reasons (extremely beyond the scope of this class) why no trick can exist.

This is a huge problem. We have an integral that we really, really want to be able to evaluate for real-world applications, but we have no hope of writing its antiderivative as something we can actually compute. In situations like this we must resort to numerically approximating the integral.

This means that we basically are never able to say what the exact value of the integral will be, but we can at least get an approximation. In fact, as we'll see, we can even go so far as to say how good our approximation is.

The simplest thing we could try to do to approximate an integral numerically is to use a simple Riemann sum. Using n rectangles of equal width, and using the left-hand edge of the rectangle to determine its height, we can approximate a definite integral as

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right) \cdot \frac{b-a}{n}.$$

Intuitively we expect these to be good approximations when n is large, but can we say anything more precise? In particular, can we say how large n needs to be in order to guarantee our integral is within some given distance of the true value?

That is, maybe we want to estimate the integral to within one one-millionth of the true value. Do we have any hope of saying how large n will need to be to guarantee that?

The main tool for answering this question is the following.

Theorem 3.5.
If $f(x)$ is a continuously differentiable function defined on $[a, b]$, then the error in approximating $\int_a^b f(x)\,dx$ by the Riemann sum above satisfies the following inequality:

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right) \cdot \frac{b-a}{n} \right| \leq \frac{R(b-a)^2}{2n}$$

where R is the maximum of $|f'(x)|$ on $[a, b]$.

Let's first try to unravel exactly what this theorem means. Here we have some unknown true quantity we're interested in, $\int_a^b f(x)\,dx$, and some approximation given by our expression with Riemann sums. The left-hand side of the inequality above is simply measuring the error in approximating the true value of the integral by the given Riemann sum. The right-hand side is an upper bound for that error on the left-hand side: it's just some quantity that we know for sure will be bigger. The exact value of the right-hand side depends on three things: the size of the interval, the number of rectangles, and the function. Intuitively, how good or bad a given approximation with some fixed number of rectangles is depends on the function, and in particular it depends on how quickly the function changes. (If our function changes really, really slowly then we shouldn't have a hard time approximating it with a few rectangles. If the function changes very quickly, however, then we'll need more and more approximations to account for that quick change in the function.) This is what the $R = \max|f'(x)|$ factor is measuring.

Example 3.27.

If $\int_{-1}^{1} e^{-x^2/2}\,dx$ is approximated by the Riemann sum

$$\sum_{i=1}^{n} e^{-\left(-1+\frac{2i}{n}\right)^2/2} \cdot \frac{2}{n},$$

then the error in this approximation is no more than

$$\frac{R(1-(-1))^2}{2n} = \frac{4R}{2n} = \frac{2R}{n}$$

where R is the maximum of

$$\left|\frac{d}{dx}e^{-x^2/2}\right| = \left|e^{-x^2/2}\cdot(-x)\right| = \left|xe^{-x^2/2}\right|$$

on $[-1, 1]$. This is a calculus maximization problem which we can easily check is maximized at $x = -1$, and so

$$R = \left|(-1)e^{-(-1)^2/2}\right| = e^{-1/2} = \frac{1}{\sqrt{e}}.$$

Thus the Riemann sum approximation is within

$$\frac{2}{n\sqrt{e}} \approx \frac{1.2131}{n}$$

of the true value.

If we wanted to estimate the integral and stay within one one-millionth, $1/10^6$, of the true value, then we would need to find the n

that guarantees $\frac{1.2131}{n}$ is less than $\frac{1}{10^6}$:

$$\begin{aligned} \frac{1.2131}{n} &< \frac{1}{10^6} \\ \implies 1.2131 \cdot 10^6 &< n \\ \implies n &> 1,213,100 \end{aligned}$$

In the previous example we saw that we required over $1,213,000$ rectangles to estimate the value of our given integral to within one one-millionth of the true value. This is a lot of rectangles, but perhaps we could modify the way we approximate the integral and get more accurate approximations. In particular, if we used a more accurate approximation, we might be able to get just as good of an estimate but without requiring as many computations.

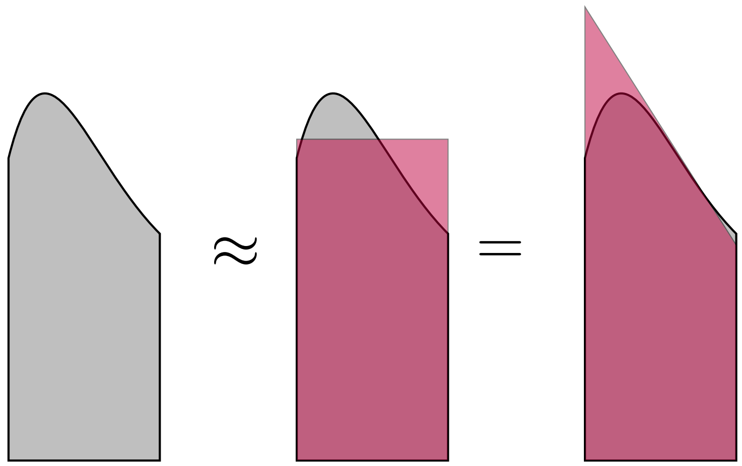
We will discuss three more methods for approximating the value of an integral numerically. One, the "midpoint rule," is very similar to our Riemann sum approximation, but with one minor change that improves our error bound. The second, the "trapezoid rule," is motivated by an observation we'll make in discussing the midpoint rule, and is essentially a Riemann sum where we use trapezoids instead of rectangles. This makes a modest change to the midpoint rule's error bound. The final method, however, which is called "Simpson's rule," is more sophisticated and offers a huge improvement in the error. If you ever compute an integral numerically on a calculator or computer, the calculator/computer is very likely using Simpson's rule.

The midpoint rule is essentially using a Riemann sum but where we used the midpoint of the rectangle's base to determine the height. That is, if our rectangle has base $[x_i, x_{i+1}]$, we will get the height by plugging the midpoint $\frac{x_i + x_{i+1}}{2}$ into the function $f(x)$.

In particular, we have

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(a + \left(\frac{b-a}{n}\right) \cdot \left(i - \frac{1}{2}\right)\right) \cdot \frac{b-a}{n}.$$

At first you might not expect this small change to make a big difference in the accuracy of our estimate, but there's a neat observation. The area of the rectangle computed in this way is equal to the area of the trapezoid whose top is tangent to the curve $y = f(x)$ at the midpoint.



Thinking of our areas as areas of these trapezoids instead of rectangles, it's pretty easy to believe this will give a much better approximation.

Theorem 3.6.
If $f(x)$ is a twice continuously differentiable function[a]*, then the error that occurs in approximating $\int_a^b f(x)\,dx$ using the midpoint rule satisfies the following inequality:*

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} \cdot \left(i - \frac{1}{2}\right)\right) \cdot \frac{b-a}{n} \right| < \frac{M(b-a)^3}{24n^2}$$

where M is the maximum value of $|f''(x)|$ on $[a,b]$.

[a] This just means that the second derivative $f''(x)$ is defined and is continuous. Virtually every function we'll care about in this class will have this property.

Example 3.28.
Approximating $\displaystyle\int_{-1}^{1} e^{-x^2/2}\,dx$ using the midpoint rule, the error is bounded by

$$\frac{M \cdot 2^3}{24n^2} = \frac{8M}{24n^2} = \frac{M}{3n^2}$$

where M is the maximum value of

$$\left|\frac{d^2}{dx^2}e^{-x^2/2}\right| = \left|\frac{x^2-1}{e^{x^2/2}}\right|$$

on $[-1,1]$, which occurs at $x=0$, and so $M = e^{-1/2}$. The error is then bounded by

$$\frac{1}{3\sqrt{e}n^2} \approx \frac{0.2022}{n^2}$$

To guarantee the error is within one one-millionth, $1/10^6$, of the true value, we require

$$\begin{aligned}\frac{0.2022}{n^2} &< \frac{1}{10^6}\\ \implies n^2 &> 202,000\\ \implies n &> \sqrt{202,000} \approx 449.441.\end{aligned}$$

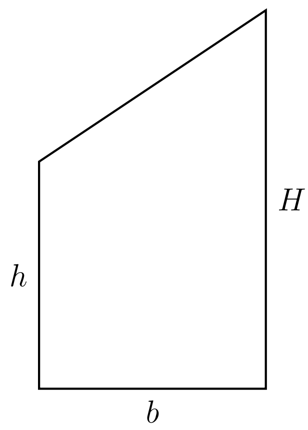
Notice that using the midpoint rule we are guaranteed that the error in our approximation is within one one-millionth of the true value if the number of rectangles used is just 450, as opposed to over one million rectangles in the normal Riemann sum!

Remark.
It's worth pointing out that our theorems are only giving us a guarantee that if we use at least "this" many rectangles, our error will be "this" small. The theorems *are not* telling us that we must use that many rectangles. I.e., it could be that our estimate using the normal Riemann sum is within one one-millionth of the true value with fewer rectangles, we just don't have any guarantees based on these theorems.

Since the midpoint rule is basically adding up the areas of certain trapezoids approximating our curve, one natural thing we might try is to just find the areas of trapezoids that touch the curve at two points. This is slightly different than the midpoint rule. In the midpoint rule we basically have areas that are equal to the area of a trapezoid tangent to the curve at one point. Depending on how much the curve wiggles around, though, these trapezoids could still have very different areas than the area under the curve. To fix this we'll instead force our trapezoids to touch two points on the curve.

That is, instead of using areas under rectangles, we'll use areas under trapezoids and we'll set our trapezoids up so that their left- and right-hand edges are both on the curve.

Let's notice that the area of a trapezoids of base b and heights h and H,



has area

$$b \cdot \frac{h+H}{2}$$

Writing out what this means in terms of sums, we are approximating the area under the graph $y = f(x)$ by

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f\left(a + \left(\frac{b-a}{n}\right) i\right)\right).$$

(What's happening here is that every term that appears as a "height" of a trapezoids appears twice, once on he left and once on the right, except the very first and very last terms which only appear once.)

Theorem 3.7.
If $f(x)$ is a twice continuously differentiable function, then the error that occurs in approximating $\int_a^b f(x)\,dx$ using the trapezoid rule satisfies the

following inequality:

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2n}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f\left(a + \left(\frac{b-a}{n}\right) i \right)\right) \right| < \frac{M(b-a)^3}{12n^2}$$

where M is the maximum value of $|f''(x)|$ on $[a, b]$.

Notice this is a modest change to the error bound from the midpoint rule.

Exercise 3.5.
Determine the number of trapezoids required to estimate the value of $\int_{-1}^{1} e^{-x^2/2}\,dx$, using the trapezoid method, to within one one-millionth of the true value.

Our calculation is almost exactly the same as for the midpoint method, in particular the value of M is the same. The only difference is that the 24 that originally appeared in our midpoint method's error bound becomes a 12. Then have

$$\begin{aligned} \frac{\frac{1}{\sqrt{e}}2^3}{12n^2} &< \frac{1}{10^6} \\ \Longrightarrow \frac{2}{3\sqrt{e}n^2} &< \frac{1}{10^6} \\ \Longrightarrow \frac{0.4044}{n^2} &< \frac{1}{10^6} \\ \Longrightarrow n &> \sqrt{404,000} \approx 635.9 \end{aligned}$$

Thus we require at least 635 trapezoids.

The last method of numerical approximation we'll mention is a bit more sophisticated. For our last method what we'll do is approximate the area under the curve not with rectangles or trapezoids, but with parabolas. That is, we'll look at triples of points on the curve at a time, and will find the parabola that goes through those three points, then compute the area under that one little parabolic piece, and add up the areas coming from each parabolic piece.

Let's first notice that the parabola that goes through three given points in the xy-plane, say (a, A), (b, B), and (c, C), is given by

$$y = \frac{(x-b)(x-c)}{(a-b)(a-c)} \cdot A + \frac{(x-a)(x-c)}{(b-a)(b-c)} \cdot B + \frac{(x-a)(x-b)}{(c-a)(c-b)} \cdot C$$

It's easy to notice that this will be parabola since each term in the expression is quadratic, and we can easily check that it passes through the given points. The terms that appear above are basically set up exactly so that the second two will cancel out when a is plugged in for x and the stuff that remains will be A. That is, plugging $x = a$ into the above gives us

$$\frac{(a-b)(a-c)}{(a-b)(a-c)} \cdot A + \frac{(a-a)(a-c)}{(b-a)(b-c)} \cdot B + \frac{(a-a)(a-b)}{(c-a)(c-b)} \cdot C = A$$

Thus (a, A) is on the curve. You can similarly check (b, B) and (c, C) are also on the curve.

Now to find the area under this piece of the parabola between a and c, we need to integrate the function above,

$$\int_a^C \left[\frac{(x-b)(x-c)}{(a-b)(a-c)} \cdot A + \frac{(x-a)(x-c)}{(b-a)(b-c)} \cdot B + \frac{(x-a)(x-b)}{(c-a)(c-b)} \cdot C \right] dx.$$

This is not a hard integral; it's just a polynomial. It is a bit tedious to write out all the details of simplifying the expression above before integrating, however, so we won't take the time to do that here. The take away, though, is that the integral will evaluate to

$$\frac{c-a}{3}\left(A + 4B + C\right)$$

Now, when we're approximating an integral $\int_a^b f(x)\,dx$, what we'll do is divide the interval $[a, b]$ up into some even number of pieces, say $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, ..., $[x_{2n-1}, x_{2n}]$. Now we'll look at parabolas that pass through three consecutive points in this partition of $[a, b]$. For example, the parabola that passes through $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$. Using these coordinates as our (a, A), (b, B), and (c, C) above we see the area under the parabola is

$$\frac{x_2 - x_0}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right).$$

This is just the first of our parabolas. For our second parabola we'll want to use the coordinates $(x_2, f(x_2))$, $(x_3, f(x_3))$, and $(x_4, f(x_4))$. Notice that we skipped over x_1 since we want our parabolas to not have any overlap. Using our formula above, the area under this parabolic piece is

$$\frac{x_4 - x_2}{3}\left(f(x_2) + 4f(x_3) + f(x_4)\right).$$

Continuing to compute these areas and add them up we obtain

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left(\sum_{i=1}^{n/2}\left[f\left(x_{2(i-1)}\right) + f\left(x_{2i}\right)\right] + 4\sum_{i=1}^{n/2} f\left(x_{2i-1}\right)\right).$$

This calculation is known as ***Simpson's rule***, and while it looks considerably more involved than our other approximations, it is very useful because of the following theorem.

Theorem 3.8.
Suppose $f(x)$ is 4-times continuously differentiable on $[a,b]$ and $\int_a^b f(x)\,dx$ is approximated using Simpson's rule and a partition of n equally-spaced parabolic pieces where n is even (i.e., there are $n/2$ parabolic pieces). Then the absolute error in this approximation is bounded above by

$$\frac{K(b-a)^5}{180n^4}$$

where K is the maximum value of $|f^{(4)}(x)|$ on the interval $[a,b]$.

In our running example of estimating $\displaystyle\int_{-1}^{1} e^{-x^2/2}\,dx$, the value of K will wind up being 3. The expression $\frac{K(b-a)^5}{180n^4}$ will then become

$$\frac{3\cdot 2^5}{180n^4}.$$

If we wish for the error to be less than one one-millionth, then we need the n that makes this expression less than $1/10^6$:

$$\begin{aligned}\frac{3\cdot 2^5}{180n^4} &< \frac{1}{10^6}\\ \implies n^4 &> \frac{96}{180}\cdot 10^6\\ \implies n &> 27.024.\end{aligned}$$

Keeping in mind n is twice the number of parabolas, this means we need only 14 parabolas to estimate the value of our integral to within one one-millionth of the true value using Simpson's rule. This is a *huge* improvement over the previous methods, and it means that we can get good estimates using a much, much smaller number of calculations, which is why most computers and calculators will use Simpson's rule to numerically estimate integrals.

3.7 Practice problems

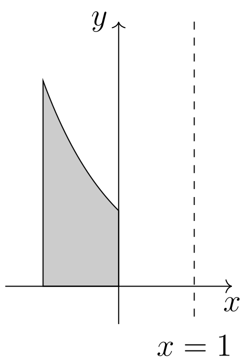
Problems about integration by parts

Problem 3.1. What is the antiderivative of $\int (4x^3 + 2x)\ln(x)\,dx$?

Problem 3.2. Compute the antiderivative of $e^{2x}\sin(3x)$.

Problem 3.3. Compute the antiderivative of $\sec^3(x)$. (Hint: At some point you will need to integrate $\sec(x)$. You can do this by multiplying and dividing $\sec(x)$ by $\sec(x) + \tan(x)$.)

Problem 3.4. Use the shell method to determine the volume of the solid obtained by rotating the region bounded by $y = e^{-x}$, $y = 0$, $x = -1$ and $x = 0$ (the shaded region below) around the line $x = 1$.



Problem 3.5. What is the antiderivative of $\ln(\sqrt{x})$? (Hint: Compare to how the antiderivative of $\ln(x)$ is computed earlier in the notes.)

Problem 3.6. What is the antiderivative of x^2e^x?

Practice problems about powers of trig functions

Problem 3.7. Find the antiderivative of $\sin^4(x)$.

Problem 3.8. Find the antiderivative of $\tan^2(x)\sin(x)$.

Practice problems about trig substitution

Problem 3.9. Compute the antiderivative of $\dfrac{x}{\sqrt{36-x^2}}$.

Problem 3.10. Compute the antiderivative of $\dfrac{x^5}{\sqrt{x^2+2}}$.

Problem 3.11. Find the antiderivative of $\dfrac{x}{\sqrt{x^2-1}}\,dx$.

Practice problems about partial fractions

Problem 3.12. Find the antiderivative of $\frac{5x-1}{x^2-1}$.

Problem 3.13. Find the antiderivative of $\dfrac{-35}{x^2+x-12}$.

Problem 3.14. Find the antiderivative of $\dfrac{4x^2+12x-4}{(x+2)^2(x-4)}$.

Problem 3.15. Find the antiderivative of $\frac{8x^2-x+3}{x^3+x}$.

Practice problems about improper integrals

Problem 3.16. Compute $\displaystyle\int_{4/\pi}^{\infty} \frac{\sec^2\left(1/x\right)}{x^2}\,dx$.

Problem 3.17. Compute $\displaystyle\int_0^1 x\ln(x)\,dx$.

Problem 3.18. Compute $\displaystyle\int_{-2}^{2} \frac{dy}{\sqrt{4-y^2}}$.

Practice problems about numerical integration

Problem 3.19. Recall that the area between the parabola through (a, A), (b, B), and (c, C) (where b is half-way between a and c) and the x-axis is given by

$$\frac{c-a}{6}\left(A+4B+C\right).$$

Use this to approximate $\displaystyle\int_0^4 \frac{x^2+x+1}{x^2+2x+2}\,dx$ using two parabolic pieces of equal width (i.e., Simpson's rule with $n = 4$). Your final answer should be a sum of fractions of integers which you do not need to simplify.

4 More Applications

The irreducible price of learning is realizing that you do not know.

JAMES BALDWIN

4.1 Arclength

In this section we want to determine the "arclength" of certain curves in the plane. That is, if our curve were to represent a road and you were to drive along that road from one end to the other, we want to know how far you would drive. Another way to think about arclength is if we were to lay a piece of string perfectly along a curve, how much string would we need?

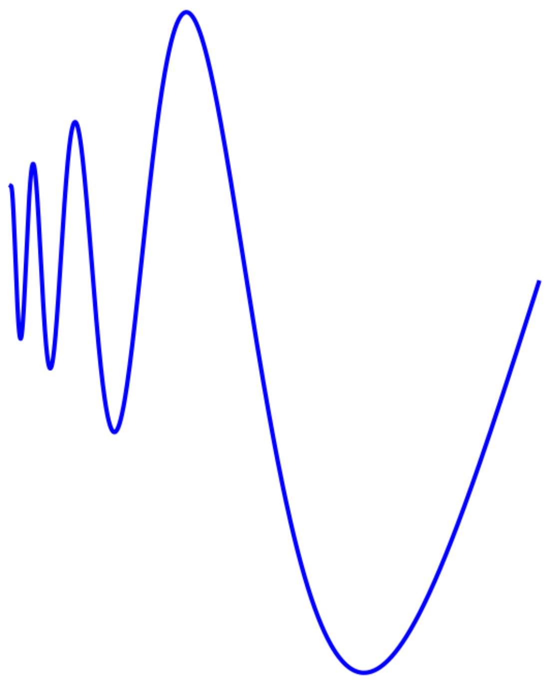


Figure 4.1: How long is this curve?

Given an arbitrary curve in the plane this seems like a pretty difficult problem. We will thus do what we always do in calculus when we have a quantity we want to calculate but aren't sure how to do it: estimate with something we can calculate, and then find a way to improve our estimate. In particular, we can pretty easily determine the length of a line segment, so what we might try to do is approximate our curve with a "broken line," which is just several line segments that are joined together at their ends. It is easy for us to determine the length of this broken line: just add the lengths of each segment!

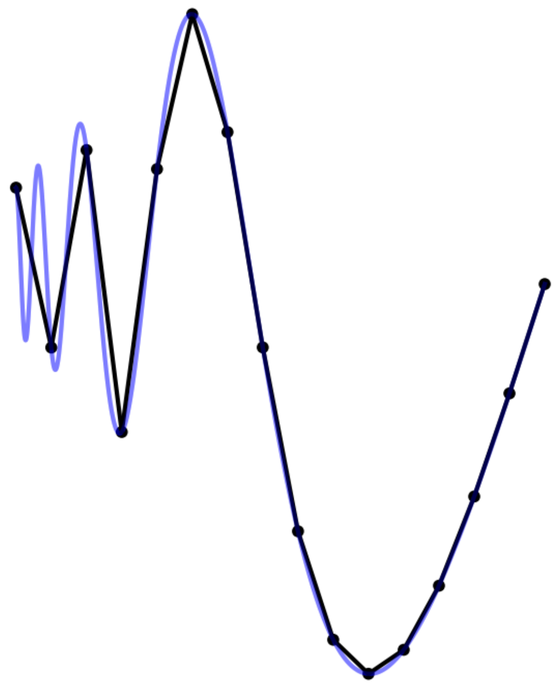


Figure 4.2: The length of this "broken line" is easier to compute.

We will see that by taking the limit as each line segment gets smaller and smaller, we get better and better estimates. Ultimately we'd like for this to be a limit of Riemann sums so we can write the value as an integral, but as we'll see below there is a slightly non-obvious step required to convert the sum we'll write down as an integral we can compute.

Remark.

For simplicity, all of the curves we will be interested will be graphs of functions. Later in the semester we'll talk about how to represent other types of curves, and in the third semester calculus course you'll see a generalization of the procedure we're described here for computing arclengths of those more general curves.

Let's begin, though, by recalling that the distance between two points (x_0, y_0) and (x_1, y_1) is given by $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$. Letting $\Delta x = x_1 - x_0$ and $\Delta y = y_1 - y_0$, we may write this as $\sqrt{\Delta x^2 + \Delta y^2}$. Of course, this is really just the Pythagorean theorem, where the length of our line segment is the hypotenuse of a right triangle with sides of length Δx and Δy, as in Figure 4.3

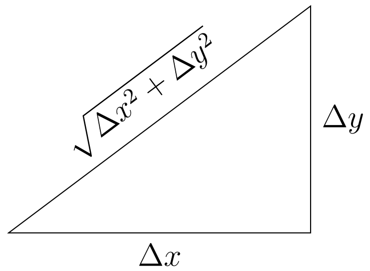


Figure 4.3: We compute length of a line segment with the Pythagorean theorem.

A single line won't give us a very good approximation to a general curve, so we need to consider something a little bit more general. A ***broken line*** is simply a finite collection of line segments concatenated together. That is, the end of one segment is the start of another segment, with the possible exception of the very first and last segments. Let's suppose the coordinates of the endpoints of our line segments are denoted (x_0, y_0), (x_1, y_1), (x_2, y_2), and so on, up to (x_n, y_n), as indicated in Figure 4.4.

Notice the length of the line segment connecting (x_{i-1}, y_{i-1}) to (x_i, y_i) is

$$\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2},$$

and so adding up the lengths of all of the line segments gives us that the length of the broken line is

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}.$$

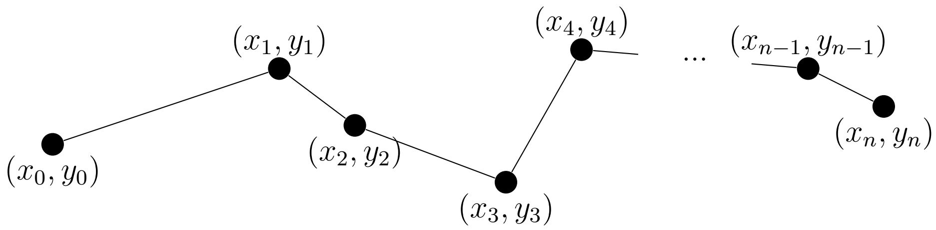


Figure 4.4: A broken line with the endpoints of each segment labeled.

So, if we wanted to estimate the length of the graph $y = f(x)$ where $a \leq x \leq b$, then we could take our x_i values in the expressions above to be points between a and b, and $y_i = f(x_i)$. For example, if we were to use $n + 1$ equally spaced points (this would give us n line segments in our broken line), we could take

$$\begin{aligned} x_i &= a + \frac{(b-a)i}{n} \\ y_i &= f(x_i) = f\left(a + \frac{(b-a)i}{n}\right) \end{aligned}$$

where $0 \leq i \leq n$. For notational simplicity we'll continue to just write x_i and y_i for the moment, but keep in mind we're just selecting some x values and then plugging those into $f(x)$ to get the corresponding y values, in order to have points on our curve.

Our goal now is to somehow recognize the expression above as a Riemann sum we can take a limit of and compute as an integral. To simplify notation a little, let's write Δx_i and Δy_i for $x_i - x_{i-1}$ and $y_i - y_{i-1}$, respectively. Then our sum above becomes

$$\sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}.$$

We would like to somehow turn this square root into something of the form "an expression in $f(x_i)$ multiplied by Δx_i" so that we could easily write down our integral. In order do this we'll have to do some simple, but not entirely obvious, manipulations to our quantity above.

Caution: The work that appears below is mostly included for the sake of completeness and fills in the details about how our estimate above becomes an integral in the limit. If these details feel a little too technical for you and you don't want to take the time to digest them, you can safely skip the discussion below and jump to Theorem 4.1

Let's first factor a Δx_i^2 from each term under our radical. That is, terms will be rewritten as

$$\begin{aligned}\sqrt{\Delta x_i^2 + \Delta y_i^2} &= \sqrt{\Delta x_i^2 + \Delta y_i^2 \cdot \frac{\Delta x_i^2}{\Delta x_i^2}} \\ &= \sqrt{\left(1 + \frac{\Delta y_i^2}{\Delta x_i^2}\right) \Delta x_i^2} \\ &= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \, \Delta x_i\end{aligned}$$

And so our sum is now

$$\sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \, \Delta x_i$$

Keeping in mind $\Delta y_i = f(x_i) - f(x_{i-1})$, this is closer to what we want, but the Δx_i that appears in the denominator of our fraction is a little bit problematic. To get rid of the Δx_i in that denominator we'll need to apply the mean value theorem. Recall that the mean value theorem says if $f(x)$ is continuously differentiable on the interval $[a, b]$, then there exists some value c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

To take advantage of the mean value theorem here, let's notice that our fraction is really equal to the following:

$$\frac{\Delta y_i}{\Delta x_i} = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}.$$

Now, supposing our function $f(x)$ is continuously differentiable, the mean value theorem applied to the interval $[x_{i-1}, x_i]$, promises us there exists some value, let's call it x_i^*, in-between x_{i-1} and x_i such that

$$f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}.$$

Thus the length of our broken line is equal to

$$\sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2} \, \Delta x_i.$$

Notice that this is only an approximation to the arclength of $y = f(x)$ that we are searching for. We get better and better approximations by moving the points in our broken line closer together, and this happens as we use more and more points in the setup above. Taking the limit as the number of points goes to infinity, we see that our approximation is really a Riemann sum for the integral

$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

This establishes the following theorem:

Theorem 4.1.
If $f(x)$ is a continuously differentiable function, then the length of the curve $y = f(x)$ where $a \leq x \leq b$ is given by

$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Remark.
In our class virtually every function we consider is continuously differentiable, so for our purposes you can avoid thinking about this condition if you feel like you don't understand it.

To start using this theorem, let's first consider a few simple examples where the arclength is something we can compute through some other method, just to verify this gives us the values we'd expect.

Example 4.1.
Compute the arclength of the line segment connecting $(0, 0)$ to $(4, 3)$ using the integral in Theorem 4.1.

We don't *need* the integral above to do this calculation, but we'll use it here just to verify this calculates the length we'd expect. First we have to write our line segment as the graph of a function. Since the line segment goes through $(0, 0)$, that graph will have the form $y = mx$ (the y-intercept is zero). We can determine m using the slope formula of rise over run (i.e., change in y-values over change in x-values). That tells us the slope is $\frac{3}{4}$, and so our line segment is $y = \frac{3}{4}x$ where $0 \leq x \leq 4$. The integral is then

$$\begin{aligned}
\text{Arclength} &= \int_0^4 \sqrt{1 + \left(\frac{d}{dx}\frac{3}{4}x\right)^2}\,dx \\
&= \int_0^4 \sqrt{1 + \left(\frac{3}{4}\right)^2}\,dx \\
&= \int_0^4 \sqrt{1 + \frac{9}{16}}\,dx \\
&= \int_0^4 \sqrt{\frac{16+9}{16}}\,dx \\
&= \int_0^4 \sqrt{\frac{25}{16}}\,dx \\
&= \int_0^4 \frac{5}{4}\,dx \\
&= \frac{5}{4}x\Big|_0^4 \\
&= \frac{5}{4}\cdot 4 - \frac{5}{4}\cdot 0 \\
&= 5
\end{aligned}$$

Notice this agrees with the value the distance formula would give us,

$$\sqrt{(4-0)^2 + (3-0)^2} = \sqrt{16+9} = \sqrt{25} = 5.$$

Example 4.2.
Find the arclength of the semicircle of radius r, thought of as the graph $y = \sqrt{r^2 - x^2}$ on the interval $[-r, r]$.

Let's notice that since this is the arclength of a semicircle, we should expect it to be half the circumference of the entire circle. This is a circle of radius r, so its circumference is $2\pi r$ and half of that is πr. Now let's verify our integral will give us the same value.

Here our function $f(x)$ that appears in $\int_a^b \sqrt{1+[f'(x)]^2}\,dx$ is $f(x) = \sqrt{r^2 - x^2}$, and the derivative is simply

$$f'(x) = \frac{-x}{\sqrt{r^2 - x^2}}.$$

We then compute the following:

$$\begin{aligned}
\text{Arclength} &= \int_{-r}^{r} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\,dx \\
&= \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx \\
&= \int_{-r}^{r} \sqrt{\frac{r^2 - x^2}{r^2 - x^2} + \frac{x^2}{r^2 - x^2}}\,dx \\
&= \int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\,dx \\
&= \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\,dx
\end{aligned}$$

To compute this integral we'll need to perform a trig substitution. We notice that the $r^2 - x^2$ that appears in the integral looks similar to the left-hand side of the trig identity $1 - \sin^2(\theta) = \cos^2(\theta)$. To account for the r we can multiply both sides of our identity by r^2 to obtain $r^2 - r^2\sin^2(\theta) = r^2\cos^2(\theta)$. Thus we will take $x = r\sin(\theta)$ to take advantage of this identity.

Using the trig substitution $x = r\sin(\theta)$, $dx = r\cos(x)\,dx$, we will rewrite our integral in terms of θ. Since we have a definite integral, we need to change the bounds from x-values to θ-values. Notice that we can solve the expression above for θ to obtain $\theta = \sin^{-1}\left(\frac{x}{r}\right)$. Now when we plug in the bounds $x = -r$ and $x = r$, we obtain new bounds $\theta = \sin^{-1}(-1)$ and $\theta = \sin^{-1}(1)$. A moment's thought reveals these can be written more simply as $\theta = -\pi/2$ and $\theta = \pi/2$.

Our integral then becomes

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \frac{r^2\cos(\theta)}{\sqrt{r^2 - r^2\sin^2(\theta)}}\,d\theta &= \int_{-\pi/2}^{\pi/2} \frac{r^2\cos(\theta)}{\sqrt{r^2\cos^2(\theta)}}\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{r^2\cos(\theta)}{r\cos(\theta)}\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} r\,d\theta \\ &= r\theta\Big|_{-\pi/2}^{\pi/2} \\ &= \frac{\pi}{2}r - \left(\frac{-\pi}{2}r\right) \\ &= \pi r\end{aligned}$$

(Notice above that r is simply a constant and θ is the variable, hence $\int r\,d\theta = r\theta$.)

Remark.
It's worth pointing out here that you likely were told the circumference of a circle of radius r was $2\pi r$, but this was simply given to you as a fact in your middle or high school math classes. If you've ever wondered *why* the circumference was this particular value, versus something else, you can think of the integration above as an explanation for this expression.

Let's now turn our attention to some examples where the arclength of our curve is not given by some simple geometric formula.

Example 4.3.
Compute the arclength of the graph $y = \frac{(x^2+2)^{3/2}}{3}$ where $0 \leq x \leq 1$.

Notice the derivative of our function is

$$\frac{dy}{dx} = \frac{1}{3} \cdot \frac{3}{2} \left(x^2+2\right)^{1/2} \cdot 2x = x\sqrt{x^2+2},$$

and so our arclength is computed as

$$\begin{aligned} \text{Arclength} &= \int_0^1 \sqrt{1 + \left(x\sqrt{x^2+2}\right)^2}\, dx \\ &= \int_0^1 \sqrt{1 + x^2\left(x^2+2\right)}\, dx \\ &= \int_0^1 \sqrt{x^4 + 2x^2 + 1}\, dx \\ &= \int_0^1 \sqrt{\left(x^2+1\right)^2}\, dx \\ &= \int_0^1 \left(x^2+1\right)\, dx \\ &= \left(\frac{x^3}{3} + x\right)\Big|_0^1 \\ &= \frac{4}{3} \end{aligned}$$

In our examples thus far our curve has been given as $y = f(x)$, but there's nothing really special about using x as the independent variable. That is, if our curve was given to us as $x = g(y)$ where $c \leq y \leq d$, we'd compute the arclength in the same way, modifying our integral appropriately to obtain

$$\text{Arclength} = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\, dy.$$

Example 4.4.
Find the arclength of $x = \frac{y^4}{4} + \frac{1}{8y^2}$ where $1 \leq y \leq 2$.

Let's first compute our derivative that appears in the integrand:

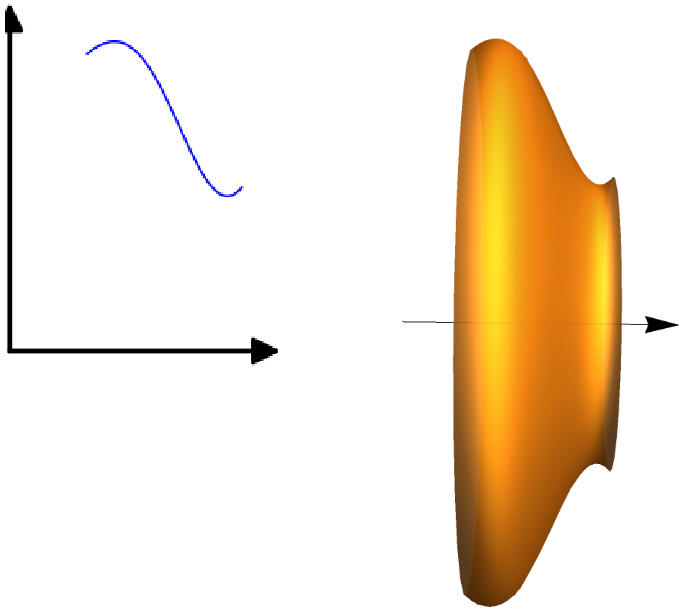
$$\frac{dx}{dy} = \frac{d}{dy}\left(\frac{y^4}{4} + \frac{1}{8y^2}\right) = \frac{4y^3}{4} + \frac{-2}{8y^3} = y^3 - \frac{1}{4y^3}.$$

Our integral is then

$$\begin{aligned}
\text{Arclength} &= \int_1^2 \sqrt{1+\left(y^3-\frac{1}{4y^3}\right)^2}\,dy \\
&= \int_1^2 \sqrt{1+y^6-\frac{1}{2}+\frac{1}{16y^6}}\,dy \\
&= \int_1^2 \sqrt{y^6+\frac{1}{2}+\frac{1}{16y^6}}\,dy \\
&= \int_1^2 \sqrt{\left(y^3+\frac{1}{4y^3}\right)^2}\,dy \\
&= \int_1^2 \left(y^3+\frac{1}{4y^3}\right)dy \\
&= \left(\frac{y^4}{4}-\frac{1}{8y^2}\right)\Bigg|_1^2 \\
&= \left(\frac{16}{4}-\frac{1}{32}\right)-\left(\frac{1}{4}-\frac{1}{8}\right) \\
&= \frac{128-1-8+4}{32} \\
&= \frac{123}{32}
\end{aligned}$$

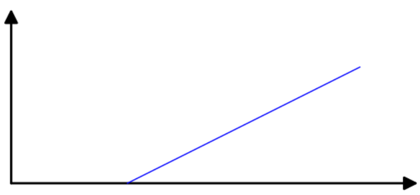
4.2 Surface area

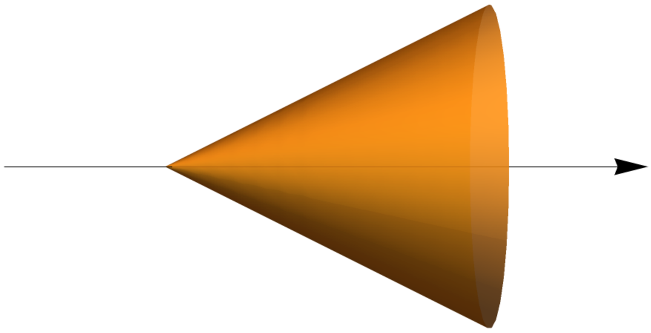
We have previously seen how to compute the volume of a solid of revolution. Today we want to determine the surface area of a surface of revolution. That is, we consider surfaces (two-dimensional objects living in three-dimensional space) constructed by rotating a curve in the plane around an axis. For us this curve will always be the graph of a function.



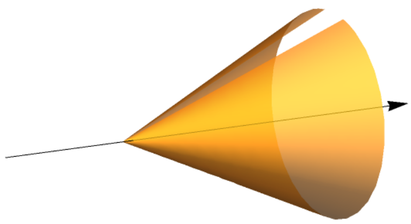
Note we're only rotating a curve, not a two-dimensional region around an axis. The resulting object is infinitesimally thin, and we are concerned with the surface area of this object. E.g., if this object were produced by bending infinitely thin sheets of plastic, how many square feet of plastic would we need?

Let's begin by considering a very simple situation: suppose our curve was some line segment being rotated around the x-axis. If that line segment touches the axis, then our surface is a cone.

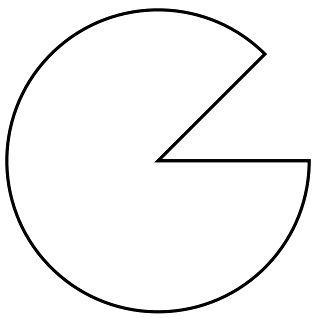




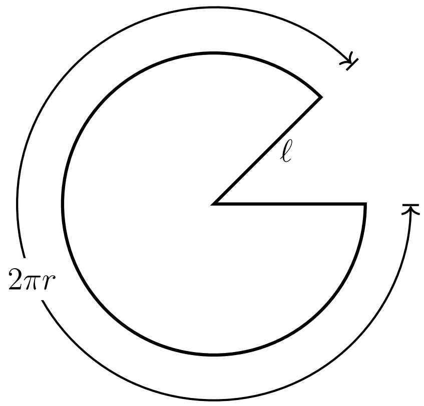
What's the surface area of this cone? To figure this out, suppose the length of our line segment is ℓ and the radius of the (missing) base of our cone is r.



If you were to cut this cone from the tip to the edge along a straight line and flatten it, you would have a sector of a disc.



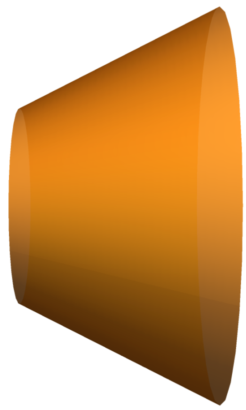
To find the area of our cone, it suffices the find the area of this sector of a disc. Notice the radius of our disc would be ℓ, and so the area of the entire disc (including the piece we're missing) would be $\pi\ell^2$. To find the area of our sector, we need to determine what proportion of our disc the sector takes up. In order to do this, let's suppose the radius of the base of the cone is r and notice that the circumference of the base of our cone is $2\pi r$, and so the circumference (arclength) of our sector is $2\pi r$. But this is some proportion θ of the total circumference of the disc.



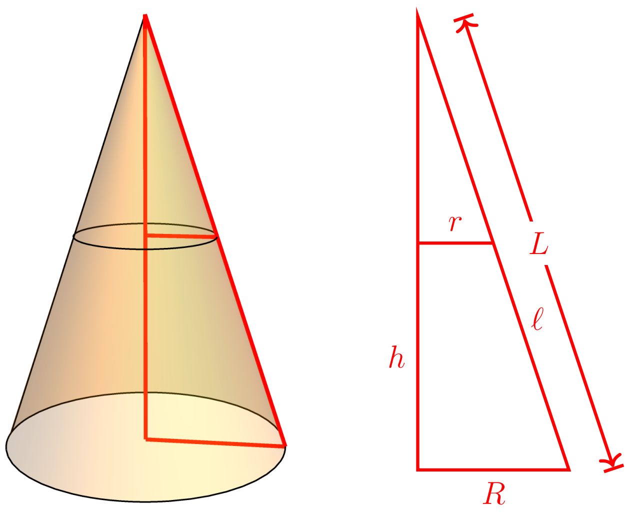
That is, we have $2\pi r = \theta 2\pi\ell$ where θ is some number between 0 and 1 that tells us what proportion of the disc we have. Solving this for θ tells us $\theta = {}^{r}/_{\ell}$. That is, our sector is proportion ${}^{r}/_{\ell}$ of the entire disc. Hence the area of our sector (the lateral surface area of the cone above) is

$$\frac{r}{\ell}\pi\ell^2 = \pi r\ell.$$

This calculation was assuming the line segment we had touched the axis of rotation at one point, but this need not happen. If we rotate a line segment not touching the axis of rotation, then we don't get a complete cone. Instead, we get a portion of a cone called a ***frustum***. Let's again let ℓ be the length of our line segment, and let's let r and R denote the smaller and larger radii of the frustum. Let's let h be the distance between the centers of the circles at the extreme ends of the frustum.



To find the area of the frustum, we'll imagine completing the frustum to get a cone. We just need to find the radii and the side lengths of these cones. We already have the radii, since they're the same as the frustum. To get the side lengths, we need to do just a little bit of trigonometry.



Letting θ be the angle at the top of the triangle above, we see that $\sin(\theta)$ can be computed as the length of the opposite side over the length of the hypotenuse using either the smaller or the larger triangle. Since these are equal (as they both equal $\sin(\theta)$), we have

$$\frac{R}{L} = \frac{r}{L-\ell}$$

We can now solve for L:

$$\begin{aligned} &\frac{R}{L} = \frac{r}{L-\ell} \\ \implies &R(L-\ell) = rL \\ \implies &RL - R\ell = rL \\ \implies &RL - rL = R\ell \\ \implies &L(R-r) = R\ell \\ \implies &L = \frac{R\ell}{R-r}. \end{aligned}$$

That is the big cone has lateral surface area

$$\pi RL = \pi\frac{R^2\ell}{R-r},$$

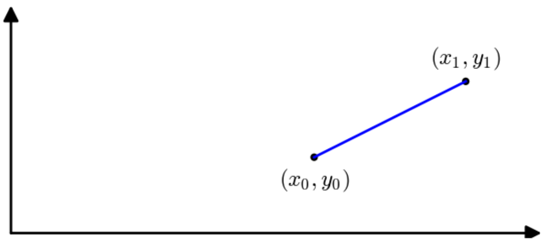
and the small cone has lateral surface area

$$\begin{aligned} \pi r(L-\ell) &= \pi r\left(\frac{R\ell}{R-r} - \ell\right) \\ &= \pi r\left(\frac{R\ell}{R-r} - \ell\frac{R-r}{R-r}\right) \\ &= \pi r\left(\frac{R\ell - R\ell + r\ell}{R-r}\right) \\ &= \frac{\pi r^2\ell}{R-r}. \end{aligned}$$

The frustum thus has lateral surface area

$$\begin{aligned} \frac{\pi R^2\ell}{R-r} - \frac{\pi r^2\ell}{R-r} &= \frac{\pi\ell}{R-r}\left(R^2 - r^2\right) \\ &= \frac{\pi\ell}{R-r}(R-r)(R+r) \\ &= \pi\ell(R+r). \end{aligned}$$

Now, supposing the points at the ends of our line segment are (x_0, y_0) and (x_1, y_1), and we are rotating around the x-axis as in the figure below,



then in our corresponding frustum we have

$$R = y_1, \\ r = y_0, \text{ and} \\ \ell = \sqrt{\Delta x^2 + \Delta y^2},$$

and so the lateral surface area of the frustum is

$$\pi (y_0 + y_1) \sqrt{\Delta x^2 + \Delta y^2}.$$

If the y-values come from a graph $y = f(x)$ and if f is continuously differentiable, then this becomes

$$\pi (f(x_0) + f(x_1)) \sqrt{(x_1 - x_0)^2 + (f(x_1) - f(x_0))^2}$$
$$= \pi (f(x_0) + f(x_1)) \sqrt{1 + \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right]^2} \cdot (x_1 - x_0).$$

By the mean value theorem there exists some x^* between x_0 and x_1 so that $f'(x^*) = \frac{f(x_1)-f(x_0)}{x_1-x_0}$, and so we can write the expression above as

$$\pi (f(x_0) + f(x_1)) \sqrt{1 + f'(x^*)^2}\, \Delta x.$$

Repeating this for each line segment in a broken line that approximates our graph $y = f(x)$, we see the approximate surface area of our surface of rotation is

$$\sum_{i=1}^{n} \pi (f(x_{i-1}) + f(x_i)) \sqrt{1 + f'(x_i^*)^2}\, \Delta x_i.$$

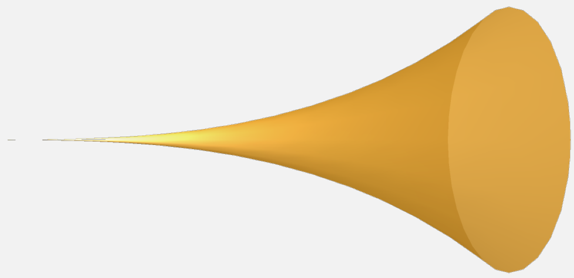
In the limit, both x_i and x_{i-1} move closer and closer together, and so we have $f(x_{i-1}) \approx f(x_i)$ since the function is continuous. Thus we can replace the above sum with

$$\sum_{i=1}^{n} 2\pi f(x_i) \sqrt{1 + f'(x_i^*)^2}\, \Delta x_i.$$

Taking the limit this gives us the integral

$$\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx.$$

Example 4.5.
Find the surface area of the surface obtained by rotating $y = x^3$ around the x axis, with $0 \le x \le 2$.



Here our $f(x)$ is $f(x) = x^3$ and so $f'(x) = 3x^2$. Our integral is thus

$$\text{Area} = \int_0^2 2\pi x^3\sqrt{1+(3x^2)^2}\,dx = 2\pi\int_0^2 x^3\sqrt{1+9x^4}\,dx.$$

Performing the substitution $u = 1+9x^4$, $du = 36x^3\,dx$, the integral becomes

$$\begin{aligned}
&\frac{2\pi}{36}\int_1^{145}\sqrt{u}\,du \\
&= \frac{2\pi}{36}\int_1^{145} u^{1/2}\,du \\
&= \frac{\pi}{18}\cdot\frac{2}{3}u^{3/2}\Big|_1^{145} \\
&= \frac{\pi}{27}\left(145^{3/2}-1\right) \\
&\approx 203.046.
\end{aligned}$$

Example 4.6.
Find the surface area of the surface obtained by rotating $y = \sqrt{1-x}$ around the x-axis with $0 \le x \le 1/2$.

In this example we have $f(x) = \sqrt{1-x}$, and so $f'(x) = \frac{-1}{2\sqrt{1-x}}$. Our integral is thus

$$\begin{aligned}
\text{Area} &= \int_0^{1/2} 2\pi\sqrt{1-x} \cdot \sqrt{1 + \left(\frac{-1}{2\sqrt{1-x}}\right)^2}\, dx \\
&= 2\pi \int_0^{1/2} \sqrt{1-x}\sqrt{1 + \frac{1}{4-4x}}\, dx \\
&= 2\pi \int_0^{1/2} \sqrt{1-x}\sqrt{\frac{4-4x+1}{4-4x}}\, dx \\
&= 2\pi \int_0^{1/2} \sqrt{1-x} \cdot \frac{\sqrt{5-4x}}{2\sqrt{1-x}}\, dx \\
&= \pi \int_0^{1/2} \sqrt{5-4x}\, dx.
\end{aligned}$$

Now we perform the substitution $u = 5-4x$, $du = -4\,dx$ and our integral becomes

$$\begin{aligned}\frac{-\pi}{4}\int_5^3 u^{1/2}\,du &= \frac{\pi}{4}\int_3^5 u^{1/2}\,du\\ &= \left.\frac{2\pi u^{3/2}}{3\cdot 4}\right|_3^5\\ &= \frac{\pi}{6}\left(5^{3/2}-3^{3/2}\right)\\ &\approx 3.133\end{aligned}$$

Example 4.7.
Find the surface area of the unit sphere.

We get the unit sphere by rotating the semicircle of radius 1 around the x-axis. The semicircle is the graph $y = \sqrt{1-x^2}$, and so

$$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}.$$

The integral giving us the surface area is then

$$\begin{aligned}\text{Area} &= \int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx\\ &= 2\pi\int_{-1}^{1}\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx\\ &= 2\pi\int_{-1}^{1}\sqrt{1-x^2}\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx\end{aligned}$$

$$
\begin{aligned}
&= 2\pi \int_{-1}^{1} \sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\,dx \\
&= 2\pi \int_{-1}^{1} \sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx \\
&= 2\pi \int_{-1}^{1} dx \\
&= 2\pi x\Big|_{-1}^{1} \\
&= 2\pi - (-2\pi) \\
&= 4\pi.
\end{aligned}
$$

Example 4.8.
Find both the volume and the surface area of *Gabriel's horn*, which is obtained by rotating $y = \frac{1}{x}$ around the x-axis for $1 \leq x < \infty$.

We will first calculate the volume. This is a straight-forward disc method problem:

$$
\begin{aligned}
\text{Volume} &= \int_{1}^{\infty} \pi\left(\frac{1}{x}\right)^2 dx \qquad\qquad = \pi \lim_{b\to\infty}\int_{1}^{b} x^{-2}\,dx \\
&= \pi \lim_{b\to\infty}\left(-x^{-1}\right)\Big|_{1}^{b}\,dx \\
&= \pi \lim_{b\to\infty}\left(1-\frac{1}{b}\right)dx \\
&= \pi
\end{aligned}
$$

For the surface area we need to compute

$$
\text{Area} = \int_{1}^{\infty} 2\pi\frac{1}{x}\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\,dx = 2\pi\int_{1}^{\infty}\frac{\sqrt{1+\frac{1}{x^4}}}{x}\,dx
$$

Let's now notice that $1+\frac{1}{x^4}$ is greater than 1, and so $\sqrt{1+\frac{1}{x^4}} > \sqrt{1} = 1$. This means

$$\frac{\sqrt{1+\frac{1}{x^4}}}{x} > \frac{1}{x}.$$

We already know, however, that $\int_1^\infty \frac{1}{x}dx$ diverges to infinity, and so the integral giving us our surface area

$$2\pi \int_1^\infty \frac{\sqrt{1+\frac{1}{x^4}}}{x}\, dx$$

must diverge to infinity as well.

Notice this means Gabriel's horn has a finite volume by an infinite surface area. That is, you could fill completely the horn up with (a finite amount of) paint, but that would not be enough to paint the outside of the horn!

Let's end by doing one example where we rotate a curve which is the graph of a function of y around the y-axis. That is, we rotate some $x = g(y)$ around the y-axis for $c \le y \le d$ to obtain our surface.

Modifying the integral for our surface area appropriately, we have

$$\text{Area} = \int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}\, dy.$$

Example 4.9.
Find the surface area of the surface obtained by rotating $y = \sqrt[3]{x}$ around the y-axis for $0 \le y \le 2$.

To apply the formula above we need to write our curve as $x = g(y)$, but this is easily done by solving $y = \sqrt[3]{x}$ for x. This gives us $x = y^3$. Our integral is then

$$\text{Area} = \int_0^2 2\pi y^3\sqrt{1+(3y^2)^2}\, dy$$

If we notice this is exactly the integral we did in Example 4.5, but with y's in place of x's, then we see the surface area is

$$\frac{\pi}{27}\left(145^{3/2}-1\right).$$

5 Sequences and Series

La mathématique est l'art de donner le même nom à des choses différentes.
Mathematics is the art of giving the same name to different things

HENRI POINCARÉ
L'avenir des mathématiques

5.1 Sequences

Before jumping into the next topic that will take up a considerable portion of the remainder of the semester, series, we turn our attention to a stepping stone called "sequences." A ***sequence*** is simply an ordered, infinite list of numbers, such as

$$2, 3, 5, 7, 9, 11, 13, 17, 19, \ldots$$

or

$$\frac{6}{1}, \frac{9}{2}, \frac{12}{3}, \frac{18}{5}, \frac{42}{6}, \frac{25}{7}, \cdots$$

We often the the n-th term of a sequence as a_n. For example, the fourth term of a sequence is denoted a_4. This subscript is sometimes called the ***index*** of the term.

Many times a sequence is specified by an expression which determines the n-th term as a function of n, such as

$$a_n = \frac{n^2+2}{3n^2+n+1}.$$

This expression tells us the general terms of the sequence

$$\frac{3}{5}, \frac{6}{15}, \frac{11}{30}, \frac{18}{59}, \cdots$$

Though sequences often start at the index $n = 1$, they don't have to. For example, we may want the sequence given by the a_n expression above to start at $n = -2$ to obtain the series

$$\frac{6}{13}, \frac{3}{3}, \frac{2}{1}, \frac{3}{5}, \frac{6}{15}, \frac{11}{30}, \cdots$$

Even though a sequence may start at any index, we'll usually have our sequences start at the index $n = 1$. If you aren't explicitly told a sequence starts somewhere else, you should assume it starts at $n = 1$.

Sometimes sequences are specified by a ***recurrence relation*** where the first few terms are given explicitly, but the subsequent terms are specified via expressions involving earlier terms. For example, consider the series where we are told

$$a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}.$$

This means the first two terms of the sequence are explicitly given to us as 1, but afterwards each term is the sum of the previous two terms. This gives us the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, ...$$

As another example, suppose are told

$$a_1 = 1, a_n = 2a_{n-1} + 1.$$

The corresponding sequence is then

$$1, 3, 7, 15, 31, 63, ...$$

As one last example of a sequence defined by a recurrence relation, consider

$$a_0 = 1, \; a_n = n \cdot a_{n-1}$$

The sequence is then

$$1, 1, 2, 6, 4, 120, 720, ...$$

Often sequences arise as successive approximations of quantities we care about. For example, the area between the graph $y = x^2$ and the interval $[0, 1]$ on the x-axis is approximated by sequences of Riemann sums. If a_n were to represent the Riemann sum with n rectangles, for example, we could then compute

$$\begin{aligned} a_n &= \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} \\ &= \frac{1}{n^3} \sum_{i=1}^{n} i^2 \\ &= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= \frac{2n^3 + 3n^2 + n}{6n^3}. \end{aligned}$$

In such a situation we want to know what values these terms in the sequence are approaching. That is, we want to have some notion of a "limit" of a sequence.

Informally, we say that L is the limit of a sequence a_n, denoted $\lim_{n\to\infty} a_n = L$, if the values a_n get "arbitrarily close" to L as n gets large. The precise definition of a limit is the following: we say L is the limit of the sequence a_n if for every $\varepsilon > 0$ there exists an $N > 0$ such that $|a_n - L| < \varepsilon$ for all $n > N$.

What this definition is conveying is that if L is the limit of our sequence, then you should get as close to L as you'd like, provided you go far enough out in the sequence. The ε that appears is saying how close you want to get, and the N is saying how far out you have to go to be guaranteed that your terms remain within ε-distance of L.

Example 5.1.

Show that $\lim_{n\to\infty} \frac{n+2}{3n+1}$ is equal to $\frac{1}{3}$ using the formal definition of the limit.

We want to show that for every $\varepsilon > 0$, we can find an $N > 0$ so that if $n > N$, then we must have

$$\left| \frac{n+2}{3n+1} - \frac{1}{3} \right| < \varepsilon.$$

I claim that taking N to be

$$N = \frac{5 - 3\varepsilon}{9\varepsilon}$$

has this property. To see this, suppose $n > N$. We then have

$$\begin{aligned} n &> \frac{5-3\varepsilon}{9\varepsilon} \\ \implies 9n &> \frac{5-3\varepsilon}{\varepsilon} \\ \implies 9n &> \frac{5}{\varepsilon} - 3 \\ \implies 9n + 3 &> \frac{5}{\varepsilon} \\ \implies \varepsilon &> \frac{5}{9n+3} \end{aligned}$$

What we have thus far is that if n is large enough (namely, if n is greater than $N = \frac{5-3\varepsilon}{9\varepsilon}$), then $\frac{5}{9n+3} < \varepsilon$. Our goal is to show that $\left|\frac{n+2}{3n+1} - \frac{1}{3}\right| < \varepsilon$, so now we are going to try to show $\frac{5}{9n+3}$ is really $\left|\frac{n+2}{3n+1} - \frac{1}{3}\right|$. This requires a little bit of arithmetic where we will try to rewrite our fraction a bit at a time to turn it into the expression we want. Let's notice

$$\begin{aligned} \varepsilon > \frac{5}{9n+3} &= \frac{6-1}{9n+3} \\ &= \frac{3n+6-3n-1}{9n+3} \\ &= \frac{3n+6}{9n+3} - \frac{3n+1}{9n+3} \\ &= \frac{3}{3} \cdot \frac{n+2}{3n+1} - \frac{3n+1}{3n+1} \cdot \frac{1}{3} \\ &= \frac{n+2}{3n+1} - \frac{1}{3}. \end{aligned}$$

Now notice that if $n \geq 1$, then $\frac{n+2}{3n+1} > \frac{1}{3}$ since $3n + 6 > 3n + 1$, and hence we have

$$\frac{n+2}{3n+1} - \frac{1}{3} = \left|\frac{n+2}{3n+1} - \frac{1}{3}\right|.$$

We have thus shown that

$$\left|\frac{n+2}{3n+1} - \frac{1}{3}\right| < \varepsilon$$

provide $n > N = \frac{5-3\varepsilon}{9\varepsilon}$.

The obvious question now is where did the $\frac{5-3\varepsilon}{9\varepsilon}$ in the problem come from? Often these types of problems are solved by "reverse-engineering" the definition. That is, we know that we want to ultimately have

$$\left|\frac{n+2}{3n+1} - \frac{1}{3}\right| < \varepsilon,$$

and so we will try to work backwards from this to determine what conditions this will impose on n. For example,

$$\begin{aligned}
&\left|\frac{n+2}{3n+1} - \frac{1}{3}\right| < \varepsilon \\
\implies &\left|\frac{3(n+2)}{3(3n+1)} - \frac{(3n+1)\cdot 1}{(3n+1)\cdot 3}\right| < \varepsilon \\
\implies &\left|\frac{3n+6-(3n+1)}{9n+3}\right| < \varepsilon \\
\implies &\left|\frac{5}{9n+3}\right| < \varepsilon \\
\implies &\frac{5}{9n+3} < \varepsilon \\
\implies &\frac{5}{\varepsilon} < 9n+3 \\
\implies &\frac{5}{\varepsilon} - 3 < 9n \\
\implies &\frac{5-3\varepsilon}{9\varepsilon} < n.
\end{aligned}$$

When a real number L satisfying the definition above exists, we say the sequence a_n ***converges*** to L. If no such real number L exists, then we say the sequence ***diverges***.

For example, the sequence $a_1 = 1$, $a_n = -a_{n-1}$ – i.e., the sequence 1, -1, 1, -1, 1, -1, ... – diverges. No choice of L will satisfy the conditions described above for all ε. In particular, if $\varepsilon = \frac{1}{2}$, then for every N there are $n > N$ such that $|a_n - L|$ is greater than ε.

We say the sequence a_n ***diverges to infinity*** if for every $M > 0$ there exists an $N > 0$ such that $a_n > M$ for all $n > N$. This simply makes precise the idea that the terms of the sequence get larger and larger without bound.

Example 5.2.
Show that the sequence $a_n = 2^n$,

$$2, 4, 8, 16, 32, 64, \dots$$

diverges to infinity.

We must show that for any $M > 0$ there exists some N such that $a_n > M$ for all $n > N$. That is, we require $2^n > M$. By taking the base-2 logarithm of each side, this becomes $n > \log_2(M)$, and so we claim that $N = \log_2(M)$ will have the property that $a_n > M$ for $n > N$. This easy to double-check: if $n > N$, then $n > \log_2(M)$ and so $a_n = 2^n > 2^{\log_2(M)} = M$. And so our sequence diverges to infinity.

We a sequence diverges to infinity we sometimes write $\lim\limits_{n\to\infty} a_n = \infty$, however this notation is somewhat misleading. In this situation the limit does not exist, but it does not exist for a particular reason: the terms get larger and larger without any upper bound.

The definition of a sequence diverging to negative infinity is similar: we say $\lim_{n\to\infty} a_n$ diverges to negative infinity if for every $M < 0$ there exists an N such that $a_n < M$ for all $n > N$.

In your first semester calculus class you learned various methods for taking the limit of a function of x, and it would be convenient if we could also use those same tools for calculating the limit of a sequence. The following theorem makes this precise.

Theorem 5.1.
If $f(x)$ is a function with the property that $a_n = f(n)$ and if $\lim_{x\to\infty} f(x)$ exists and equals L, then $\lim_{n\to\infty} a_n$ exists and equals L as well.

Example 5.3.

Consider the sequence

$$a_n = \frac{4n^2 + 2n - 1}{8n^2 + 3n + 5}.$$

Notice the terms of this sequence are given by evaluating

$$f(x) = \frac{4x^2 + 2x - 1}{8x^2 + 3x + 5}$$

at integer values of n. Thus if $\lim_{x\to\infty} f(x)$ were to exist, then the limit of our sequence must exist and equal the same value. We can compute this limit, however, by using l'Hôpital's rule:

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{4x^2 + 2x - 1}{8x^2 + 3x + 5} \\ &\overset{\mathscr{L}}{=} \lim_{x\to\infty} \frac{8x + 2}{16x + 3} \\ &\overset{\mathscr{L}}{=} \lim_{x\to\infty} \frac{8}{16} \\ &= \frac{1}{2}\end{aligned}$$

and thus

$$\lim_{n\to\infty} a_n = \frac{1}{2}.$$

Notice, though, that $\lim_{n\to\infty} a_n$ may exist even if $\lim_{x\to\infty} f(x)$ does not!

Example 5.4.
Consider the squence $a_n = \cos(2\pi n)$. Notice $\lim_{x\to\infty} \cos(2\pi x)$ does not exist, but $\lim_{n\to\infty} \cos(2\pi n) = 1$ since every term of the sequence is 1!

Just as we have limit laws for computing limits of functions, we have corresponding laws for computing limits of sequences:

Theorem 5.2.
Suppose that a_n and b_n are two sequences such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. (Neither L nor M is $\pm\infty$.) Then

- $\lim\limits_{n\to\infty} (a_n \pm b_n) = L \pm M$,
- $\lim\limits_{n\to\infty} (a_n \cdot b_n) = LM$,
- $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ *provided* $M \neq 0$, *and*
- $\lim\limits_{n\to\infty} ka_n = kL$ *for all constants* k.

There is also a version of the sandwich theorem (aka squeeze theorem) for sequences.

Theorem 5.3 (The sandwich theorem for sequences)**.**
If a_n, b_n, and c_n are three sequences with the property that $a_n \leq b_n \leq c_n$ for all $n > N$, for some value of N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ as well.

In our next example we will use the sandwich theorem for sequences to compute a certain limit, but first we make one definition. The ***factorial*** of a non-negative integer n, denoted $n!$, is defined to be 1 if $n = 0$, and is otherwise defined as $n! = n \cdot (n-1)!$. For example,

$$\begin{aligned}
0! &= 1\\
1! &= 1 \cdot 0! = 1\\
2! &= 2 \cdot 1! = 2\\
3! &= 3 \cdot 2! = 6\\
4! &= 4 \cdot 3! = 24\\
5! &= 5 \cdot 4! = 120\\
&\vdots
\end{aligned}$$

It's worth noting that if $n > 0$, then we can also write $n!$ as

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

For instance, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

Example 5.5.
Compute the limit $\lim_{n\to\infty} \frac{n^2}{(n+1)!}$.

To apply the sandwich theorem, we want the "middle sequence" to be the sequence in question. That is, we want $b_n = \frac{n^2}{(n+1)!}$. We then need to find sequences a_n and c_n such that $a_n \leq b_n \leq c_n$ for every n, and the a_n and c_n sequences have the same limit.

Let's notice that for every value of $n \geq 2$ we must have

$$\frac{1}{n!} < \frac{n^2}{(n+1)!}.$$

This is not completely obvious, but let's notice that if $n \geq 2$ then we must have $n+1 < n^2$ and so

$$\begin{aligned} & n+1 < n^2 \\ \implies & (n+1)n! < n^2 \cdot n! \\ \implies & (n+1)! < n^2 \cdot n! \\ \implies & \frac{1}{n!} < \frac{n^2}{(n+1)!}. \end{aligned}$$

Thus we will take $a_n = \frac{1}{n!}$.

For our c_n sequence let's just notice that $n^2 < (n+1)^2$ and so $\frac{n^2}{(n+1)!} < \frac{(n+1)^2}{(n+1)!}$. We will take our c_n sequence as $c_n = \frac{(n+1)^2}{(n+1)!}$.

It's easy to see that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n!} = 0$$

as we are have ever larger values in the denominator.

To apply the sandwich theorem we need that our c_n sequence also goes to zero, and right now this may not be so clear. Let's notice we can rewrite the terms of the c_n sequence, however, as

$$c_n = \frac{(n+1)^2}{(n+1)!} = \frac{n+1}{n!} = \frac{n}{n!} + \frac{1}{n!} = \frac{1}{(n-1)!} + \frac{1}{n!}$$

It is easy to see each of these terms goes to zero, however, and so their sum also goes to zero.

Thus we have $\lim_{n\to\infty} \frac{n^2}{(n+1)!} = 0$ since

$$\frac{1}{n!} \leq \frac{n^2}{(n+1)!} \leq \frac{(n+1)^2}{(n+1)!}$$

and

$$\lim_{n\to\infty} \frac{1}{n!} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)!} = 0.$$

We say that a sequence a_n is ***bounded above*** if there exists a number M such that $a_n < M$ for every n. Similarly, we say a_n is ***bounded below*** if there is an m such that $m < a_n$ for every n. If a sequence is simultaneously bounded above and below, then we simply say the sequence is bounded.

Theorem 5.4.
If a sequence is convergent, then it must be bounded.

Corollary 5.5.
If a sequence is unbounded, then it is not convergent.

Example 5.6.

The sequence given by

$$a_n = \frac{(-1)^n \cdot n^2}{n+1}$$

that is, the sequence

$$\frac{1}{2}, \frac{-4}{3}, \frac{9}{4}, \frac{-25}{6}, \ldots$$

is unbounded and so does not converge.

(To see this sequence is unbounded, we need to show that for any M we can find an n such that $\frac{n^2}{n+1} > M$, but this is really just an algebra problem. We could rewrite the above as $n^2 > M(n+1) = Mn + M$, and thus we want $n^2 - Mn + M > 0$. We can find the roots of this equation for any given M using the quadratic formula, and then slightly modify n to satisfy the inequality.)

Notice that just because a sequence is bounded does not imply the sequence converges. For example, the sequence $a_n = (-1)^{n+1}$,

$$1, -1, 1, -1, 1, -1, \ldots$$

is bounded, but does not converge.

There are some special cases, however, where knowing a sequence is bounded together with one other condition will imply convergence.

We say a sequence a_n is ***increasing*** if $a_n \leq a_{n+1}$,

$$a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots ;$$

we say the sequence is ***decreasing*** if $a_n \geq a_{n+1}$,

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots .$$

We say a sequence is ***monotonic*** if it is either increasing or decreasing, whichever it happens to be.

Theorem 5.6.
Every bounded, monotonic sequence converges.

That is, if you can show that a sequence is bounded and is either increasing or decreasing, then the sequence *must* converge. Determining what the sequence converges to can be difficult sometimes, but at least you can say that the sequence will converge.

Theorem 5.7.
Consider the sequence given by

$$a_1 = \sqrt{5}, a_n = \sqrt{5 \cdot a_{n-1}}.$$

That is, the terms of our sequence are

$$\sqrt{5},\ \sqrt{5\sqrt{5}},\ \sqrt{5\sqrt{5\sqrt{5}}},\ \sqrt{5\sqrt{5\sqrt{5\sqrt{5}}}},\ \cdots$$

Let's notice that this sequence is bounded. To see this, first note that $\sqrt{5} < 5$*. Now notice that* $a_n = \sqrt{5 \cdot a_{n-1}} = \sqrt{5} \cdot \sqrt{a_{n-1}}$*, and so if* $a_{n-1} < 5$ *we must have*

$$a_n = \sqrt{5} \cdot \sqrt{a_{n-1}} < \sqrt{5} \cdot \sqrt{5} = 5.$$

That is, each $a_n < 5$ *and so the sequence is bounded.*

This sequence is also increasing: since $5 > a_{n-1}$*, we must have*

$$a_n = \sqrt{5a_{n-1}} > \sqrt{a_{n-1} \cdot a_{n-1}} = \sqrt{a_{n-1}^2} = a_{n-1},$$

and so $a_n > a_{n-1}$*.*

Since the sequence is bounded and monotonic, it must converge.

We will end with one theorem that will be very useful later in the semester:

Theorem 5.8.
Let r *be a fixed constant, and consider the sequence* $a_n = r^n$*. This sequence converges if* $|r| < 1$ *or if* $r = 1$*, but diverges otherwise.*

5.2 Series

"Series" are simply infinite sums and appear throughout mathematics. As we'll see soon, many functions can be represented as series, and this representation greatly simplifies the study of these functions. To begin, though, we just consider summing individual values.

Given any sequence a_n, we would like to consider the sum of all the terms of the sequence,

$$\sum_{n=1}^{\infty} a_n.$$

This infinite sum is called a ***series***. The first and most obvious question that comes to mind is how do we make sense of an infinite sum like this? We simply do what we always do in calculus: approximate something we care about with something simpler. In this case, we approximate the infinite sum with finite sums which we will call the ***partial sums*** of the series.

To be more precise, the ***N-th partial sum*** of the series $\sum_{n=1}^{\infty} a_n$ is the sum of the first N terms of the series, and denoted S_N. In general,

$$S_N = \sum_{n=1}^{N} a_n.$$

The first few partial sums of a series are given below:

$$\begin{aligned}
S_1 &= \sum_{n=1}^{1} a_n = a_1 \\
S_2 &= \sum_{n=1}^{2} a_n = a_1 + a_2 \\
S_3 &= \sum_{n=1}^{3} a_n = a_1 + a_2 + a_3 \\
&\vdots \\
S_N &= \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N.
\end{aligned}$$

For example, consider the series $\sum_{n=1}^{\infty} ({}^{1}\!/_{2})^n$. The first four partial sums of this series are

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\ S_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} \end{aligned}$$

Notice that the partial sums of a series form a sequence. We say that the series $\sum_{n=1}^{\infty} a_n$ ***converges to*** L, and write $\sum_{n=1}^{\infty} a_n = L$, if the sequence of partial sums S_N converges to L. If the sequence of partial sums diverges, then we say the corresponding series ***diverges*** as well.

Remark.
Notice that we have already discussed what it means for a sequence to converge or diverge, so we are translating this idea of a series converging/diverging into this language of sequences that we already understand.

Example 5.7.
Does the series $\sum_{n=1}^{\infty} ({}^{1}\!/_{2})^n$ converge? And if so, what does the series converge to?

This question is tantamount to asking what does the sequence of partial sums associated to this series converge to. To answer that it would be nice if we had a simpler way to write down the terms of that sequence of partial sums.

By definite, the N-th term of the sequence of partial sums for this series is

$$S_N = \sum_{n=1}^{N} \left(\frac{1}{2}\right)^N.$$

It may not be very clear at this moment what will happen with this sequence as N goes to infinity, and so we would like to rewrite this expression for S_N as something that will be easier to work with.

In order to get an easier to work with expression, let's make the observation that $\frac{1}{2}S_N$ is equal to

$$\frac{1}{2} \cdot \sum_{n=1}^{N} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{N} \left(\frac{1}{2}\right)^{n+1}$$

Now also notice that we could write $\frac{1}{2}S_N$ as $S_N - \frac{1}{2}S_N$. Combining these two expressions together we have the following:

$$\begin{aligned}\frac{1}{2}S_N &= S_N - \frac{1}{2}S_N \\ &= \sum_{n=1}^{N} \left(\frac{1}{2}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n+1} \\ &= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{N-1}} + \frac{1}{2^N}\right) - \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^N} + \frac{1}{2^{N+1}}\right)\end{aligned}$$

Now notice that after we distribute the negative, everything in this last expression will appear twice: once as a positive value and once as a negative value, except the very first and last terms. That is, every term but the very first and very last in the above expression will cancel, leaving us with

$$\frac{1}{2}S_N = \frac{1}{2} - \frac{1}{2^{N+1}}.$$

Multiplying through by 2, we then have

$$S_N = 1 - \frac{1}{2^N}.$$

Notice this agrees with our earlier computation of the first four values of S_N:

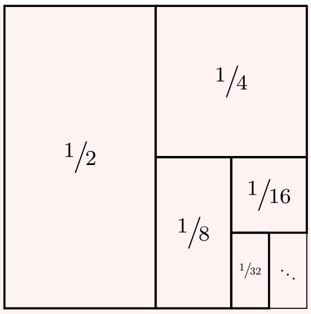
$$S_1 = 1 - \frac{1}{2} = \frac{1}{2}$$
$$S_2 = 1 - \frac{1}{4} = \frac{3}{4}$$
$$S_3 = 1 - \frac{1}{8} = \frac{7}{8}$$
$$S_4 = 1 - \frac{1}{16} = \frac{15}{16}$$

We really do have the same values we had before, but they're written in a way that makes it easier for us to calculate the limit. In particular we see

$$\lim_{N\to\infty} S_N = \lim_{n\to\infty} \left(1 - \frac{1}{2^N}\right) = 1$$

and thus our series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is equal to 1.

Remark.
There is another way to think about the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ which is a little more geometrical. Imagine taking a 1×1 square, whose area is obvious 1, and dividing it in half into two pieces. Now divide one of the remaining pieces in two, and then one of those remaining smaller pieces in two. In this way you obtain a collection of rectangles of areas $1/2$, $1/4$, $1/8$, and so on. This infinite collection of rectangles fits back together to give you the entire square which had area 1, so the infinite sum of these areas must equal 1 as well.



Of course, if the limit of partial sums does not exist, then we say that the serives ***diverges***. If the sequence of partial sums goes to infinity, then we say the series ***diverges to infinity***.

In general, for a series to converge we *must* have all of the terms of the series go to zero. Intuitively, if the terms converged to something else (say, 1) then our series would basically be adding up that value infinitely often (e.g., $1 + 1 + 1 + \cdots$) and we would expect this to diverge.

Theorem 5.9.
If $\sum a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$. *Conversely, if* $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum a_n$ *diverges.*

Remark.
Notice in the statement of the theorem above we simply write our series as $\sum a_n$ instead of the more correct $\sum_{n=1}^{\infty} a_n$. While many of our sums will start at $n = 1$, they don't strictly have to: all of our notions of partial sums, convergence, divergence, etc., would carry over just fine if our series started at $n = 3$ or $n = -17$ or some other finite value. In some of our statements of theorems we will thus simply write $\sum a_n$ to emphasize that the important thing is that we have an infinite sum, and where the series begins doesn't really matter.

In other theorems, however, we will explicitly need to know the series starts at $n = 1$ or $n = 0$ for some of our calculations, and when that happens we will explicitly say where the series starts in our notation.

Example 5.8.
The series $\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$ must diverge as $\lim_{n\to\infty} \frac{n-1}{2n+1} = \frac{1}{2} \neq 0$.

Even though we require the individual terms of a series to go to zero in order to have convergence, this is not enough to guarantee convergence. (Sometimes mathematicians will express this by saying the condition is necessary but not sufficient. It is necessary that the terms go to zero to have convergence, but it is not sufficient.) The following example shows one particular case where the terms of the series go to zero, yet the series does not converge.

Example 5.9.
Consider the ***harmonic series*** which is the following:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

The terms of this series certainly go to zero, yet the series diverges to infinity because of the following observation.

Notice that the first and second terms of the series are greater than or equal to $1/2$. The next two terms are each greater than or equal to $1/4$. The next four terms are greater than or equal to $1/8$. The next eight terms are greater than or equal to $1/16$, and so on:

$$\underbrace{1}_{\geq 1/2} + \underbrace{\frac{1}{2}}_{\geq 1/2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 1/4 + 1/4 = 1/2} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 1/8 + 1/8 + 1/8 + 1/8 = 1/2} + \cdots$$

Noticing these groups of terms add up to something that is at least $1/2$, we see that our series must be larger than the series that simply adds up $1/2$ infinitely many times, and thus diverges to infinity.

It can be tricky in general to determine if a series converges or diverges, but there are a few special cases where it is easy. One important example are the geometric series.

A series of the form

$$\sum_{n=0}^{\infty} kr^n$$

k and r are constants is called a ***geometric series***. Geometric series are nice because we can find a nice, simple, closed form expression for their partial sums. Mimicking what we did in Example 5.7, we can find an expression for the N-th partial sum as follows:

$$\begin{aligned}
S_N - rS_N &= \sum_{n=0}^{N-1} kr^n - r\sum_{n=0}^{N-1} kr^n \\
&= \sum_{n=0}^{N-1} kr^n - \sum_{n=0}^{N-1} kr^{n+1} \\
&= \left(k + kr + kr^2 + \cdots + kr^{N-2} + kr^{N-1}\right) - \left(kr + kr^2 + kr^3 + \cdots + kr^{N-1} + kr^N\right) \\
&= k - kr^N = k(1 - r^N)
\end{aligned}$$

Now we can rewrite the left-hand side of the first expression as $S_N(1-r)$, then divide the $1 - r$ over to obtain

$$S_N = \frac{k(1 - r^N)}{1 - r}$$

Now, we are interested in taking the limit of these partial sums as N goes to infinity. Since only one portion of our expression relies on N, though, we easily see

$$\sum_{n=0}^{\infty} kr^n = \lim_{N\to\infty} S_N = \frac{k\left(1 - \lim_{N\to\infty} r^N\right)}{1 - r}$$

Notice that r^N converges to zero if $|r| < 1$ and diverges if $|r| > 1$. Thus

$$\sum_{n=0}^{\infty} kr^n = \frac{k}{1-r} \text{ if } |r| < 1$$
$$\sum_{n=0}^{\infty} kr^n \text{ diverges to infinity if } |r| > 1$$

If $r = 1$, then the above derivation doesn't apply because we would divide by zero when we divide by $1-r$ above. Notice in this, however, our series becomes simply $k+k+k+\cdots$ which diverges for any $k \neq 0$. When $r = -1$, the series becomes $k-k+k-k+k-k+\cdots$ which again diverges (the terms do not go to zero, for example) when $k \neq 0$. Combining these together we have proven the following theorem.

Theorem 5.10.
The geometric series $\sum_{n=0}^{\infty} kr^n$ *with* $k \neq 0$ *converges if and only if* $|r| < 1$, *and in particular converges to* $k/(1-r)$.

Example 5.10.
The geometric series $\sum_{n=0}^{\infty} 8 \cdot \left(\frac{1}{3}\right)^n$ has $k = 8$ and $r = 1/3$, thus converges to

$$\frac{8}{1-\frac{1}{3}} = \frac{8}{\left(\frac{2}{3}\right)} = \frac{3}{2} \cdot 8 = \frac{24}{2} = 12.$$

Example 5.11.
The geometric series $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ has $k = 1$ and $r = 1/2$, and so con-

verges to
$$\frac{1}{1-\frac{1}{2}} = 2.$$

We can often manipulate series so that our sums start at another index. This is helpful, in particular, for geometric series because our result in Theorem 5.10 requires the series to start at $n = 0$.

Often these manipulations work by adding extra terms or subtracting unwanted terms. For example, consider the series
$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n.$$
We can not immediately apply our result from Theorem 5.10 here since the series does not start at $n = 0$. In particular, this series is lacking a term compared to the corresponding series starting at $n = 0$. Thus we can rewrite this in terms of the serires which begins at $n = 0$ by subtracting off the unnecessary term:
$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^0$$
Now we can apply the formula for geometric series above, and simply subtract off $\left({}^1\!/_2\right)^0 = 1$, giving us
$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^0 \\ &= \frac{1}{1-\frac{1}{2}} - 1 \\ &= 2 - 1 \\ &= 1.\end{aligned}$$

In general, when using the formula from Theorem 5.10 above, we *must* have the series start at $n = 0$. If you want to compute the value of a geometric series that starts somewhere other than $n = 0$, then you need to manipulate the series so that it starts at $n = 0$, apply the formula, and then remove any unnecessary terms or add back any required terms.

In the example below we consider a geometric series which starts at $n = -2$, and our strategy will be to write this as the series starting at $n = 0$, plus the $n = -1$ and the $n = -2$ terms.

Example 5.12.

Compute $\displaystyle\sum_{n=-2}^{\infty} \frac{3}{5^n}$.

Let's begin by writing out the first several terms of this series:

$$\frac{3}{5^{-2}} + \frac{3}{5^{-1}} + \frac{3}{5^0} + \frac{3}{5^1} + \frac{3}{5^2} + \cdots$$

Notice that we can think of this as the series which begins at $n = 0$, but has two additional terms tacked on:

$$\frac{3}{5^{-2}} + \frac{3}{5^{-1}} + \underbrace{\frac{3}{5^0} + \frac{3}{5^1} + \frac{3}{5^2} + \cdots}_{=\sum_{n=0}^{\infty} \frac{3}{5^n}}$$

Thus we can use Theorem 5.10 on that portion fo our series, and we simply need to add on the two extra terms:

$$\begin{aligned}
\sum_{n=-2}^{\infty} \frac{3}{5^n} &= \frac{3}{5^{-2}} + \frac{3}{5^{-1}} + \sum_{n=0}^{\infty} \frac{3}{5^n} \\
&= 3 \cdot 25 + 3 \cdot 5 + \frac{3}{1 - {}^1\!/_5} \\
&= 75 + 15 + \frac{3}{{}^4\!/_5} \\
&= 90 + 3 \cdot \frac{5}{4} \\
&= 90 + \frac{15}{4} \\
&= \frac{360 + 15}{4} \\
&= \frac{375}{4}
\end{aligned}$$

It can sometimes be helpful to think of one series as a sum of two other series, as we will see in a moment. First, though, let's notice that in general we can manipulate series in some of the same ways that we manipulate integrals: breaking up sums/differences, and pulling out constant factors.

Theorem 5.11.
If $\sum_{n=k}^{\infty} a_n$ *and* $\sum_{n=k}^{\infty} b_n$ *both converge, say*

$$\sum_{n=k}^{\infty} a_n = L \quad \text{and} \quad \sum_{n=k}^{\infty} b_n = M$$

then the series $\sum_{n=k}^{\infty} (a_n \pm b_n)$ *converges to* $L \pm M$*:*

$$\sum_{n=k}^{\infty} (a_n \pm b_n) = \sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n = L \pm M.$$

If c *is any constant, then the series* $\sum_{n=k}^{\infty} ca_n$ *converges to* cL*:*

$$\sum_{n=k}^{\infty} ca_n = c\sum_{n=k}^{\infty} a_n = cL.$$

These two properties are sometimes called the *linearity properties* of series.[1]

We can often use the linearity properties of Theorem 5.11 to help us take more complicated series and break them up into simpler pieces, as in the following example.

Example 5.13.
Compute $\sum_{n=0}^{\infty} \frac{2 \cdot 5^n + 4 \cdot 3^n}{15^n}$.

Let's first notice that this *is not* a geometric series. Taking advatange of Theorem 5.11, though, we can write our series as the

[1] "Linearity" may sound like a strange term, since it's not clear what these properties have to do with lines, but the language comes from a branch of mathematics called linear algebra where functions having these two properties, splitting up sums and pulling out constants, are particularly important.

sum of two geometric series. To see this, let's perform a little bit of arithmetic on the terms of our series:

$$\begin{aligned}\frac{2\cdot 5^n + 4\cdot 3^n}{15} &= \frac{2\cdot 5^n}{15^n} + \frac{4\cdot 3^n}{15^n}\\ &= 2\cdot\left(\frac{5}{15}\right)^n + 4\cdot\left(\frac{3}{15}\right)^n\\ &= \frac{2}{3^n} + \frac{4}{5^n}\end{aligned}$$

Now we can compute our original series as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{2\cdot 5^n + 4\cdot 3^n}{15^n} &= \sum_{n=0}^{\infty}\left(\frac{2}{3^n} + \frac{4}{5^n}\right)\\ &= \sum_{n=0}^{\infty}\frac{2}{3^n} + \sum_{n=0}^{\infty}\frac{4}{5^n}\\ &= \frac{2}{1 - {}^1\!/_3} + \frac{4}{1 - {}^1\!/_5}\\ &= \frac{2}{{}^2\!/_3} + \frac{4}{{}^4\!/_5}\\ &= 2\cdot\frac{3}{2} + \frac{4\ 5}{\cdot\ 4}\\ &= 3 + 5\\ &= 8\end{aligned}$$

5.3 Series with only positive terms

Determining if a given series diverges or not is often difficult, but if we're in the special case where all of the terms of the series are positive, then we have some tools called "divergence tests" which can at least tell us whether a series converges or not, even if the test can't tell us what the series converges to. Later we'll see there are also some convergence tests for series where the terms aren't necessarily positive.

Let's first notice that if every term a_n that appears in a series $\sum a_n$ is positive, then the sequence of partial sums *must* be increasing since we're always adding positive values onto the partial sums. As we'd seen earlier, if S_N is the N-th partial sum of $\sum a_n$, then we have $S_N = S_{N-1} +$

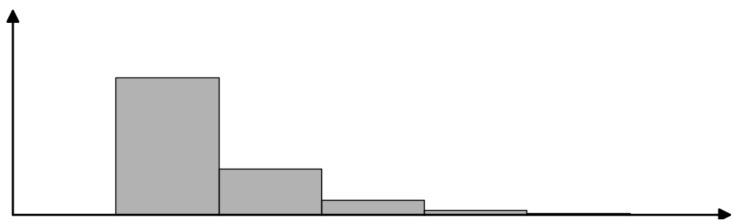
a_N. If $a_N > 0$, though, then we have

$$\begin{aligned} & a_N > 0 \\ \Longrightarrow & S_{N-1} + a_N > S_{N-1} \\ \Longrightarrow & S_N > S_{N-1}. \end{aligned}$$

Since the sequence of partial sums is increasing, it must converge if it's bounded and diverge to infinity if it is not bounded. That is, if all the a_n terms of the series $\sum a_n$ are positive, determining whether the series converges or not is tantamount to determiining if the sequence of partial sums is bounded.

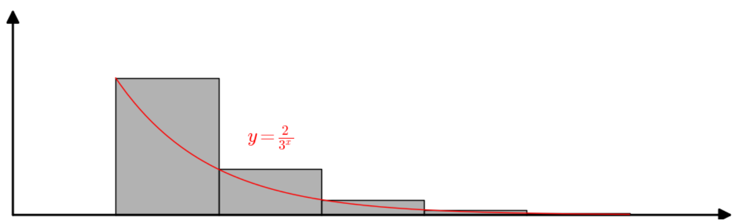
To understand the first convergence test we'll introduce, let's draw the following picture: For the series $\sum_{n=1}^{\infty} a_n$ where each $a_n > 0$, let's draw a rectangle in the plane for each term where the base of the rectangle is the interval $[n, n+1]$ on the x-axis, and the height of the rectangle is a_n.

For example, for the series $\sum_{n=1}^{\infty} \frac{2}{3^n}$, we consider the following:



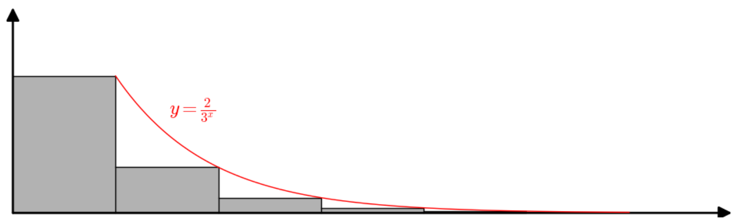
The value of $\sum_{n=1}^{\infty} \frac{2}{3^n}$ is thus the same as the sum of the areas of these rectangles. To determine if the series converges or diverges, we need to see if that area is finite or infinite. We know that areas under curves are calculated by integrals, so let's try to relate the series we care about to an integral. The reason we want to do this is that we know how to calculate lots of integrals, so it'd be nice if we could use some of those tools to help us understand series.

Consider the function $f(x) = \frac{2}{3^x}$. The area under the graph $y = f(x)$ over $[1, \infty)$ on the x-axis fits into our above picture as follows:



Notice that the area is smaller than the area we care about, so if that area was infinite, the area we care about would be infinite too: If $\int_1^\infty \frac{2}{3^x}\,dx$ diverges to infinity, then $\sum_{n=1}^\infty \frac{2}{3^n}$ diverges to infinity as well.

If the integral $\int_1^\infty \frac{2}{3^x}\,dx$ was finite, however, that doesn't necessarily mean $\sum_{n=1}^\infty \frac{2}{3^n}$ is finite since the areas of the rectangles are larger than the area under the curve. What we need to do to fix this is consider a curve that sits above our rectangles. Here there's a little trick: if we slide all of our rectangles one unit to the left, then our earlier curve $y = \frac{2}{3^x}$ over $[1, \infty)$ sits over the rectangles corresponding to a_2, a_3, a_4, ...



Since the area of the first rectangle is some finite value, whatever it happens to be, adding or subtracting it doesn't affect whether the series converges or diverges. But what we have now, just from our picture above, is that $\sum_{n=2}^\infty \frac{2}{3^n} < \int_1^\infty \frac{2}{3^x}\,dx$. That is, if $\int_1^\infty \frac{2}{3^x}\,dx$ is finite, then $\sum_{n=2}^\infty \frac{2}{3^n}$ is finite as well, but that implies our original sum $\sum_{n=1}^\infty \frac{2}{3^n}$ must also be finite.

Combining this with our earlier observation that if $\int_1^\infty \frac{2}{3^x}\,dx$ was infinite, then $\sum_{n=1}^\infty \frac{2}{3^n}$ would be infinite as well, we see that $\sum_{n=1}^\infty \frac{2}{3^n}$ converges if and only if $\int_1^\infty \frac{2}{3^x}\,dx$ converges.

Of course, we'd like to generalize this to other series, and this is what the next theorem, called ***the integral test***, allows us to do.

Theorem 5.12 (The integral test)**.**
If $f(x)$ is a positive, decreasing, continuous function defined on $[1, \infty)$, then the series $\sum_{n=1}^\infty a_n$ where $a_n = f(n)$ converges (or diverges) if and only if $\int_1^\infty f(x)\,dx$ converges (or diverges).

Example 5.14.

Does the series $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+n+1}$ converge?

Let's notice that the terms of this series are given by $a_n = f(n)$ where $f(x) = \frac{2x+1}{x^2+x+1}$. This function is defined and continuous everywhere (the denominator $x^2 + x + 1$ has no real roots), and to see that it is decreasing we need to consider its derivative. We simply calculate

$$f'(x) = \frac{(x^2+x+1)\cdot 2 - (2x+1)(2x+1)}{(x^2+x+1)^2}$$

$$= \frac{2x^2+2x+2-4x^2-4x-1}{(x^2+x+1)^2}$$

$$= \frac{-2x^2-2x+1}{(x^2+x+1)^2}.$$

Notice $f'(x) < 0$ for all $x \geq 1$, so $f(x)$ is decreasing. Thus, by the integral test, our original series converges or diverges if and only if the integral $\int_1^{\infty} f(x)\, dx$ does, and so we simply need to compute $\int_1^{\infty} \frac{2x+1}{x^2+x+1}\, dx$.

To do this, let's perform the substitution $u = x^2 + x + 1$, $du = (2x+1)dx$ so the integral becomes

$$\begin{aligned}
\int_1^{\infty} \frac{2x+1}{x^2+x+1}\, dx &= \int_3^{\infty} \frac{du}{u} \\
&= \lim_{b\to\infty} \int_3^{b} \frac{du}{u} \\
&= \lim_{b\to\infty} \ln|u| \Big|_3^b \\
&= \lim_{b\to\infty} \ln(b) - \ln(3) \\
&= \infty
\end{aligned}$$

Since the integral diverges, our series $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+n+1}$ must diverge as well.

Example 5.15.
Does the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converge or diverge?

Consider the function $f(x) = \frac{1}{x^3} = x^{-3}$. This is positive, continuous, and decreasing on $[1, \infty)$. Thus $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (or diverges) if and only if $\int_1^\infty \frac{dx}{x^3}$ converges (or diverges). Thus we only need to compute the following improper integral:

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^3} &= \lim_{b\to\infty} \int_1^b x^{-3}\,dx \\
&= \lim_{b\to\infty} \left.\frac{-x^{-2}}{2}\right|_1^b \\
&= \lim_{b\to\infty} \left(\frac{-1}{2b} - \frac{-1}{2}\right) \\
&= \lim_{b\to\infty} \left(\frac{1}{2} - \frac{1}{2b}\right) \\
&= \frac{1}{2}.
\end{aligned}$$

Thus $\int_1^\infty \frac{dx}{x^3}$ converges and so by the integral test our series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges as well.

In general, series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

where p is a positive constant are called ***p*-series** and we can use the integral test to prove the following:

Theorem 5.13 (p-series test)**.**
The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Example 5.16.

- The series $\sum_{n=1}^{\infty} \sqrt{n^{-3}}$ converges as it's a p series ($\sqrt{n^{-3}} = \frac{1}{n^{3/2}}$) with $p > 1$.
- The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges as it's a p-series with $p \leq 1$. (Here $p = 1/2$.)

The idea behind the integral test was that we compared the series we cared about to an integral we could compute. We could also compare a series to another, easier to work with, series.

Consider, for example, the series

$$\sum_{n=1}^{\infty} \frac{9n}{(3n+1)^3}.$$

Let's notice that for each n we must have

$$\frac{9n}{(3n+1)^3} < \frac{9n}{(3n)^3}$$

as the denominator on the left is greater than the denominator on the right. That means that if $\sum_{n=1}^{\infty} \frac{9n}{(3n)^3}$ converges, then we should expect our original series to converge as well, since each term of our original series is smaller than the corresponding term of this new series. Notice, however, that we may write

$$\frac{9n}{(3n)^3} = \frac{9n}{27n^3} = \frac{1}{3n^2} = \frac{1}{3} \cdot \frac{1}{n^2}.$$

Thus

$$\sum_{n=1}^{\infty} \frac{9n}{(3n)^3} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and this series converges by the p-series test. Thus our original series must also converge.

As another example, consider

$$\sum_{n=1}^{\infty} \frac{2}{\frac{n}{3} - 1}.$$

Notice that

$$\frac{n}{3} - 1 < \frac{n}{3} < n,$$

and so

$$\frac{1}{\frac{n}{3} - 1} > \frac{1}{n} \text{ if } n > 2$$

and hence

$$\frac{2}{\frac{n}{3} - 1} > \frac{1}{n} \text{ if } n > 2$$

The right-hand side of this inequality, though, corresponds to the terms of the harmonic series, and we know the harmonic series diverges. Thus the terms of our series are larger than the terms of another series we know diverges to infinity – at least once we get past the first two terms. Ignoring the first few terms, which is just going to give us some finite number, we should expect our series to diverge. Adding back these few finitely-many terms we are missing does not change the fact the series still diverges to infinity.

Remark.
In general, the convergence or divergence of a series is unchanged if we add or remove finitely-many terms to the series.

The reasoning behind the examples above essentially proves the following, called the ***direct comparison test***.

Theorem 5.14 (Direct comparison test)**.**
Suppose $\sum a_n$ and $\sum b_n$ are two series such that there exists some number $N > 0$ such that $0 < a_n \leq b_n$ for all $n > N$.

- *If $\sum a_n$ diverges, then $\sum b_n$ diverges as well.*
- *If $\sum b_n$ converges, then $\sum a_n$ converges as well.*

Notice that when we're concerned with the convergence or diverges we can always ignore any finite number of terms at the start of the series.

The only thing that matters is what the series does "in the long run." In fact, when comparing two series we can compare the limits of the terms.

Theorem 5.15 (Limit comparison test)**.**
Suppose $\sum a_n$ *and* $\sum b_n$ *are two series of positive terms and the limit of the ratios* $\frac{a_n}{b_n}$ *exists and equals* L*:*

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

- *If* $L > 0$*, then* $\sum a_n$ *converges (or diverges) if and only if* $\sum b_n$ *converges (or diverges).*
- *If* $L = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*
- *If* $L = \infty$ *and* $\sum a_n$ *converges, then* $\sum b_n$ *diverges too.*

Intuitively, what the limit comparison test is doing is comparing the sizes of terms of the sequences. If the ratio $\frac{a_n}{b_n}$ is a finite, positive number then we should think the terms are "basically" growing at the same rate, so if one series converges or diverges, the other series should do the same thing. If the ratio $\frac{a_n}{b_n}$ goes to zero, though, that should mean that the b_n terms are growing much faster than the a_n terms, and if b_n series converges, the a_n series with smaller terms should too.

Example 5.17.
Does the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^6+1}}$ converge?

To use the limit comparison test, we need something to compare our series to. Since we'll be taking the limit as n goes to infinity, let's notice that for very, very large values of n the "$+1$" that appears in the square root of our series matters very little; for large n's we should think of our terms as being approximately $\frac{1}{n^2}$:

$$\frac{n}{\sqrt{n^6+1}} \approx \frac{n}{\sqrt{n^6}} = \frac{n}{n^3} = \frac{1}{n^2}.$$

Thus, let's compare our series to $\sum \frac{1}{n^2}$.

Letting $a_n = \frac{n}{\sqrt{n^6+1}}$ and $b_n = \frac{1}{n^2}$ for the limit comparison test, we're interested in the limit:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\left(\frac{n}{\sqrt{n^6+1}}\right)}{\left(\frac{1}{n^2}\right)} &= \lim_{n\to\infty} \frac{n^3}{\sqrt{n^6+1}} \\
&= \lim_{n\to\infty} \frac{n^3}{\sqrt{n^6+1}} \cdot \frac{1/n^3}{1/n^3} \\
&= \lim_{n\to\infty} \frac{n^3}{\sqrt{n^6+1}} \cdot \frac{1/n^3}{\sqrt{1/n^6}} \\
&= \lim_{n\to\infty} \frac{n^3/n^3}{\sqrt{n^6/n^6 + 1/n^6}} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{1 + 1/n^6}} \\
&= 1
\end{aligned}$$

Thus, by the limit comparison test, both of our series

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^6+1}} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converge or they both diverge. Since $\sum \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, however, this series converges, and so the limit comparison test tells us our original series converges as well.

5.4 Absolute convergence, conditional convergence, and alternating series

We saw in the last section that there are some convergence tests for determining if a series converges or diverges, but the test we learned thus far have required that the terms of the series to all be positive, which is a pretty significant restriction. However, there is one "cheap" trick for associating a series of all positive terms to an arbitrary series: just take the absolute value of each term.

For example, consider the series

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)}{3^n} &= \frac{\sin\left(\frac{3\pi}{4}\right)}{3}+\frac{\sin\left(\frac{5\pi}{4}\right)}{9}+\frac{\sin\left(\frac{7\pi}{4}\right)}{27}+\frac{\sin\left(\frac{9\pi}{4}\right)}{81}+\frac{\sin\left(\frac{11\pi}{4}\right)}{243}+\cdots\\ &= \frac{\sqrt{2}/2}{3}+\frac{-\sqrt{2}/2}{9}+\frac{-\sqrt{2}/2}{27}+\frac{\sqrt{2}/2}{81}+\frac{\sqrt{2}/2}{243}+\cdots\\ &= \frac{\sqrt{2}}{2}\left(\frac{1}{3}-\frac{1}{9}-\frac{1}{27}+\frac{1}{81}+\frac{1}{243}-\frac{1}{729}-\cdots\right)\end{aligned}$$

This is "basically" the geometric series with $k = \sqrt{2}/2$ and $r = 1/3$, but the signs flip every other term: after the first term we have two negative terms, then two positive terms, then two negative terms, then two positive terms, and so on.

Notice that if we were to take the absolute value of each term we would have

$$\sum_{n=1}^{\infty}\left|\frac{\sin\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)}{3^n}\right| = \frac{\sqrt{2}/2}{3}+\frac{\sqrt{2}/2}{9}+\frac{\sqrt{2}/2}{27}+\frac{\sqrt{2}/2}{81}+\frac{\sqrt{2}/2}{243}+\cdots$$

and this new series of positive terms is very easy for us to compute:

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{\sin\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)}{3^n}\right| &= \sum_{n=1}^{\infty}\frac{\sqrt{2}/2}{3^n}\\ &= \sum_{n=0}^{\infty}\frac{\sqrt{2}/2}{3^n}-\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}/2}{1-1/3}-\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}/2}{2/3}-\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}2}{\cdot}\frac{3}{2}-\frac{\sqrt{2}}{2}\\ &= \frac{3\sqrt{2}-2\sqrt{2}}{4}\\ &= \frac{\sqrt{2}}{4}\end{aligned}$$

It would be nice if we could relate this series of all positive terms to the original series we started with. In particular, it would be nice if convergence of $\sum |a_n|$ told us something about the convergence of $\sum a_n$.

Before describing this relationship, we make a definition. We will say that the series $\sum a_n$ is ***absolutely convergent*** if the corresponding series of absolute values of terms, $\sum |a_n|$ converges. The reason we are interested in absolute convergence is the following theorem:

Theorem 5.16.
If a series is absolutely convergent, then it is convergent. I.e., if $\sum |a_n|$ converges, then $\sum a_n$ must converge as well.

Thus, by Theorem 5.16, our earlier series $\sum {}^{(\sin(n\pi/2+\pi/4))}/_{3^n}$ must converge since it is absolutely convergent: the series $\sum \left|{}^{(\sin(n\pi/2+\pi/4))}/_{3^n}\right|$ converges.

Example 5.18.
Does the series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converge?

If this series is absolutely convergent, then it is convergence. Notice, though, that since $|\sin(n)| \leq 1$ we have

$$\left|\frac{\sin(n)}{n^2}\right| \leq \frac{1}{n^2}$$

and by the direct comparison test, $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right|$ converges as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Thus the series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ is absolutely convergent, and hence convergent.

While convergence of $\sum |a_n|$ implies convergence of $\sum a_n$, in general the opposite is false: $\sum a_n$ may converge even if $\sum |a_n|$ diverges, as the next example shows.

Example 5.19.

The ***alternating harmonic series*** is the series

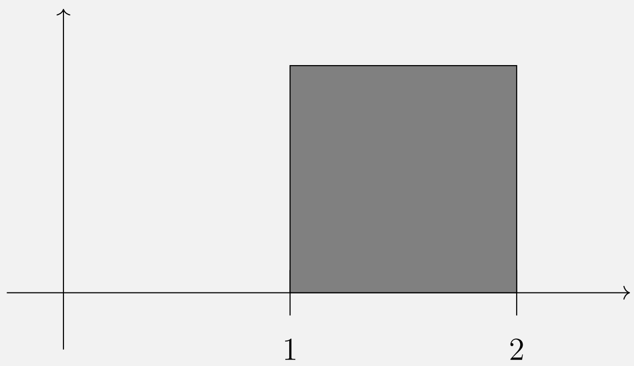
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

The sum of absolute values of these terms gives the harmonic series which we had seen diverges,

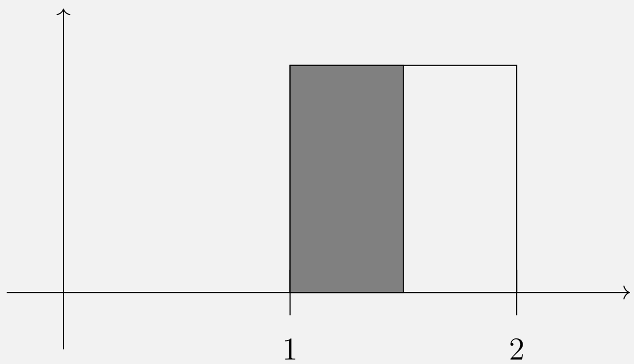
$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$$

We claim, however, that the alternating harmonic series converges, and will briefly sketch why this is the case by showing geometrically that the partial sums of terms in the alternating harmonic series can be thought of as a very special type of Riemann sum.

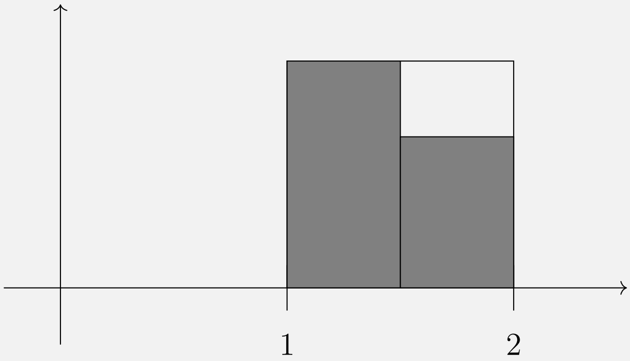
We begin by imagining a 1×1 square in the plane whose base is the interval $[1, 2]$ on the x-axis.



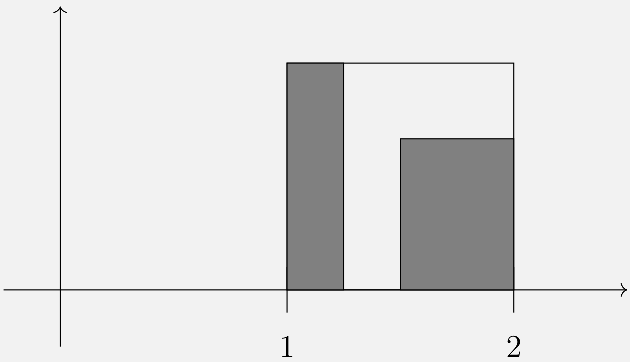
The area of the square is of course 1. Now remove the right-hand half of the square to get a rectangle of area $1 - {}^{1}\!/_{2}$:



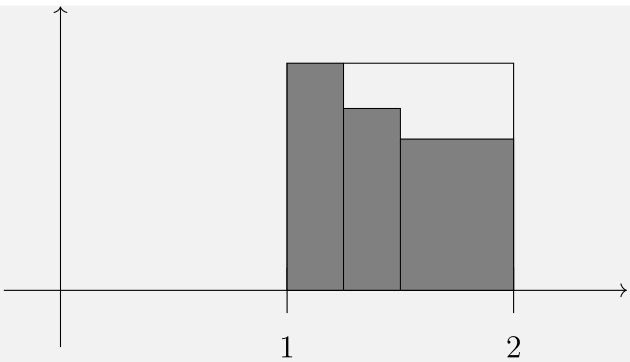
Now we add back a rectangle of height $\frac{2}{3}$ and width $\frac{1}{2}$ onto the region where we removed a rectangle in the previous step. Notice this rectangle has area $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$, and so the total area of the shaded region below is $1 - {}^1/_2 + {}^1/_3$:



We then removing the rectangle with base $[{}^5/_4, {}^3/_2]$ on the x-axis. Notice the rectangle we remove has width ${}^1/_4$ and height 1, so we are removing an area of ${}^1/_4$. Thus the area of the shaded region below is $1 - {}^1/_2 + {}^1/_3 - {}^1/_4$:



Now we add back a rectangle with base $[{}^5/_4, {}^3/_2]$ and height ${}^4/_5$, and the area of the shaded region below is $1 - {}^1/_2 + {}^1/_3 - {}^1/_4 + {}^1/_5$



We can continue in this way, removing a rectangle of area $1/6$, then adding back a rectangle of area $1/7$, removing a rectangle of area $1/8$, adding a rectangle of $1/9$, and so on, while staying inside the initial 1×1 square we started with. Since the area of the square is finite, the sum we are computing must be finite as well.

In fact, each time we add a rectangle to the area we have removed, we have a collection of rectangles which give a Riemann sum approximation to the integral of $1/x$ over $[1, 2]$, and so the limit of the sequence of partial sums of the alternating harmonic series converges to $\int_1^2 \frac{dx}{x}$, hence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2).$$

Remark.
Later we will see another way of justifying the alternating harmonic series converges to $\ln(2)$ by replacing this strange Riemann sum argument by an argument involving the *Taylor series* expansion for $\ln(1 + x)$, but that will have to wait until after we've defined Taylor series.

The main take-away from the above example is that absolute convergence is a *stronger* condition than convergence since absolute convergence implies convergence, but the converse is not true.

If $\sum a_n$ converges but $\sum |a_n|$ diverges, as in Example 5.19 above, then we say the series $\sum a_n$ is ***conditionally convergent***. Generally, absolute convergence is more desirable than conditional convergence because, for one thing, we have some tests for convergence that require us to have a series of positive terms. Another important, but slightly technical, reason has to do with re-ordering the terms of our series. It is not intuitive, but if a series is only conditionally convergent, then its terms can be re-ordered so that the series converges to *any* real value you would like. This is an extremely strange and disconcerting fact (known as *the Riemann series theorem*). For finite sums we know $A+B+C = A+C+B = B+C+A = ...$; any rearrangement of the values being summed gives us the same value. However, for infinite sums this is not true unless the series is absolutely convergent!

Example 5.20.
We will show that the terms of the alternating harmonic series, which is conditionally convergent by Example 5.19, can be rearranged to sum to another value.

Let us write out all the values of the series, which we have seen converges to $\ln(2)$ and write

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

If multiply both sides of this equation by 2 we have

$$2\ln(2) = 2 - 1 + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \cdots$$

Now, "half" of these terms will simplify. In particular, each term with an even denominator now has two in its numerator, so we can rewrite it. For example, $\frac{2}{4}$ we can rewrite as $\frac{1}{2}$, and $\frac{2}{6}$ we can rewrite as $\frac{1}{3}$, and so on. Thus

$$2\ln(2) = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \cdots$$

Everything we have done up to this point is totally fine: all we've done is multiply by 2 and then simplify some of our fractions. Now comes the possibly questionable step of rearranging the terms. Let's notice that *if we could rearrange terms in the series and not change the* *value of the series*, then we could combine our terms $\frac{2}{3}$ and $\frac{-1}{3}$ to obtain $\frac{1}{3}$. Similarly, there will be a $\frac{-2}{10}$ later in our series which we can write as $\frac{-1}{5}$ and combine with the $\frac{2}{5}$ that appears above to obtain $\frac{1}{5}$. Doing this for all terms (combining terms with the same denominator after simplifying), as well as writing $2 - 1 = 1$ at the start of our series, we would have

$$2\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$

But we already determined in Example 5.19 that the right-hand side converges to $\ln(2)$, thus we arrive at

$$2\ln(2) = \ln(2) \implies 2 = 1.$$

Since $2 \neq 1$, there must be an incorrect step in our work above. The only unjustified step, though, was our assumption that we could rearrange terms in an infinite sum without changing what the sum converges to, so that assumption must be incorrect.

Bernhard Riemann took the example above a step further and showed that in fact you can take *any* conditionally convergent series and rearrange the terms so that the series converges to *any* number you would like. That is, there is some way to rearrange the terms of

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

so that the series converges to π or -7 or $\sqrt{e}$, or any other crazy number you'd like! Thus the value of a conditionally convergent series is highly dependent on how you list the terms, which is something we generally do not like. For absolutely convergent series, though, there is no ambiguity: any rearrangement of terms in an absolutely convergent series will still sum to the same value.

Theorem 5.17.
If $\sum a_n$ is an absolutely convergent series, then any rearrangement of the terms of the series will give a series that converges to the same value.

In the alternating harmonic series we had terms that kept switching back and forth from positive to negative to positive to negative, ... Series like this are called ***alternating series*** and can be written in the form

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n \text{ or } \sum_{n=1}^{\infty}(-1)^{n}b_n$$

where each $b_n > 0$. (The difference between the two series above boils down to whether the first term will be positive or negative.)

Recall that in general having terms that shrink down to zero is necessary but not sufficient to know that a series converges. For alternating series, however, this *is* good enough:

Theorem 5.18 (Alternating series test)**.**
If $\sum(-1)^n b_n$ *(or* $\sum(-1)^{n-1}b_n$*) is an alternating series where* $b_1 > b_2 > \cdots$ *and* $\lim_{n\to\infty} b_n = 0$*, then the series must converge.*

Example 5.21.
The series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2/3}}$ is conditionally convergent: it converges because it's an alternating series whose absolute value of terms goes to zero as n goes to infinity, but it *is not* absolutely convergent because the series absolute values of these terms is a p-series with $p < 1$.

So, one of the reasons we like alternating series is it is very easy to determine if the series will converge or not. Another very important reason is that it is extremely easy to estimate the value of the series.

Theorem 5.19.
If $\displaystyle\sum_{n=1}^{\infty}(-1)^n b_n$ *(or* $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}b_n$*) is an alternating series with* $b_1 > b_2 >$

$\cdots$ *and* $b_n \to 0$, *then for each* N *we have*

$$\left| S_N - \sum_{n=1}^{\infty} (-1)^n b_n \right| < b_{N+1}.$$

That is, the $(N+1)$-st term of the series gives an upper bound on how close the N-th partial sum is to the limit of the series. This gives us an effective tool for estimating the value that a series converges to.

For the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2/3}}$, for example, we can easily compute that the 1000-th partial sum is

$$S_{1000} = 27.5574$$

By the above theorem, this is within

$$\frac{1}{1001^{2/3}} \approx 0.00999$$

of the true value of the series.

Example 5.22.
What value of N guarantees the partial sum S_N of the alternating series is within one one-millionth of the true value of the series?

Here the b_n values $\frac{1}{n}$, and by our above theorem we only need to find the value of N such that $b_{N+1} < 10^{-6}$:

$$\begin{aligned} b_{N+1} &< \frac{1}{10^6} \\ \implies \frac{1}{N+1} &< \frac{1}{10^6} \\ \implies 10^6 &< N+1 \\ \implies 10^6 - 1 &< N. \end{aligned}$$

Thus for the alternating harmonic series, any $N > 999,999$ will give an estimate S_N within 10^{-6} of the true value of the series.

5.5 The ratio and root tests

In this section we'll introduce two more tests for helping us determine if a given series converges or not. The first test, called "the ratio test," is similar to the limit comparison test we had seen earlier but, unlike the limit comparison test, we don't need to find another another series to compare our given series against. Instead, we basically just compare terms of our series to the next term in the series and see how quickly the terms are growing or shrinking.

Theorem 5.20 (The ratio test)**.**
Suppose a series $\sum a_n$ *has terms such that the limit* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ *exist, and call this value* ρ*.*

- *If* $\rho < 1$*, then the series* $\sum a_n$ *is absolutely convergent.*
- *If* $\rho > 1$*, then the series* $\sum a_n$ *diverges.*
- *If* $\rho = 1$*, then the test is inconclusive.*

The idea behind the ratio test is that we're seeing how quickly the a_n terms of the series are growing or shrinking. If $\rho < 1$, then the terms of the series are shrinking quickly enough that the series converges. In fact, because we're looking atthe absolute values of the ratios, the series is absolutely convergent. If, on the other hand, $\rho > 1$, then that means the terms are growing and the series must diverge. The $\rho = 1$ case is the tricky situation; there are examples of series where $\rho = 1$ and the series converges, but also examples where $\rho = 1$ and the series diverges. Thus if $\rho = 1$ we basically don't have enough information to determine if the series converges or diverges or not, and so we say the test is inconclusive.

Remark.
Just because the ratio test is inconclusive *does not* mean that we can't determine whether the series converges or not, it just means the ratio test isn't the right tool for the job.

Example 5.23.
Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converge?

To answer this question we'll apply the ratio test with $a_n = \frac{n^3}{3^n}$. Notice that $a_{n+1} = \frac{(n+1)^3}{3^{n+1}}$. Thus we want to consider the limit

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{\left((n+1)^3/3^{n+1}\right)}{\left(n^3/3^n\right)} \\
&= \lim_{n\to\infty} \frac{3^n \cdot (n+1)^3}{3^{n+1} \cdot n^3} \\
&= \lim_{n\to\infty} \frac{3^n \cdot (n+1)^3}{3 \cdot 3^n \cdot n^3} \\
&= \lim_{n\to\infty} \frac{(n+1)^3}{3n^3} \\
&= \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{3n^3} \\
&= \frac{1}{3}
\end{aligned}$$

Thus the series converges by the ratio test. (The ratio test actually tells us the series is absolutely convergent, but here all the terms were already positive anyway.)

Example 5.24.
Does the series

$$\sum_{n=1}^{\infty} \frac{2^n \cdot 5^{n+1}}{n^2 + 1}$$

converge?

Here $a_n = \frac{2^n 5^{n+1}}{n^2+1}$ and $a_{n+1} = \frac{2^{n+1}5^{n+2}}{(n+1)^2+1}$, thus we consider the limit

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{\left(\frac{2^{n+1}5^{n+2}}{(n+1)^2+1}\right)}{\left(\frac{2^n5^{n+1}}{n^2+1}\right)}\\ &= \lim_{n\to\infty}\frac{2^{n+1}5^{n+2}}{2^n5^{n+1}}\cdot\frac{n^2+1}{(n+1)^2+1}\\ &= \lim_{n\to\infty}\frac{2^{n+1}5^{n+2}}{2^n5^{n+1}}\cdot\frac{n^2+1}{n^2+2n+2}\\ &= \lim_{n\to\infty} 2\cdot 5\cdot\frac{n^2+1}{n^2+2n+1}\\ &= 10\end{aligned}$$

As $10 > 1$, the series diverges by the ratio test.

Example 5.25.
We have seen that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ while the alternating harmonic series converges, but notice that in either case the ratio test is inconclusive as our limits of ratios of consecutive terms are

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{1/(n+1)}{1/n}\right| &= \lim_{n\to\infty}\frac{n}{n+1} = 1, \text{ and}\\ \lim_{n\to\infty}\left|\frac{(-1)^n/(n+1)}{(-1)^{n-1}/n}\right| &= \lim_{n\to\infty}\frac{n}{n+1} = 1.\end{aligned}$$

As one final convergence test, we have the "root test" below which is especially useful for series $\sum a_n$ where a_n involves a power of n.

Theorem 5.21 (The root test)**.**
Suppose $\sum a_n$ is a series where $\lim_{n\to\infty}\sqrt[n]{|a_n|}$ exists or is infinite.

- *If $\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$, then the series $\sum a_n$ is absolutely convergent.*

- *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ *or is infinite, then* $\sum a_n$ *is divergent.*
- *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, *then the root test is inconclusive.*

Example 5.26.
Does the series following series converge?

$$\sum_{n=1}^{\infty} \left(\frac{-n^2 - 3}{2n^2 + n} \right)^n$$

Applying the root test we have

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{-n^2-3}{2n^2+n} \right)^n \right|} &= \lim_{n\to\infty} \sqrt[n]{\left| \frac{-n^2-3}{2n^2+n} \right|^n} \\
&= \lim_{n\to\infty} \left| \frac{-n^2-3}{2n^2+n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)(n^2+3)}{2n^2+n} \right| \\
&= \lim_{n\to\infty} \frac{n^2+3}{2n^2+n} \\
&= \frac{1}{2}
\end{aligned}$$

As this limit is less than 1, the series is absolutely convergent.

Example 5.27.
Does the following series converge?

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n} \right)^{n^2}$$

Applying the root test we have

$$\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{\left|1-\frac{1}{n}\right|^{n^2}} &= \lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)^{n^2}\right]^{1/n} \\
&= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} \\
&= \lim_{n\to\infty} e^{\ln\left(\left(1-\frac{1}{n}\right)^{n}\right)} \\
&= e^{\lim_{n\to\infty} n\cdot\ln\left(1-\frac{1}{n}\right)} \\
&= e^{\lim_{n\to\infty} \frac{\ln(1-n^{-1})}{n^{-1}}} \\
&\overset{\mathscr{L}}{=} e^{\lim_{n\to\infty} \frac{\left(1/(1-n^{-1})\right)\cdot n^{-2}}{-n^{-2}}} \\
&= e^{\lim_{n\to\infty} \frac{-1}{1-1/n}} \\
&= e^{-1} < 1
\end{aligned}$$

Thus the series converges by the root test.

6 Power series, Taylor polynomials, and Taylor series

> *"Obvious" is the most dangerous word in mathematics.*
>
> E. T. BELL

At this point we have built up some knowledge about series, but we haven't really applied series to solve any interesting problems. Even if you feel confident and comfortable with the material we have introduced, you may very well wonder what the "point" is, or why we'd study series in calculus.

In this chapter we will start to answer these questions about what series have to do with calculus and why we care about series. In Section 6.1, we will review some very basic facts about polynomials and then consider "power series" which can be thought of as infinite degree polynomials. In Section 6.2 we show how we can associate polynomials to "nice" functions that give good approximations to those functions, reminiscient of the linearizations you learned about in your first semester calculus class. Finally, in Section 6.3 we combine the previous sections together by considering "Taylor series," which are the infinite-degree analogues of the polynomials discussed in Section 6.2.

6.1 Power series

Recall that a polynomial is an expression of the form

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_2 x^2 + c_1 x + c_0$$

where the c_i are constants called the ***coefficients*** of the polynomial, and (assuming $c_n \neq 0$) we call n the ***degree*** of the polynomial. For example, $3x^2 - 34x + 96$ is a polynomial of degree two.

In calculus we really like functions that are defined by polynomials because they are very easy to work with; it's really easy to both integrate and differentiate polynomials. More fundamentally, though, they are functions that we can effectively evaluate. That is, if $f(x)$ is a function defined by some polynomial, say $f(x) = 7x^2 - 2x + 3$, then we can actually sit down with pencil and paper and compute quantities such as $f(-2)$ or $f(3.6)$. For other types of functions, however, it's not nearly as clear how to actually evaluate the function: if $f(x) = \sqrt{e^x}$, for instance, how on Earth do you actually compute $f(-2)$ or $f(3.6)$?

A naive answer to how to evaluate these functions would be "use a calculator or computer," but then we may ask how does the calculator or computer perform that calculation? A human being designed and built that machine, so someone somewhere had to "tell" the computer how to perform those calculations, and so in principle someone could do those calculations by hand, and our question is "how?"

The point here is that polynomials are nice for a variety of reasons, but they are also a pretty restrictive class of functions. There are lots and lots of functions, like $\sqrt{x}$, e^x, or $\sin(x)$, which are not polynomials. Our goal in this chapter will be to use series to introduce a sort of generalization of polynomials that have a lot of the nice properties of polynomials (such as being very easy to integrate or differentiate), but which aren't as restrictive.

Before jumping directly into that generalization, though, let's mention one convenient property of polynomials that you might not be aware of: For every number a, every polynomial $f(x) = c_n x^n + \cdots + c_1 x + c_0$ can be written in the form $f(x) = d_n(x-a)^n + \cdots + d_1(x-a) + d_0$.

For example, the polynomial

$$f(x) = 3x^2 - 34x + 36$$

can be rewritten as

$$f(x) = 3(x-5)^2 - 4(x-5) + 1.$$

By expanding the the last expression and combining like-terms, you will see that these two polynomials really are the same thing. However, the second form is convenient to use sometimes because it is very easy to read off quantities such as $f(5)$, $f'(5)$, and $f''(5)$. To see this, let's compute $f(5)$ using the first and second expressions for $f(x)$ above:

$f(5) = 3 \cdot 5^2 - 34 \cdot 5 + 96 = 75 - 170 + 96 + 1 = 1$ (Using the first expression.)
$f(5) = 3 \cdot (5-5)^2 - 4 \cdot (5-5) + 1 = 1$ (Using the second expression.)

The first derivative we can compute as

$f'(x) = 6x - 34$ (Using the first expression.)
$f'(x) = 6(x-5) - 4$ (Using the second expression.)

thus

$$f'(5) = 6 \cdot 5 - 34 = -4 \text{ (Using the first expression.)}$$
$$f'(5) = 6 \cdot (5 - 5) - 4 = -4 \text{ (Using the second expression.)}$$

And for the second derivative we simply have $f''(x) = 6$ using either expression, and so $f''(5) = 6$.

The key observation is that the coefficients 3, -4, and 1 that appear in the expression

$$f(x) = 3(x - 5)^2 - 4(x - 5) + 1$$

are very closely related to the values of $f''(5)$, $f'(5)$, and $f(5)$. This will be come very useful later.

In general, to convert a polynomial $c_n x^n + \cdots + c_1 x + c_0$ into the expression $d_n(x-a)^n + \cdots d_1(x-1) + d_0$, we have to solve a system of equations. For example, to find the 3, -4, and 1 above we hvae to write

$$3x^2 - 34x + 96 = A(x - 5)^2 + B(x - 5) + C$$

we can expand the terms on the right-hand side, giving us

$$Ax^2 - 10Ax + 25A + Bx - 5B + C,$$

and combining like-terms we can write this as

$$Ax^2 + (-10A + B)x + 25A - 5B + C.$$

As this is supposed to equal $3x^2 - 34x + 96$, we have the following system of equations:

$$\begin{aligned} A &= 3 \\ -10A + B &= -34 \\ 25A - 5B + C &= 96 \end{aligned}$$

and solving this system of equations yields the coefficients above.

Example 6.1.
Rewrite $6x^2 + 4x + 5$ in the form $A(x - 2)^2 + B(x - 2) + C$.

First we expand and combine like-terms of $A(x-2)^2+B(x-2)+C$ to write it as

$$\begin{aligned} &Ax^2 - 4Ax + 4A + Bx - 2B + C \\ =&Ax^2 + (-4A + B)x + 4A - 2B + C \end{aligned}$$

Now we set up our system of equations,

$$\begin{aligned} A &= 6 \\ -4A + B &= 4 \\ 4A - 2B + C &= 5 \end{aligned}$$

The first equation instantly tells us that $A = 6$, and so the second equation becomes

$$\begin{aligned} & -4 \cdot 6 + B = 4 \\ \implies & -24 + B = 4 \\ \implies & B = 28 \end{aligned}$$

We can now easily solve the third equation:

$$\begin{aligned} 4 \cdot 6 - 2 \cdot 28 + C = 5 & \\ & \implies 24 - 56 + C = 5 \\ & \implies -32 + C = 5 \\ & \implies C = 37 \end{aligned}$$

Hence our polynomial may be written as

$$6(x-2)^2 + 28(x-2) + 37.$$

When a polynomial is written in this way, the coefficients are closely related to the value of the function and its derivatives at $x = a$.

Proposition 6.1.
If $f(x)$ is a polynomial expressed as

$$f(x) = \sum_{i=0}^{d} d_i (x-a)^i,$$

then $d_i = \dfrac{f^{(i)}(a)}{i!}$.

Remark.
Recall that $f^{(n)}(x)$ is notation for the n-th derivative of f, and $n!$, pronounced "n factorial," is the quantity

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

and by convention we define $0! = 1$.

For example, the proposition above says that we can find the coefficients required to rewrite our polynomial simply by evaluating the derivative. For our $f(x) = 6x^2 + 4x + 5$ example above, we have $f'(x) = 12x + 4$ and $f''(x) = 12$. Notice we can evaluate $f(2) = 37$, $f'(2) = 28$, and $f''(2) = 12$ to obtain

$$\begin{aligned} f(x) &= \frac{f''(2)}{2!}(x-2)^2 + \frac{f'(2)}{1!}(x-2)^1 + \frac{f(2)}{0!}(x-2)^0 \\ &= \frac{12}{2}(x-2)^2 + \frac{28}{1}(x-2) + \frac{37}{1} \cdot 1 \\ &= 6(x-2)^2 + 28(x-2) + 37 \end{aligned}$$

Remark.
You should basically interpret Proposition 6.1 as giving you a shortcut to the solution to the system of linear equations described earlier.

We now define a ***power series*** as an expression of the form

$$\sum_{i=0}^{\infty} c_i(x-a)^i = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

The "a" that appears above is called the ***center*** of the series.

A power series thus a kind of "infinite degree" polynomial. If we want to use this expression to define a function,

$$f(x) = \sum_{i=0}^{\infty} c_i(x-a)^i$$

then the very first question we should ask ourserlves is what is the domain of this function; what values of x "make sense" in the expression above, meaning for what values of x will the above expression result in a convergent series?

For example, consider the function below defined by a power series,

$$f(x) = \sum_{i=0}^{\infty} \left(\frac{1}{5}\right)^i (x-3)^i$$

Here our coefficients c_i are simply $\left(\frac{1}{5}\right)^i$, and the center of the power series is 3. Does this expression "make sense" at $x = 4$? What about at $x = 12$?

If we try to evalute $f(4)$ by replacing all of the x's in our power series expression above by 4's, then we have

$$\begin{aligned}\sum_{i=0}^{\infty} \left(\frac{1}{5}\right)^i \cdot (4-3)^i &= \sum_{i=0}^{\infty} \left(\frac{1}{5}\right)^i \\ &= \frac{1}{1 - {}^1\!/_5} \\ &= \frac{1}{{}^4\!/_5} \\ &= \frac{5}{4}\end{aligned}$$

where the last few steps followed from noticing $\sum_{i=1}^{\infty} \left(\frac{1}{5}\right)^i$ is a geometric series. Thus our expression above "makes sense" for $x = 4$, and so we should say 4 is in the domain of our function and $f(4) = \frac{5}{4}$.

If we try to evaluate the expression above at $x = 12$, though, we obtain

$$\sum_{i=0}^{\infty} \left(\frac{1}{5}\right)^i (12-3)^i = \sum_{i=0}^{\infty} \left(\frac{9}{5}\right)^i.$$

However, this series diverges since it is a geometric series with $r > 1$. Since the expression defining $f(x)$ doesn't converge to a finite value at $x = 12$, we are forced to conclude that $x = 12$ *is not* in the domain of this function.

More generally, once we've picked a value of x to plug into our series,

$$\sum_{i=0}^{\infty} \left(\frac{1}{5}\right)^i (x-3)^i,$$

then $x-3$ becomes some number, and we really have a geometric series with $r=\frac{x-3}{5}$. We know that geometric series converge if $|r|<1$, and so this expression will converge if

$$\begin{aligned} &\left|\frac{x-3}{5}\right|<1 \\ \Longrightarrow\ &|x-3|<5 \\ \Longrightarrow\ &-5<x-3<5 \\ \Longrightarrow\ &-2<x<8. \end{aligned}$$

Thus, the interval $(-2,8)$ is definitively in the domain of our function. Notice that points in the interval $(-\infty,-2]\cup[8,\infty)$ are definitively outside of the domain of the function since for these values we have

$$\begin{aligned} &|x-3|>5 \\ \Longrightarrow\ &\left|\frac{x-3}{5}\right|>1 \\ \Longrightarrow\ &\sum_{i=0}^{\infty}\left(\frac{x-3}{5}\right)^i \text{ diverges.} \end{aligned}$$

Thus the domain of

$$f(x)=\sum_{i=0}^{\infty}\left(\frac{x-3}{5}\right)^i$$

is $(-2,8)$.

In general, a power series

$$\sum_{n=0}^{\infty}c_n(x-a)^n$$

may converge for some values of x and diverge for other values. Above we noticed that our series was really a geometric series "in disguise," but that was kind of special: power series are not necessarily geometric series after a value is plugged in for x. So, how can we determine where our power series converges or diverges in general? Recall that the ratio test promises a series $\sum a_n$ converges absolutely if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|<1$ and diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$.

For the power series $\sum c_n(x-a)^n$, this means we are interested in the limit

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right|=\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|\cdot|x-a|$$

Notice the $|x-a|$ factor does not depend on n, and so we can write the limit as

$$|x-a| \cdot \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|.$$

Let's momentarily suppose this limit is a finite, non-zero number which we'll denote by $\frac{1}{R}$. Then the series converges absolutely if

$$\begin{aligned} &|x-a| \cdot \frac{1}{R} < 1 \\ \Longrightarrow\ &|x-a| < R \\ \Longrightarrow\ &-R < x-a < R \\ \Longrightarrow\ &a-R < x < a+R. \end{aligned}$$

And so the series converges absolutely for all x in the interval $(a-R, a+R)$. Similarly, the series diverges by the ratio test if $|x-a| \cdot \frac{1}{R} > 1$ which means $x < a-R$ or $x > a-R$; the series diverges for all x in $(-\infty, a-R) \cup (a+R, \infty)$.

The points $x = a-R$ and $x = a+R$, the endpoints of the interval above, are trickier. At those points the limit is 1, so the ratio test is inconclusive: the series may converge or diverge at these points and we have to "manually" test each endpoint with some other test.

Regardless of whether the endpoints converge or not, note the power series definitely converges absolutely if $-R < x-a < R$. That is, if x is within distance R from a. This region of guaranteed convergence around a is called the ***radius of convergence*** of the power series. Since a is in the middle of the region of guaranteed convergence, it is called the ***center*** of the series. Adding whichever endpoints give convergence gives us the ***interval of convergence*** of the series.

In the case of our earlier power series

$$\sum_{i=0}^{\infty} \left(\frac{1}{5}\right)^i (x-3)^i$$

the center of the series is 3, the radius of convergence is 5, and the interval of convergence is $(-2, 8)$.

Example 6.2.
Find the center, radius of convergence, and interval of convergence

of the series

$$\sum_{n=1}^{\infty} \frac{1}{n} (3x+9)^n .$$

To find the center, we need to write our series as $\sum c_n(x-a)^n$. To do that here, let's factor a 3 out as follows:

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n} (3x+9)^n \\ =&\sum_{n=1}^{\infty} \frac{1}{n} (3(x+9))^n \\ =&\sum_{n=1}^{\infty} \frac{3^n}{n} (x+3)^n \end{aligned}$$

Thus our "c_n" coefficients are $\frac{3^n}{n}$, and $x-a$ is equal to $x+3$, thus the center a is $a=-3$.

For the radius of convergence we need to compute the limit

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| &= \lim_{n\to\infty} \frac{3^{n+1}/(n+1)}{3^n/n} \\ &= \lim_{n\to\infty} \frac{3^{n+1}\cdot n}{3^n\cdot(n+1)} \\ &= \lim_{n\to\infty} \frac{3n}{n+1} \\ &= 3 \end{aligned}$$

Keep in mind this is one over the radius, and so our radius is $1/3$. That is, the series definitively converges in the interval

$$\left(-3-\frac{1}{3}, -3+\frac{1}{3}\right) = \left(\frac{-10}{3}, \frac{-8}{3}\right).$$

Now to find the interval of convergence we also need to check if the series converges at the endpoints of $\frac{-8}{3}$ and $\frac{-10}{3}$ or not.

At $x = \frac{-8}{3}$, our series becomes

$$\sum_{n=1}^{\infty} \frac{3^n}{n} \left(\frac{-8}{3} + 3 \right)^n = \sum_{n=1}^{\infty} \frac{3^n}{n} \cdot \left(\frac{1}{3} \right)^n = \sum_{n=1}^{\infty} \frac{1}{n}.$$

Since this is the harmonic series, this diverges.

At $x = \frac{-10}{3}$ we have

$$\sum_{n=1}^{\infty} \frac{3^n}{n} \left(\frac{-10}{3} + 3 \right)^n = \sum_{n=1}^{\infty} \frac{3^n}{n} \left(\frac{-1}{n} \right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Since this is the alternating harmonic series, this series converges.

Thus the interval of convergence of our power series is $\left[\frac{-10}{3}, \frac{8}{3}\right)$.

The ratio test is our primary tool for determining the radius of convergence of a series. Many times this radius will be a finite, positive number meaning we have convergence inside some finite interval and convergence outside the interval, and the endpoints of the interval have to be "manually" checked. However, there are a few other possibilities that can occur.

For example, consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

By the ratio test this series will be absolutely convergent for all values of x satisfying

$$\lim_{n\to\infty} \left| \frac{\left(x^{n+1}/(n+1)!\right)}{\left(x^n/n!\right)} \right| < 1.$$

When we calculate this limit, however, we have

$$\lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \cdot \frac{x^{n+1}}{x^n} \right| = |x| \cdot \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

That is, this limit is zero *for all* choices of x, and since $0 < 1$, this series must be absolutely convergent for all x; the interval of convergence is the entire real line $(-\infty, \infty)$ and the radius of convergence is infinite.

As another example, consider the series

$$\sum_{n=0}^{\infty} n!(x-3)^n.$$

If we wish to find the interval of convergence of this series, we first need to determine all of the x values where

$$\lim_{n\to\infty} \left| \frac{(n+1)!(x-3)^{n+1}}{n!(x-3)} \right| < 1.$$

Notice, though,

$$\lim_{n\to\infty} \left| \frac{(n+1)!(x-3)^{n+1}}{n!(x-3)^n} \right| = |x-3| \cdot \lim_{n\to\infty} (n+1).$$

This limit will below up to infinity for all values of x *except* $x = 3$: when $x = 3$ each term of the sequence

$$\frac{(n+1)!(x-3)^{n+1}}{n!(x-3)^n}$$

is zero, and so the corresponding series is absolutely convergent. Thus the interval of convergence for this series simply the single point $x = 3$, which as an interval is $[3, 3]$. The radius of convergence is zero.

In general, a power series will always converge at its center since

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

will have every term except the first term, c_0, vanish when $x = a$. Note too that if $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, then the coefficient c_0 is exactly $f(a)$, regardless of what other points may happen to be in the interval of convergence.

Example 6.3.
What is the center, radius of convergence, and the interval of con-

vergence for the power series

$$\sum_{n=0}^{\infty} \frac{(2x+3)^{2n}}{4^{2n+1}} = \frac{1}{4} + \frac{(2x+3)^2}{4^3} + \frac{(2x+3)^4}{4^5} + \frac{(2x+3)^6}{4^7} + \cdots$$

Let's first notice that we may rewrite this series as

$$\sum_{n=0}^{\infty} \frac{2^{2n}}{4 \cdot 4^{2n}} \left(x + \frac{3}{2}\right)^{2n} = \sum_{n=0}^{\infty} \frac{1}{4} \left(\frac{1}{2}\right)^{2n} \left(x + \frac{3}{2}\right)^{2n}$$

From this we see that the center of the series is $\frac{-3}{2}$.

For the radius of convergence we apply the ratio test:

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{\frac{1}{4} \cdot \left(\frac{1}{2}\right)^{2(n+1)} \left(x + \frac{3}{2}\right)^{2(n+1)}}{\frac{1}{4} \cdot \left(\frac{1}{2}\right)^{2n} \left(x + \frac{3}{2}\right)^{2n}} \right| &= \lim_{n\to\infty} \left| \frac{\left(\frac{1}{2}\right)^{2n+2} \left(x + \frac{3}{2}\right)^{2n+2}}{\left(\frac{1}{2}\right)^{2n} \left(x + \frac{3}{2}\right)^{2n}} \right| \\ &= \left(\frac{1}{2}\right)^2 \left| x + \frac{3}{2} \right|^2\end{aligned}$$

Thus our series converges absolutely when the following inequalities are satisfied:

$$\begin{aligned} & \frac{1}{4} \left| x + \frac{3}{2} \right|^2 < 1 \\ \implies & \left| x + \frac{3}{2} \right|^2 < 4 \\ \implies & \left| x + \frac{3}{2} \right| < 2 \\ \implies & -2 < x + \frac{3}{2} < 2 \\ \implies & -\frac{3}{2} - 2 < x < -\frac{3}{2} + 2 \\ \implies & \frac{-7}{2} < x < \frac{1}{2} \end{aligned}$$

So our power series converges absolutely for all x in the interval $\left(\frac{-7}{2}, \frac{1}{2}\right)$, and the radius of convergence is 2. To find the interval of convergence we must "manually" check if the series converges at the endpoints $x = \frac{-7}{2}$ and $x = \frac{1}{2}$.

At $x = \frac{-7}{2}$, the series becomes

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{1}{4} \cdot \left(\frac{1}{2}\right)^{2n} \left(\frac{-7}{2} + \frac{3}{2}\right)^{2n} \\ =&\sum_{n=0}^{\infty} \frac{1}{4} \left(\frac{1}{2}\right)^{2n} \left(\frac{-4}{2}\right)^{2n} \\ =&\sum_{n=1}^{\infty} \frac{1}{4}(-1)^{2n} \\ =&\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots\end{aligned}$$

thus the series diverges.

At $x = \frac{1}{2}$ we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{1}{4} \left(\frac{1}{2}\right)^{2n} \left(\frac{1}{2} + \frac{5}{2}\right)^{2n} \\ =&\sum_{n=0}^{\infty} \frac{1}{4} \left(\frac{1}{2}\right)^{2n} \cdot 3^{2n} \\ =&\sum_{n=0}^{\infty} \frac{1}{4} \left(\frac{3}{2}\right)^{2n} \\ =&\sum_{n=0}^{\infty} \frac{1}{4} \left(\frac{9}{4}\right)^{n}\end{aligned}$$

which is also a divergent geometric series. Thus the interval of convergence is just $(-7/2, 1/2)$.

We had previously mentioned that power series are sort of like infinite-degree polynomials, and one of the nice things about polynomials is that they are easy to integrate and differentiate. The following theorem says this is basically true for power series as well.

Theorem 6.2.

Suppose $f(x)$ *is determined by the power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *which has* *radius of convergence* $R > 0$*. Then inside the interval* $(a - R, a + R)$ *we have the following:*

- f *is continuous;*
- f *is infinitely differentiable*[a]*;*
- $f'(x)$ *is given by the power series obtained by differentiating* $\sum c_n(x-a)^n$ *term-by-term and this series has radius of converge* R *as well:*

$$\begin{aligned} f'(x) &= \frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n \\ &= \sum_{n=0}^{\infty} \frac{d}{dx} c_n(x-a)^n \\ &= \sum_{n=0}^{\infty} c_n n(x-a)^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1)c_{n+1}(x-a)^n \end{aligned}$$

- *the antiderivative* $F(x)$ *is given by a power series obtained by anti-differentiating* $\sum c_n(x-a)^n$ *term-by-term and this also has radius of convergence* R*:*

$$\begin{aligned} F(x) &= \int \sum_{n=0}^{\infty} c_n(x-a)^n \, dx \\ &= \sum_{n=0}^{\infty} \sum c_n(x-a)^n \, dx \\ &= \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} + C \\ &= \sum_{n=1}^{\infty} \frac{c_n(x-a)^n}{n} + C \end{aligned}$$

[a] Being "infinitely differentiable" simply means that *all* the derivatives of f exist: the first derivative, second derivative, third derivative, etc.

Example 6.4.
If $f(x) = \sum_{n=0}^{\infty} \left(\frac{x-3}{5}\right)^n$, then we had seen before the radius of convergence of this series is 5. The derivative is then

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} n \left(\frac{x-3}{5}\right)^{n-1} \\ &= \sum_{n=1}^{\infty} n \left(\frac{x-3}{5}\right)^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1) \left(\frac{x-3}{5}\right)^{n} \end{aligned}$$

and this series also has radius of convergence 5.

The antiderivative is

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\frac{x-3}{5}\right)^{n+1} + C \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{x-3}{5}\right)^{n} + C \end{aligned}$$

and this series also has radius of convergence 5.

That power series are very easy to integrate and differentiate has numerous applications in calculus. For instance, this can be helpful in solving certain differential equations which arise in areas as diverse as economics and engineering. We haven't discussed differential equations this semester simply for the purposes of time, so you will have to take this on faith until you take a course in differential equations, but power series can be used to make otherwise difficult problems much easier, precisely because they are so easy to integrate and differentiate.

6.2 Taylor polynomials

We had previously mentioned that functions defined by polynomials can be "efficiently computed;" i.e., you can do the computation by hand, or have a computer do the computation for you, since we have "recipes" (algorithms) for how to add, subtract, multiply, and divide, and this is all that's required to evaluate a polynomial. Lots of functions we care about are not polynomials, however. To work with those functions, such as e^x, $\sin(x)$, and $\sqrt{x}$, what we'll do instead is approximate the function with a polynomial.

You had previously seen a simple version of this in your first semester of calculus where you learned about linearization. Recall that the linearization of a differentiable function $f(x)$ at a point a is given by

$$L(x) = f(a) + f'(a) \cdot (x - a)$$

Notice this is really just the function whose graph is the tangent line to $y = f(x)$ at $x = a$.

Another way of saying this is that $L(x)$ is the polynomial of degree one which agrees with $f(x)$ and $f'9x)$ at $x = a$. I.e., $L(a) = f(a)$ and $L'(a) = f'(a)$. Recall that we think of $L(x)$ as an approximation to $f(x)$ near zero. To get a better approximation we can look at higher degree polynomials. Motivated by the observation that the linear approximation agrees with the first derivative, we might ask that an n-th degree polynomial approximating $f(x)$ agrees with the first n derivatives of $f(x)$.

To be more precise, we define the ***n-th order Taylor polynomial of $f(x)$ centered at $x = a$***, denoted $T_n(x)$, to be the degree n polynomial such that

$$\begin{aligned} T_n(a) &= f(a) \\ T_n'(a) &= f'(a) \\ T_n''(a) &= f''(a) \\ &\vdots \\ T_n^{(n)}(a) &= f^{(n)}(a) \end{aligned}$$

Writing $T_n(x)$ in the form

$$T_n(x) = \sum_{i=0}^{n} c_i (x-a)^i = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n$$

we could explicitly compute each of $T_n(x)$, $T_n'(x)$, $T_n''(x)$ and so on, and we'd find

$$\begin{aligned}
T_n(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots + c_n(x-a)^n \\
T_n'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots + nc_n(x-a)^{n-1} \\
T_n''(x) &= 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3\cdot c_4(x-a)^2 + \ldots + n(n-1)c_n(x-a)^{n-2} \\
T_n'''(x) &= 3\cdot 2\cdot c_3 + 4\cdot 3\cdot 2\cdot c_4(x-a) + \ldots + n(n-1)(n-2)(x-a)^{n-3} \\
&\vdots \\
T_n^{(n)}(x) &= n!c_n.
\end{aligned}$$

Evaluating these at $x = a$, we see that In general, $T_n^{(i)}(a) = i!c_i$, and so our assumption that the i-th derivative of $T_n(x)$ at $x = a$ evaluates with the i-th derivative of $f(x)$ at $x = a$ tells us

$$\begin{aligned}
i!c_i &= f^{(i)}(a) \\
\Longrightarrow c_i &= \frac{f^{(i)}(a)}{i!}
\end{aligned}$$

Thus the n-th degree Taylor polynomial of $f(x)$ centered at $x = a$ is

$$\begin{aligned}
T_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i
\end{aligned}$$

Example 6.5.
Find the third order Taylor polynomial for $f(x) = \sqrt{x}$ centered at $x = 1$.

Using the formula derived above, we need to compute the first three derivatives of $f(x) = \sqrt{x}$, and then evaluate these at $x = 1$. We simply compute

$$\begin{aligned}
f(x) &= x^{1/2} = \sqrt{x} & f(1) &= 1 \\
f'(x) &= \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} & f'(1) &= \frac{1}{2} \\
f''(x) &= \frac{-1}{4}x^{-3/2} = \frac{-1}{4\sqrt{x^3}} & f''(1) &= \frac{-1}{4} \\
f'''(x) &= \frac{3}{8}x^{-5/2} = \frac{3}{8\sqrt{x^5}} & f'''(1) &= \frac{3}{8}
\end{aligned}$$

The Taylor polynomial is thus

$$\begin{aligned} T_3(x) &= 1 + \frac{1}{2}(x-1) + \frac{-1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3 \\ &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{3}{48}(x-1)^2 \end{aligned}$$

The main point of Taylor polynomials is that they give us approximations to the function near the center, and we can actually explicitly compute those approximations. For instance, using the approximation $T_3(x)$ for $\sqrt{x}$ in Example 6.5 we may approximate square roots near $x = 1$ by simply plugging those values into $T_3(x)$:

$$\begin{aligned} \sqrt{1.2} &\approx T_3(1.2) \\ &= 1 + \frac{1}{2}(1.2-1) - \frac{1}{8}(1.2-1)^2 + \frac{3}{48}(1.2-1)^3 \\ &= 1 + \frac{1}{2} \cdot 0.2 - \frac{1}{8} \cdot 0.004 + \frac{3}{48} \cdot 0.008 \\ &= 1.0955 \end{aligned}$$

$$\begin{aligned} \sqrt{2} &\approx T_3(2) \\ &= 1 + \frac{1}{2} - \frac{1}{8} + \frac{3}{48} \\ &= 1.4375 \end{aligned}$$

Example 6.6.
Find the fourth-order Taylor polynomial of $f(x) = e^x$ centered at $x = 0$.

Notice that since every derivative of e^x is simply e^x, each coefficient in $T_4(x)$ is just $\frac{1}{i!}$, and so we have

$$\begin{aligned} T_4(x) &= \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \end{aligned}$$

We can thus approximate e^x for values of x near 0 by simply plugging into the polynomial $T_4(x)$ computed in Example 6.6.

$$e^{0.5} \approx 1 + 0.5 + \frac{0.5}{2} + \frac{0.125}{6} + \frac{0.0625}{24} = 1.64844$$
$$e = e^1 \approx 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.70833$$

These are only approximations to $f(x)$, ao reasonable question is how good are these approximations. To answer that question we need the following, which is essentially a more sophisticated version of the mean value theorem.

Theorem 6.3 (Taylor's remainder theorem)**.**
If $f(x)$ is $N+1$-times continuously differentiable and $T_n(x)$ is the n-th order Taylor polynomial for $f(x)$ centered at $x = a$, then for every value of x there exists a number τ_x between a and x such that

$$f(x) = T_n(x) + \frac{f^{(n+1)}(\tau_x)}{(n+1)!}(x-a)^{n+1}$$

The quantity $\frac{f^{(n+1)}(\tau_x)}{(n+1)!}(x-a)^{n+1}$ appearing in Theorem 6.3 is sometimes called the ***remainder*** of the n-th order Taylor polynomial and denoted $R_n(x)$.

Notice that the remainder edepends on the value τ_x, but the theorem does not give us any hint about how to compute τ_x – it only promises us that such a τ_x exists. This is very similar to the mean value theorem from your first semester calculus course. In fact, if we take n to be zero, then $T_n(x)$ is simply $f(a)$ and the expression above in Taylor's remainder theorem becomes

$$f(x) = T_0(x) + \frac{f^{(0+1)}(\tau_x)}{(0+1)!}(x-a)^{0+1} = f(a) + f'(\tau_x)(x-a)$$
$$\implies f(x) - f(a) = f'(\tau_x)(x-a)$$
$$\implies f'(\tau_x) = \frac{f(x)-f(a)}{x-a}$$

It is in this sense that Taylor's remainder theorem is a generalization of the mean value theorem for higher-order derivatives.

Notice that by moving $T_n(x)$ to the left-hand side of the equation that appears in Taylor's remainder theorem, we have an expression for the error in approximating $f(x)$ by $T_n(x)$,

$$f(x) - T_n(x) = \frac{f^{(n+1)}(\tau_x)}{(n+1)!}(x-a)^{n+1}.$$

Thus to bound the error in approximating $f(x)$ by $T_n(x)$, we need to bound this expression on the right-hand side. So, as a corollary to Taylor's remainder theorem we obtain the following:

Corollary 6.4.
If $\left|f^{(n+1)}(x)\right| < K$ *for all* x *in the interval* $[c, d]$, *then for every* $c \leq x \leq d$, *we have*

$$|f(x) - T_n(x)| \leq \frac{K}{(n+1)!}|x-a|^{n+1}$$

This corollary gives us an effective tool in saying exactly how "good" an approximation $T_n(x)$ is to $f(x)$ – at least for x's in the interval where we can bound $|f^{(n+1)}(x)|$.

Example 6.7.
What is the maximum amount of error that can appear in approximating $f(x) = \sin(x)$ by the third-order Taylor polynomial $T_3(x)$ centered at $x = 0$ for all x in the interval $(-\pi/2, \pi/2)$.

Notice that the remainder for $T_3(x)$ is

$$R_3(x) = \frac{\left.\frac{d^4}{dx^4}\right|_{\tau_x} \sin(x)}{4!}(x-0)^4$$

Since the fourth derivative of $\sin(x)$ is simply $\sin(x)$, we have $|\sin(x)| \leq 1$ for all x. Since we are explicitly concerned with x's in the interval

$(-\pi/2, \pi/2)$, however, we also know $|x| < \pi/2$, and thus

$$\begin{aligned}|R_3(x)| &= \left|\frac{\sin(\tau_x)}{24}x^4\right| \\ &= \frac{|\sin(\tau_x)|}{24}\cdot|x|^4 \\ &< \frac{1}{24}\left(\frac{\pi}{2}\right)^4 \\ &= \frac{\pi^4}{384} \\ &\approx 0.2537\end{aligned}$$

Thus the worst possible error that can occur by approximating $\sin(x)$ by $T_3(x)$ for x's between $-\pi/2$ and $\pi/2$ is no more than 0.2537.

Perhaps the more interesting question, though, would be what value of n guarantees that the Taylor polynomial approximating $f(x)$ is within some desired accuracy, at least for all x-values in some given interval.

Example 6.8.
What n guarantees that the n-th order Taylor polynomial for $\sin(x)$ centered at $a = 0$ approximates $\sin(x)$ to within 10^{-6} of the true value for $-\pi/2 < x < \pi/2$?

We want to find the n that guarantees $|R_n(x)| < 10^{-6}$ when $|x| < \pi/2$. That is,

$$\begin{aligned} &|R_n(x)| < 10^{-6} \\ \implies &\frac{\left|\frac{d}{dx^n}\Big|_{\tau_x}\sin(x)\right|}{(n+1)!}|x|^{n+1} < 10^{-6}\end{aligned}$$

Since every derivative of $\sin(x)$ is $\pm\sin(x)$ or $\pm\cos(x)$, all of which are bounded by 1, and we explicitly only care about $|x| < \pi/2$, it suffices to find the n such that

$$\frac{1}{(n+1)!}\left|\frac{\pi}{2}\right|^{n+1} < 10^{-6}.$$

The factorial makes it difficult for us to algebraically solve for n, so here we must resort to trying various values of n until the expression on the left is less than 10^{-6}. Fortunately, factorials grow *very* quickly and – while tedious to do by hand – on a computer it is extremely easy to determine this inequality is satisfied when $n \geq 13$. Thus, we are guaranteed that the n-th order Taylor approximation to $\sin(x)$, centered at 0, accurate to within one one-millionth of the true value for x in $(-\pi/2, \pi/2)$ provided n is at least 13.

6.3 Taylor series

In Section 6.2 we saw that functions could be approximated by polynomials. Notice that the error in this approximation degreases as the degree of the polynomial increases. Intuitively, the higher the degree of a polynomial, the more the polynomial's graph is allowed to bend and change directions, and so presumably is able to "hug" the graph of the original function. To be a little more precise, let's suppose that we knew *all* of the original function's derivatives were bounded by some value K at least for x's in the interval $(a - R, a + R)$. For the x's in that interval we then have $|x - a| < R$. By Theorem 6.3, the n-th order Taylor polynomial of $f(x)$ centered at $x = a$ would then satisfy the following inequality for all values of n:

$$|f(x) - T_n(x)| < \frac{K}{(n+1)!}|x-a|^{n+1} < K\frac{R^{n+1}}{(n+1)!}.$$

Now notice that for any value value of R we would have

$$\frac{R^{n+1}}{(n+1)!} = \frac{R}{1} \cdot \frac{R}{2} \cdot \frac{R}{3} \cdot \ldots \cdot \frac{R}{n-1} \cdot \frac{R}{n} \cdot \frac{R}{n+1}.$$

We would like to use this to say that the error in the approximation will go to zero as n goes to infinity. To see this, first notice that as n gets larger and larger, we eventually have $n > R$, meaning $\frac{R}{n} < 1$, $\frac{R}{n+1} < 1$, $\frac{R}{n+2} < 1$, and so on. If we let N be the smallest whole number larger than R (e.g., if R was 12.37, then we'd take $N = 13$), we could write

$$\begin{aligned}\frac{R^{n+1}}{(n+1)!} &= \frac{R}{1} \cdot \frac{R}{2} \cdot \frac{R}{3} \cdot \ldots \cdot \frac{R}{N-1} \cdot \frac{R}{N} \cdot \frac{R}{N+1} \cdot \ldots \cdot \frac{R}{n+1} \\ &= \frac{R^{N-1}}{(N-1)!} \cdot \frac{R}{N} \cdot \frac{R}{N+1} \cdot \ldots \cdot \frac{R}{n+1}\end{aligned}$$

But notice $1 > \frac{R}{N} > \frac{R}{N+1} > \cdots > \frac{R}{n_1}$, thus

$$\frac{R^{n+1}}{(n+1)!} < \frac{R^{N-1}}{(N-1)!} \cdot \left(\frac{R}{N}\right)^{n-N}$$

But $\frac{R^{(N-1)}}{(N-1)!}$ is just some constant, and since $\frac{R}{N}$ is a constant less than 1, we see that as n goes to infinity this quantity gets arbitrarily small, and so

$$\lim_{n\to\infty} |f(x) - T_n(x)| = 0.$$

This, provided that the derivative of f is bounded for all x in the interval $(a-R, a+R)$, means the Taylor polynomials approach the true value of x for each x in $(a-R, a+R)$, as the degree n increases. As we increase the degree of the polynomial to infinity, however, our polynomial becomes a power series called the "Taylor series" of the function.

That is, the ***Taylor series*** of a function $f(x)$ is the power series that appears as the limit of the Taylor polynomials for $f(x)$ as the degree goes to infinity. Using our Taylor polynomial formula this is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Just as with any other power series, we have to worry about what the interval of convergence of this series is. However, the above discussion shows that for values of x within the interval of convergence, our series *equals* $f(x)$. That is, we can think of our function as being defined by the Taylor series provided we are in the interval of convergence.

Example 6.9.
Find the Taylor series for $f(x) = e^x$ centered at $a = 0$ and determine its interval of convergence.

Notice that every derivative is $f^{(n)}(x) = e^x$ and when evaluated at $x = 0$ this gives us $f^{(n)}(0) = e^0 = 1$. Thus the Taylor series is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

To find the interval of convergence, we first apply the ratio test to find the radius of convergence:

$$\lim_{n\to\infty}\left|\frac{\left(\frac{x^{n+1}}{(n+1)!}\right)}{\left(\frac{x^n}{n!}\right)}\right| = |x|\cdot\lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Thus the series convergence everywhere – the radius of convergence is infinite – and so the interval of convergence is $(-\infty, \infty)$.

Just to recap: the example above tells us that for every value of x we have

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

When a Taylor series is centered at $a = 0$, as in the example above, it is sometimes referred to as a ***Maclaurin series***.

Example 6.10.
Find the Maclaurin series for $f(x) = \ln(1+x)$ and determine its interval of convergence.

We must compute our derivatives:

$$\begin{aligned}
\frac{d}{dx}\ln(1+x) &= \frac{1}{1+x} = (1+x)^{-1}\\
\frac{d^2}{dx^2}\ln(1+x) &= \frac{d}{dx}(1+x)^{-1} = -(1+x)^{-2}\\
\frac{d^3}{dx^3}\ln(1+x) &= \frac{d}{dx} - (1+x)^{-2} = 2(1+x)^{-3}\\
\frac{d^4}{dx^4}\ln(1+x) &= \frac{d}{dx}2(1+x)^{-3} = -6(1+x)^{-4}.
\end{aligned}$$

Notice that each time we differentiate we pull down a negative power, combining it with what we already have. This gives us factorials that alternate between positive and negative. In general,

$$\frac{d^n}{dx^n}\ln(1+x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$$

Evaluating this at $x = 0$, the n-th coefficient of our series will be

$$\frac{(-1)^{n-1}(n-1)!}{n!} = (-1)^{n-1}\frac{1}{n}$$

and so our Maclaurin series is

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n}.$$

Now we apply the ratio test to determine the radius of convergence:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}/(n+1)}{(-1)^{n-1}x^n/n}\right| &= \lim_{n\to\infty}\frac{n|x|^{n+1}}{(n+1)|x|^n}\\ &= \lim_{n\to\infty}\frac{n}{n+1}|x|\\ &= |x|\end{aligned}$$

Thus the series converges when $|x| < 1$, and so our interval of convergence contains $(-1, 1)$. Now we must check the endpoints.

At $x = -1$ our series becomes

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{(-1)^n}{n} = \sum_{n=0}^{\infty}(-1)^{2n-1}n.$$

Notice that $2n - 1$ is always odd, and -1 raised to an odd power is always -1, so the series may be written

$$\sum_{n=0}^{\infty}\frac{-1}{n} = -\sum_{n=0}^{\infty}\frac{1}{n}$$

and this diverges since it is the (negative of the) harmonic series.

At $x = 1$ our series becomes

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{1}{n}.$$

This is an alternating series and so converges by the alternating series test.

Thus the Maclaurin series of $\ln(1+x)$ is

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

and this series converges for all x in $(-1,1]$.

Remark.
Notice that, in fact, when we evaluate the Maclaurin series for $\ln(1+x)$ at $x=1$ we get the alternating harmonic series. Thus the altnerating harmonic series converges to $\ln(1+1)=\ln(2)$.

Example 6.11.
Find the Maclaurin series for $f(x)=\frac{1}{1-x}$ and determine its interval of convergence.

To determine our series, we simply need to compute the derivatives of $f(x)=\frac{1}{1-x}$, evaluate these at zero, and use the formula for the Taylor series above.

$$\begin{aligned}\frac{d}{dx}\frac{1}{1-x} &= \frac{d}{dx}(1-x)^{-1} = (1-x)^{-2}\\ \frac{d^2}{dx^2}\frac{1}{1-x} &= \frac{d}{dx}(1-x)^{-2} = 2(1-x)^{-3}\\ \frac{d^3}{dx^3}\frac{1}{1-x} &= \frac{d}{dx}2(1-x)^{-3} = 6(1-x)^{-4}.\end{aligned}$$

In general, $\frac{d^n}{dx^n}\frac{1}{1-x}=n!(1-x)^{-(n+1)}$. When evaluated at $x=0$ this simply becomes $n!$ are we see that our Maclaurin series is just

$$\sum_{n=0}^{\infty}x^n.$$

The radius of convergence can be computed using the ratio test or the root test. Here the root test is very convenient, since it tells us that we want to find the x's that satisfy $\lim_{n\to\infty} \sqrt[n]{|x^n|} < 1$. But of course,

$$\lim_{n\to\infty} \sqrt[n]{|x^n|} = \lim_{n\to\infty} \sqrt[n]{|x|^n} = |x|$$

and so we want $|x| < 1$. That is, the radius of convergence is 1 and we are guaranteed convergence inside the interval $(-1, 1)$. We still need to check the endpoints by plugging them into our series, but of course these are simply

$$\sum_{n=0}^{\infty}(-1)^n \text{ and } \sum_{n=0}^{\infty} 1^n,$$

both of which diverge. Thus the interval of convergence of our Maclaurin series is $(-1, 1)$.

Notice that the function $\frac{1}{1-x}$ is defined for all x *except* $x = 1$, but the series above only makes sense for x in the interval $(-1, 1)$. That is, we are only justified in saying

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

when x is between -1 and 1. This doesn't mean that $\frac{1}{1-x}$ can not be written as a power series for other values of x, it just means it can't be the power series $\sum_{n=0}^{\infty} x^n$ we have above.

Example 6.12.
Find the Taylor series of $\frac{1}{1-x}$ centered at $x = -2$.

As in our earlier example above, the derivatives are given by

$$\frac{d^n}{dx^n}\frac{1}{1-x} = n!(1-x)^{-(n+1)}.$$

Evaluating this at $x = -2$ we have $n!3^{-(n+1)}$. And so our power series is

$$\sum_{n=0}^{\infty} \frac{n! 3^{-(n+1)}}{n!} (x+2)^n = \sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x+2}{3} \right)^n .$$

This is of course a geometric series with $r = \frac{x+2}{3}$, and so the series converges if $\left| \frac{x+2}{3} \right| < 1$, meaning $|x+2| < 3$, or $-5 < x < 1$.

Thus

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}} \text{ for } x \text{ in } (-5, 1).$$

As the examples above show, the function $\frac{1}{1-x}$ can be written as multiple different power series with different intervals of convergence. The power series representation that you want to use in some particular problem depends on the value of x you care about.

Sometimes we can compute a Taylor or Maclaurin series for a function by manipulating a function whose series expression is already known. For example, we determined the Maclaurin series for $\frac{1}{1-x}$ is equal to $\sum_{n=0}^{\infty} x^n$ if $-1 < x < 1$. If we replace x by $3x$ in $\frac{1}{1-x}$ we get $\frac{1}{1-3x}$. Performing the same substitution in our power series gives us $\sum_{n=0}^{\infty} (3x)^n$. The original series converges if $-1 < x < 1$, and so our new series converges if $-1 < 3x < 1$, or $\frac{-1}{3} < x < \frac{1}{3}$. That is

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n \text{ if } \frac{-1}{3} < x < \frac{1}{3}.$$

Similarly, replacing x by $-x^2$ we would have

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} \left(-x^2 \right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ if } -1 < -x^2 < 1.$$

Writing $-1 < -x^2 < 1$ as $x^2 < 1$, this simply means we still have convergence in the interval $-1 < x < 1$.

Now, notice that since the expressions above are equal, at least provided $-1 < x < 1$, their antiderivatives must be equal for $-1 < x < 1$. That is,

$$\int \frac{1}{1+x^2} \, dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx.$$

However, we know that the antiderivative of $\frac{1}{1+x^2}$ is $\tan^{-1}(x) + C$, and we can integrate our power series above term-by-term to obtain

$$\tan^{-1}(x)+C = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\, dx + C = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C \text{ if } -1 < x < 1.$$

Taking the unique antiderivative whose output at $x = 0$ is zero, we then have

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ if } -1 < x < 1.$$

Notice we thus have obtained the Maclaurin series for $\tan^{-1}(x)$ *without* having to compute $\frac{d^n}{dx^n} \tan^{-1}(x)$, which would require considerable amount of work!

7 Parametric Curves and Polar Coordiantes

There is no real ending. It's just the place where you stop the story.

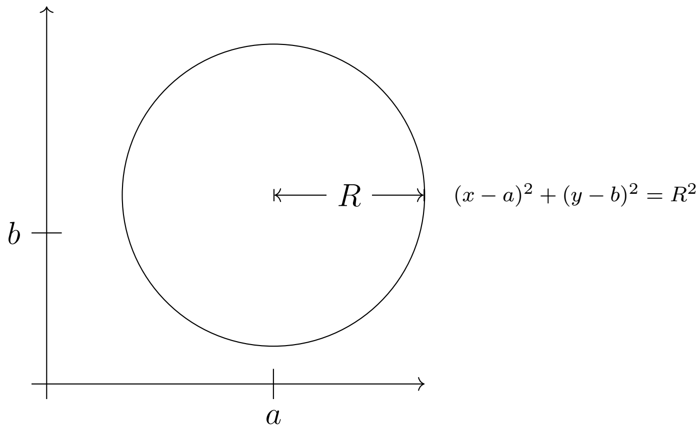
FRANK HERBERT

For the last little bit of the semester we will study two topics which at first may seem very different or unrelated to what we have studied thus far in this class: *parametric curves* and *polar coordinates*. These two topics are often very useful in more advanced courses, and discussing them here is really setting the stage for material that is covered in the third semester calculus class.

As we will see, parametric curves give us another way to describe curves in the plane, and in particular it allows us to represent curves that are not simply graphs of functions. Many applications in computer graphics rely heavily on parametric curves (and the related idea of parametric surfaces discussed in multivariable calculus). Polar coordinates are another way of representing points in the plane where we record an angle and a distance from the origin, instead of the distance to the x- and y-axes. Certain problems are often much easier to describe in polar coordinates than they are in our familiar Cartesian coordinate system.

7.1 Parametric curves

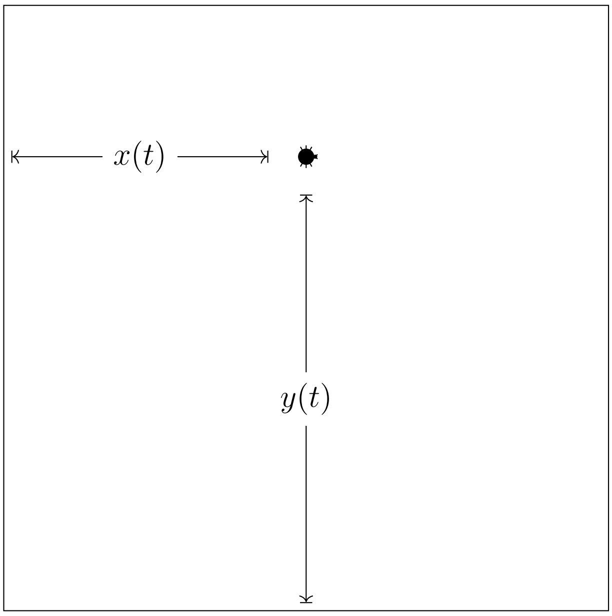
Most of the curves we have seen in this class have been graphs of functions, $y = f(x)$, but we have also seen a few curves defined implicitly by an equation, such as the a circle:



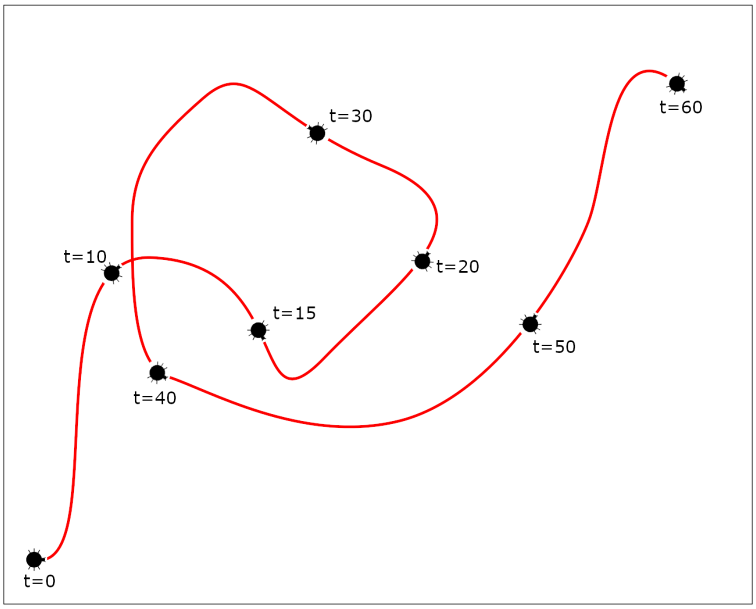
In many applications, though, it can be hard to work with curves given in such a way. Graphs are very limited since they must pass the vertical line test, and implicitly-defined equations can be difficult to use because there's no obvious way to determine all of the points. For example, how would you determine all of the points on the curve defined by $y^2 + y = x^3 + x^2$?

It is often desirable to have a way of explicitly describing every point on a curve, and this can be accomplished by "parametrizing" the curve. This means we have two functions which tell us the x-coordinates and y-coordinates of points on the curve.

Imagine, for example, you watched a bug crawling on the ceiling and at each moment in time t (say, t is the number of seconds that have elapsed since you first began observing the bug) you recorded how far the bug was from one edge of the ceiling, maybe you call this value $x(t)$, and also how far the bug was from another edge, call this $y(t)$:



If you kept track of both of these quantities, $x(t)$ and $y(t)$, for all t (or for all t in some interval, say $0 \leq t \leq 60$), you could then trace out the path the bug walked on the ceiling.

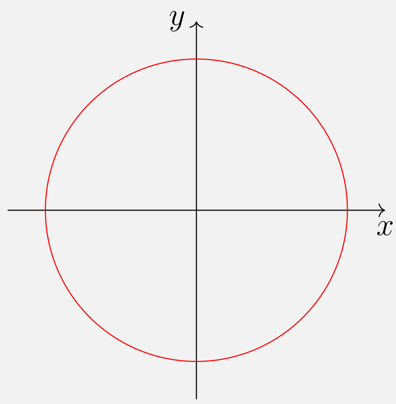


Remark.
Watching a bug walk on a ceiling like this is, supposedly, what lead René Descartes to the discovery of the Cartesian coordinate system, our familiar (x, y) coordinates for the plane.

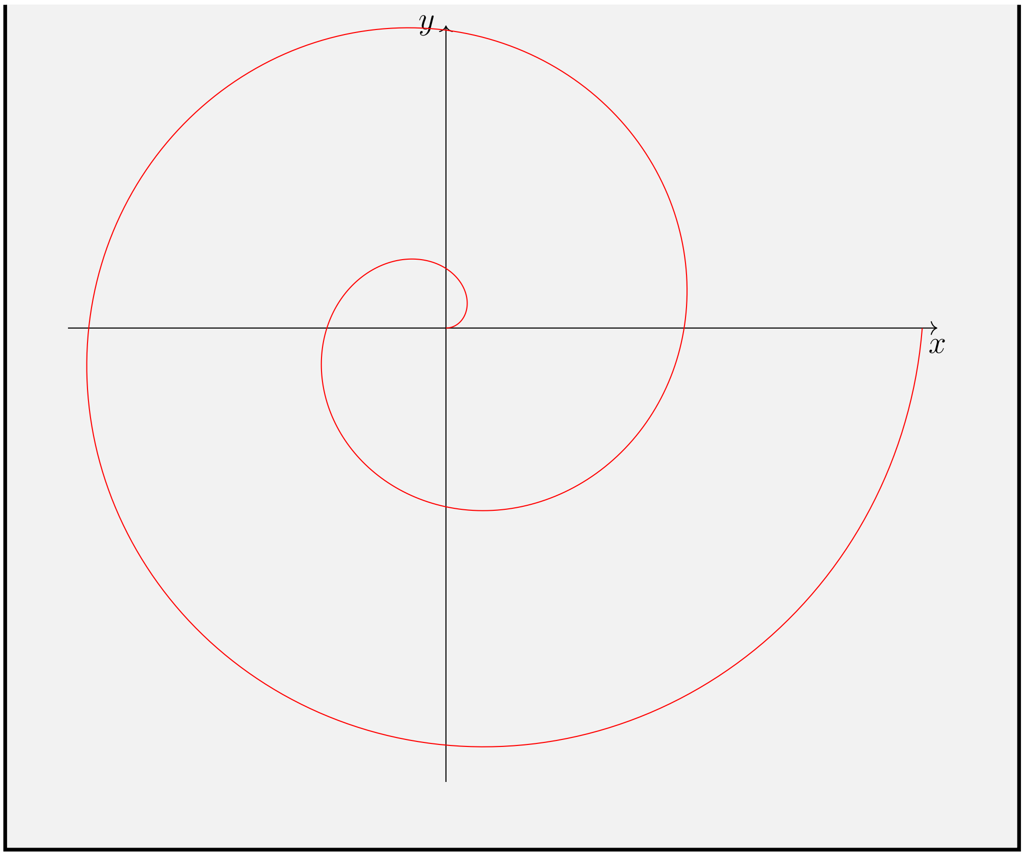
In general, a pair of functions $x(t)$ and $y(t)$ define a ***parametric curve***, which is the set of all points in the plane of the form $(x(t), y(t))$ for some choice of t.

Notice that the graph of a function $y = f(x)$ can easily be parametrized by $x(t) = t$ and $y(t) = f(t)$. However, much more interesting curves can be described. Let's start with some simple examples.

Example 7.1.
Using $x(t) = \cos(t)$ and $y(t) = \sin(t)$ for $0 \leq t \leq 2\pi$ gives a parametrization of the unit circle, essentially by the definition of sine and cosine.



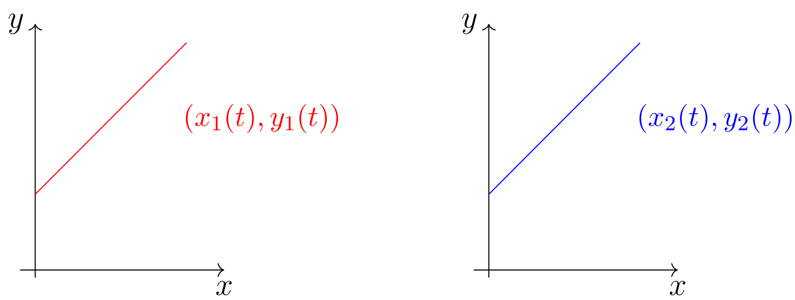
Example 7.2.
The curve parametrized by $x(t) = t\cos(t)$, $y(t) = t\sin(t)$, $0 \leq t \leq 4\pi$ is a spiral emanating from the origin which winds around twice.



Notice that there may be multiple different ways to parametrize the same curve. For example, the parametrizations

$$
\begin{aligned}
(x_1(t), y_1(t)) &= (t, 2t+1) & 0 \le t \le 1 \\
(x_2(t), y_2(t)) &= (2t, 4t+1) & 0 \le t \le 1/2
\end{aligned}
$$

both give the line segment from $(0, 1)$ to $(1, 3)$.



Yet another way of parametrizing the same line segment would be

$$(x_3(t), y_3(t)) = (\sin(t^2), 2\sin(t^2) + 1),\ 0 \le t \le \frac{\sqrt{\pi}}{2}.$$

All of these give the same set of points in the plane, but in slightly different ways. This is because a parametrization actually gives you not just a curve, but a way of moving across the curve. In particular, if we imagine a particle moving along our curve at position $(x(t), y(t))$ at time t, the particle's speed depends on which parametrization we use.

To determine the speed of the particle at some particular moment in time t, let's approximate the speed by comparing the particle's position at time t to its position at some nearby moment in time, say $t+h$. Over a change in time of h, the particle's position changes from $(x(t), y(t))$ to $(x(t+h), y(t+h))$ is distance

$$\sqrt{(x(t+h) - x(t))^2 + (y(t+h) - y(t))^2}$$

away. Since the time at it took to get from one point to the other is h, the speed is approximately

$$\frac{\sqrt{(x(t+h) - x(t))^2 + (y(t+h) - y(t))^2}}{h}.$$

This is only an approximation to the speed and we get better and better approximations by using smaller and smaller h's. Taking the limit as h goes to zero tells us that the instantaneous speed of the particle at time t is

$$\begin{aligned}
&\lim_{h\to 0} \frac{\sqrt{(x(t+h) - x(t))^2 + (y(t+h) - y(t))^2}}{h}. \\
=&\lim_{h\to 0} \sqrt{\frac{(x(t+h) - x(t))^2 + (y(t+h) - y(t))^2}{h^2}} \\
=&\lim_{h\to 0} \sqrt{\left(\frac{x(t+h) - x(t)}{h}\right)^2 + \left(\frac{y(t+h) - y(t)}{h}\right)^2} \\
=&\lim_{h\to 0} \sqrt{\left(\lim_{h\to 0}\frac{x(t+h) - x(t)}{h}\right)^2 + \left(\lim_{h\to 0}\frac{y(t+h) - y(t)}{h}\right)^2} \\
=&\sqrt{x'(t)^2 + y'(t)^2}
\end{aligned}$$

Example 7.3.
The speed of a particle moving along the spiral above, $(x(t), y(t)) = (t\cos(t), t\sin(t))$, at time t is

$$\begin{aligned}
&\sqrt{\left(\frac{d}{dt}t\cos(t)\right)^2 + \left(\frac{d}{dt}t\sin(t)\right)^2}\\
=&\sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2}\\
=&\sqrt{\cos^2(t) - 2t\cos(t)\sin(t) + t^2\sin^2(t) + \sin 62(t) + 2t\cos(t)\sin(t) + t^2\cos^2(t)}\\
=&\sqrt{1+t^2}
\end{aligned}$$

So at time $t = {}^{\pi}/_{2}$, for example, the speed is $\sqrt{1 + {}^{\pi^2}/_{4}} \approx 2.0351$; and at time $t = 4$ the speed is $\sqrt{17} \approx 4.123$.

Keeping in mind that the speed is the change of position, we can use our notion of speed to help us determine the arclength of the curve above. In particular, suppose that a curve is parametrized by $(x(t), y(t))$ with $a \le t \le b$. Let $s(\tau)$ denote the arclength of the portion of the curve parametrized with this same parametrization, but only over the interval of time $a \le t \le \tau$. Notice that $s(a) = 0$ and $s(b)$ is the arclength of the curve. The speed defined above is then exactly $s'(t)$, that is

$$s'(t) = \sqrt{x'(t)^2 + y'(t)^2},$$

since velocity is the derivative of position. Keeping in mind that $s(b)$ is the arclength of the curve and $s(a)$ is zero (by definition of the function $s(t)$ above), the fundamental theorem of calculus then tells us the following:

$$\begin{aligned}
\text{Arclength} &= s(b)\\
&= s(b) - 0\\
&= s(b) - s(a)\\
&= \int_a^b s'(t)\,dt\\
&= \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt.
\end{aligned}$$

Thus by integrating $\sqrt{x'(t)^2 + y'(t)^2}$ we obtain the arclength of the curve.

Notice in particular that if our curve is the graph of a function $y = f(x)$ for $a \leq x \leq b$, then we can use the parametrization $(x(t), y(t)) = (t, f(t))$, $a \leq t \leq b$ and the above integral becomes

$$\int_a^b \sqrt{1 + f'(t)^2}\, dt = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

and we recover the arclength formula we had earlier.

Remark.
Since $s'(t) = \sqrt{x'(t)^2 + y'(t)^2}$, the differential of the function s is

$$ds = s'(t)\, dt = \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

and so sometimes the arclength is written as simply $\int_a^b ds$.

Example 7.4.
Compute the arclength of the spiral parametrized by $(t\cos(t), t\sin(t))$ for $0 \leq t \leq 4\pi$.

As we had calculated in Example 7.3,

$$s'(t) = \sqrt{1 + t^2}$$

and so we simply need to compute the integral

$$\int_0^{4\pi} \sqrt{1 + t^2}\, dt.$$

Now, using the trig substitution $t = \tan(\theta)$, $dt = \sec^2(\theta)d\theta$ the integral becomes

$$\int_{\tan^{-1}(0)}^{\tan^{-1}(4\pi)} \sqrt{1 + \tan^2(\theta)}\sec^2(\theta)\, d\theta = \int_0^{\tan^{-1}(4\pi)} \sec^3(\theta)\, d\theta.$$

Earlier in the course we had computed that

$$\int \sec^3(\theta) d\theta = \frac{1}{2}\left(\sec(\theta)\tan(\theta) + \ln|\sec(\theta)+\tan(\theta)|\right) + C$$

(This can be computed using integration by parts with $u = \sec(\theta)$ and $dv = \sec^2(\theta)\, d\theta$.) Plugging this into our arclength calculation above, we have that the arclength is given by the expression

$$\frac{1}{2}\left(\sec(\tan^{-1}(4\pi))\tan(\tan^{-1}(4\pi)) + \ln|\sec(\tan^{-1}(4\pi)) + \tan(\tan^{-1}(4\pi))|\right) - \frac{1}{2}\left(\sec(0)\tan(0) + \right.$$

We note that if $\tan(\theta) = 4\pi$, then by simple trigonometry we can determine that

$$\sec(\theta) = \frac{1}{\sqrt{1+16\pi^2}}.$$

Now our expression above becomes

$$\frac{1}{2}\left(\sqrt{1+16\pi^2}\cdot 4\pi + \ln|\sqrt{1+16\pi^2}+4\pi|\right) - \frac{1}{2}\left(0 + \ln|1|\right)$$

Of course, the second term is simply zero, and so the first term gives the arclength of the spiral, which is approximately 80.8192.

At a given point on the curve parametrized by $(x(t), y(t))$ we may be interested in the line tangent to the curve at that point. To determine the slope, we may approximate the tangent line by the secant line through $(x(t), y(t))$ and $(x(t+h), y(t+h))$ which has slope

$$\frac{y(t+h)-y(t)}{x(t+h)-x(t)}.$$

If we multiple and divide this quantity by $1/h$ over itself, then we can write the slope as

$$\frac{\left(\frac{y(t+h)-y(t)}{h}\right)}{\left(\frac{x(t+h)-x(t)}{h}\right)}.$$

Taking the limit as h goes to zero, the slope of the tangent line is thus

$$\frac{y'(t)}{x'(t)}.$$

Example 7.5.
Determine the equation of the line tangent to the spiral parametrized by $(t\cos(t), t\sin(t))$ at $t = \pi/4$.

To write the equation of the line we need the coordinates of a point on the line and the slope. The coordinates we get by simply plugging $t = \pi/4$ into our parametrization, which gives us

$$\left((\frac{\pi}{4}\cos\left(\frac{\pi}{4}\right), \frac{\pi}{4}\sin\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi\sqrt{2}}{8}, \frac{\pi\sqrt{2}}{8}\right).$$

To determine the slope we use the formula above. First we differentiate both of $x(t)$ and $y(t)$, giving us

$$x'(t) = \cos(t) - t\sin(t)$$
$$y'(t) = \sin(t) + t\cos(t).$$

Plugging $t = \pi/4$ in gives us

$$x'(\pi/4) = \frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{8} = \frac{4\sqrt{2} - \pi\sqrt{2}}{8}$$
$$y'(\pi/4) = \frac{\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{8} = \frac{4\sqrt{2} + \pi\sqrt{2}}{8}$$

The slope of the line is then

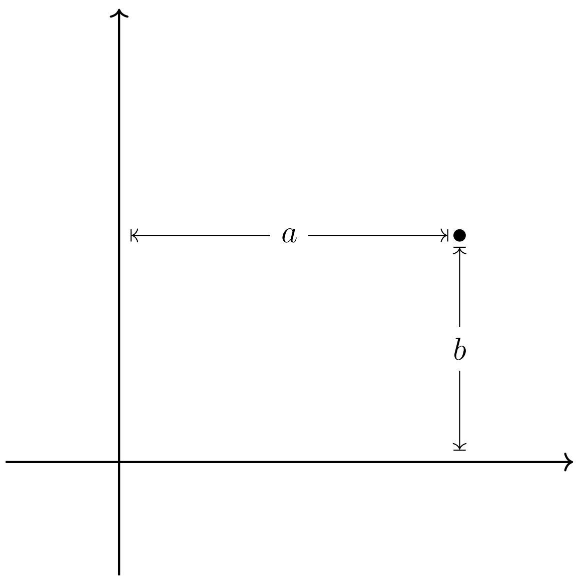
$$\frac{y'(\pi/4)}{x'(\pi/4)} = \frac{4+\pi}{4-\pi}.$$

The equation of the line is thus

$$y - \frac{\pi\sqrt{2}}{8} = \frac{4+\pi}{4-\pi}\left(x - \frac{\pi\sqrt{2}}{8}\right).$$

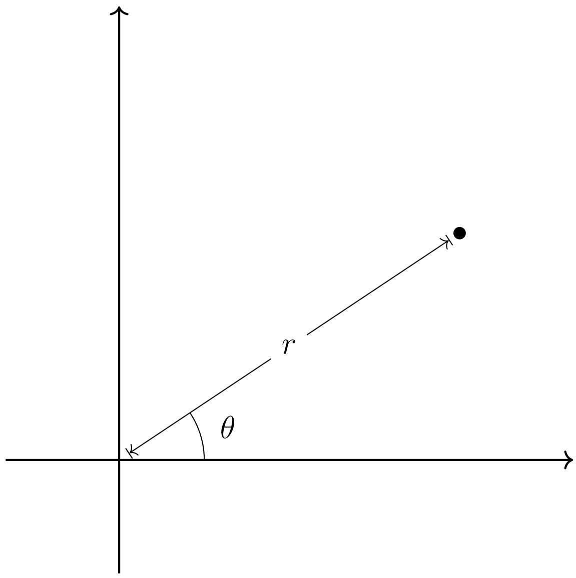
7.2 Polar Coordinates

We usually represent points in the plane in Cartesian coordinates, where we have two axes meeting at $90°$ and record how far we are to the left/right and above/below the axes by two numbers.

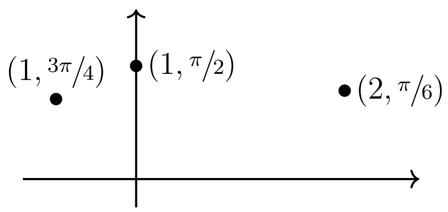


This isn't the only way we can represent points, however. There are other coordinate systems which are useful in some types of problems. We'll end our discussion of calculus this semester by introducing a new coordinate system called ***polar coordinates***.

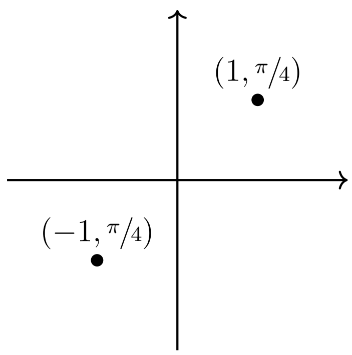
The polar coordinates of a point in the plane consist of two numbers, denoted r and θ, and are usually presented as an ordered pair (r, θ). The r represents the distance from the point to the origin, and θ is the angle from the positive x-axis. (Usually θ will be measured in radians.)



For example, each point marked below is given in these (r, θ) coordinates where r is the distance from the point to the origin, and θ is the angle measured above the positive x-axis.



By convention, if r is a negative number, we interpret this as giving the point $(-r, \theta)$, but rotated $180°$ about the origin.



It's actually very easy to go back and forth between these new polar coordinates and our familiar Cartesian coordinates.

Cartesian to polar

Given a point (x, y) in Cartesian coordinates, the corresponding point in polar coordinates is (r, θ) where

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Polar to Cartesian

Given a point (r, θ) in polar coordinates, the corresponding Cartesian coordinates are (x, y) where

$$x = r\cos(\theta) \qquad y = r\sin(\theta).$$

Exercise 7.1.
Justify the conversions above using simple trigonometry and the Pythagorean theorem.

For example, if $(x, y) = (2\sqrt{3}, 6)$, then the same point in polar coordinates is (r, θ) where

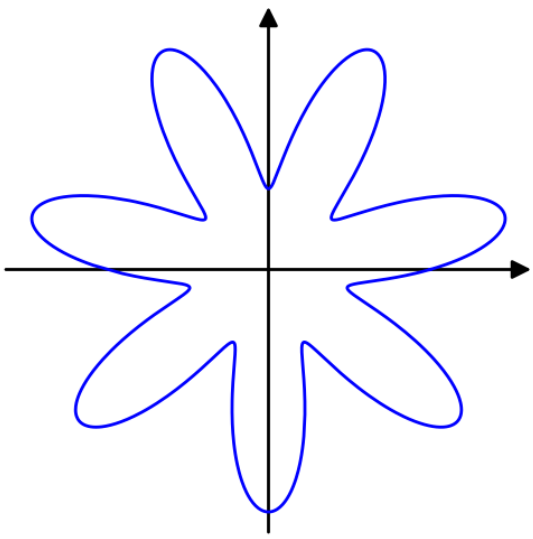
$$r = \sqrt{(2\sqrt{3})^2 + y^2} = \sqrt{4 \cdot 3 + 36} = \sqrt{48} = 4\sqrt{3}$$
$$\theta = \tan^{-1}(6/2\sqrt{3}) = \tan^{-1}(\sqrt{3}) = \pi/6$$

The point $(r, \theta) = (2, 2\pi/3)$ in polar coordinates is represented in Cartesian coordinates by (x, y) where

$$x = 2\cos(2\pi/3) = 2 \cdot 1/2 = 1$$
$$y = 2\sin(2\pi/3) = 2 \cdot \sqrt{3}/2 = \sqrt{3}.$$

So, Cartesian coordinates and polar coordinates really carry the same amount of information, they're just two different ways of represent points in the plane. However, certain objects can be represented in one coordinate system more easily than in the other coordinates.

For example, a circle of radius 3 centered at the origin in Cartesian coordinates is given by $x^2 + y^2 = 9$. In polar coordinates, though, this is simply $r = 3$. The set of (r, θ) points satisfying $r = 2 + \sin(7\theta)$ gives a curve called a *seven-petaled flower*



In Cartesian coordinates we'd have to use parametrizations to create such a curve. Notice in polar this easily parametrized by $\theta(t) = t$, $r(t) = 2+\sin(7t)$. To parametrize this in Cartesian coordinates we'd need to take advantage of our conversion formulas $x = r\cos(\theta)$, $y = r\sin(\theta)$ to obtain

$$x(t) = (2 + \sin(7t))\cos(t)$$
$$y(t) = (2 + \sin(7t))\sin(t).$$

In general, we can convert parametric curves in Cartesian to polar and vice versa fairly easily, but *can not* generally write graphs $y = f(x)$ as graphs $r = g(\theta)$, or the other way around. The flower above, for example, is given as a graph where r equals a function of θ, but can not be written as y equals a function of x since this curve fails vertical line test.

To represent a Cartesian graph in polar, for example the graph $y = x^4 - 5x^3 + 5x^2$, we can first write the curve parametrically as

$$(x(t), y(t)) = (t, t^4 - 5t^3 + 5t^2),$$

and we can now convert to polar using our formulas above,

$$r(t) = \sqrt{x(t)^2 + y(t)^2} = \sqrt{t^2 + (t^4 - 5t^3 + 5t^2)^2}$$
$$\theta(t) = \tan^{-1}\left(\frac{y(t)}{x(t)}\right) = tan^{-1}\left(t^3 = t^2 + 5t\right).$$

Though we can convert parametric curves between polar and Cartesian coordinates like this, it is sometimes helpful to do more direct conversions. For example, let's try to plot $r = 8\cos(\theta)$ by first getting a corresponding Cartesian equation. Notice that $\cos(\theta)$ is only a factor of r away from being $r\cos(\theta)$, i.e., x in Cartesian coordinates. So, let's multiply through by r to obtain

$$r^2 = 8r\cos(\theta).$$

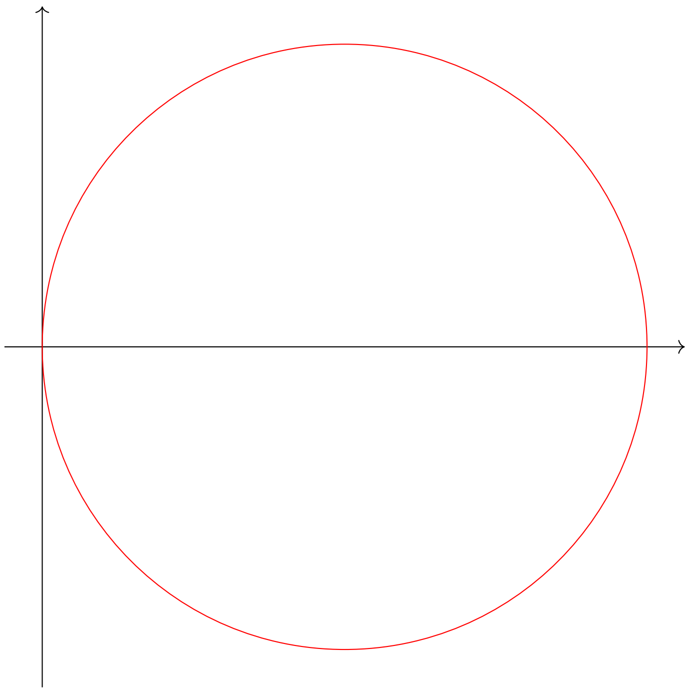
Now notice $r^2 = x^2 = y^2$ and $r\cos(\theta) = x$, so the equation becomes,

$$x^2 + y^2 = 8x$$
$$\implies x^2 - 8x + y^2 = 0$$

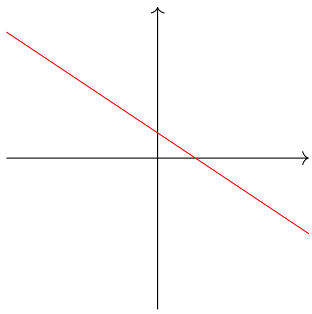
Now we can try to complete the square for $x^2 - 8x$ to obtain

$$x^2 - 8x + 16 + y^2 = 16$$
$$\implies (x - 4)^2 + y^2 = 16$$

And this is the familiar equation of a circle of radius 4 centered at $(4, 0)$.



As another example, let's graph $r = {}^{1}/_{(2\cos(\theta)+3\sin(\theta))}$ by first rewriting it in Cartesian coordinates. Notice if we cross-multiply this becomes $2r\cos(\theta)+3r\sin(\theta) = 1$, which we easily convert to Cartesian coordinates $2x+3y = 1$, or $y = \frac{-2}{3}x + \frac{1}{3}$, and thus we have a line!



Using our knowledge of parametric curves, we can find the slope of tangent lines to curves expressed in polar coordinates. In particular, recall the slope of a parametric curve $(x(t), y(t))$ is given by ${}^{y'(t)}/_{x'(t)}$. Given $r = f(\theta)$, we may parametrize this in polar as

$$(r(t), \theta(t)) = (f(t), t).$$

Now we can convert to a Cartesian parametrization by

$$(x(t), y(t)) = (f(t)\cos(t), f(t)\sin(t)).$$

Using the expression ${}^{y'(t)}/_{x'(t)}$ above, the slope becomes

$$\frac{f'(t)\sin(t) + f(t)\cos(t)}{f'(t)\cos(t) - f(t)\sin(t)}.$$

Keeping in mind $\theta = t$ is our (polar) parametrization, we have the slope of the tangent line to $r = f(\theta)$ is

$$\frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}.$$

Example 7.6.
Find the equation (in Cartesian coordinates) of the line tangent to the rose with four leaves given in polar coordinates by $r = 3\sin(2\theta)$ at the point $({}^{3\sqrt{3}}/_{2}, {}^{\pi}/_{6})$ (in polar).

The slope of the tangent line at a point on this curve is given by

$$\frac{\left(\frac{d}{d\theta}3\sin(2\theta)\right)\sin(\theta) + 3\sin(2\theta)\cos(\theta)}{\left(\frac{d}{d\theta}3\sin(2\theta)\right)\cos(\theta) - 3\sin(2\theta)\sin(\theta)} = \frac{6\cos(2\theta)\sin(\theta) + 3\sin(2\theta)\cos(\theta)}{6\cos(2\theta)\cos(\theta) - 3\sin(2\theta)\sin(\theta)}$$

At $\theta = {}^{\pi}/_{6}$ this becomes ${}^{5}/_{\sqrt{3}}$.

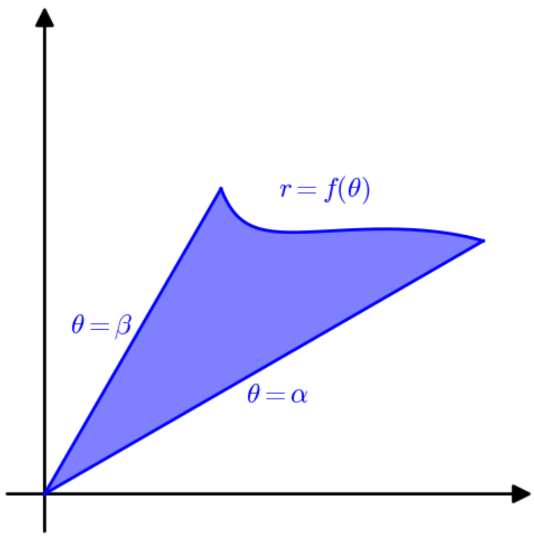
The Cartesian coordinates of our point are

$$(x, y) = \left(\frac{3\sqrt{3}}{2}\cos\left(\frac{\pi}{6}\right), \frac{3\sqrt{3}}{2}\sin\left(\frac{\pi}{6}\right)\right) = \left(\frac{9}{4}, \frac{3\sqrt{3}}{4}\right)$$

so the equation of the tangent line is

$$y - \frac{3\sqrt{3}}{4} = \frac{5}{\sqrt{3}}\left(x - \frac{9}{4}\right).$$

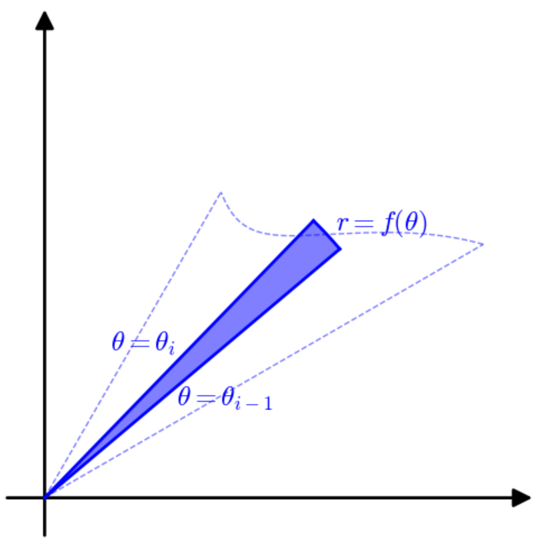
Let's end our discussion of polar coordinates by discussing how to find the area of a region enclosed by $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$.



By partitioning the interval $[\alpha, \beta]$ into pieces,

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \theta_n = \beta,$$

we will approximate our region by circular wedges between $\theta = \theta_{i-1}$ and $\theta = \theta_i$ of radius $f(\theta_i^*)$ for some θ_i^* between θ_{i-1} and θ_i.



We need to add up the areas of all sectors obtained in this way to get an approximation to the area we are interested in. Let's notice that a circle of radius R has area πR^2, and so a wedge accounting for proportion p

($0 \le p \le 1$) of the circle has area $\pi R^2 p$. If the angle of our sector is $\Delta\theta$, then our proportion of the circle is $p = {}^{\Delta\theta}/_{2\pi}$, so the area of the sector is

$$\pi R^2 \frac{\Delta\theta}{2\pi} = \frac{1}{2} R^2 \Delta\theta.$$

In our situation the angle is $\Delta\theta_i = \theta_i - \theta_{i-1}$, and the radius is $f(\theta_i^*)$, so our approximation to the area is

$$\sum_{i=1}^{n} \frac{1}{2} f\left(\theta_i^*\right) \Delta\theta_i$$

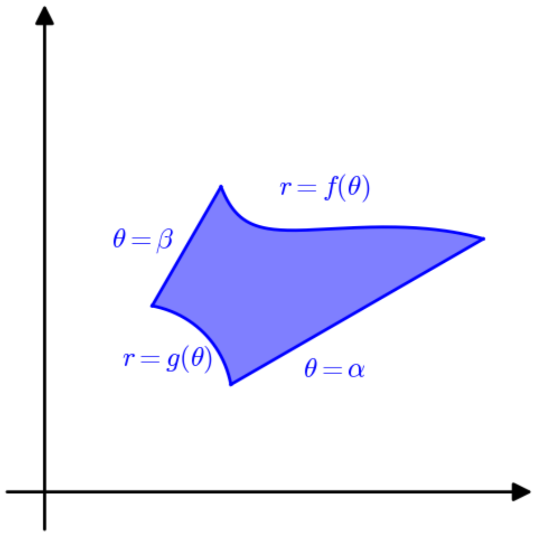
The area of our region is then

$$\int_\alpha^\beta \frac{1}{2} f(\theta)^2 \, d\theta.$$

Example 7.7.
The area of a single leaf (corresponding to $0 \le \theta \le {}^{\pi}/_{2}$) of the rose $r = 3\sin(2\theta)$ is

$$\begin{aligned}
\int_0^{\pi/2} \frac{1}{2} \left(3\sin(2\theta)\right)^2 \, d\theta &= \frac{9}{2} \int_0^{\pi/2} \sin^2(2\theta) \, d\theta \\
&= \frac{9}{2} \int_0^{\pi/2} \frac{1 - \cos(4\theta)}{2} \, d\theta \\
&= \frac{9}{4} \left(\theta - \frac{1}{4}\sin(4\theta) \right) \Big|_0^{\pi/2} \\
&= \frac{9}{4} \left(\left(\frac{\pi}{2} - \frac{1}{4}\sin(2\theta) \right) - \left(0 - \frac{1}{4}\sin(\theta) \right) \right) \\
&= \frac{9}{4} \cdot \frac{\pi}{2} = \frac{9\pi}{8}
\end{aligned}$$

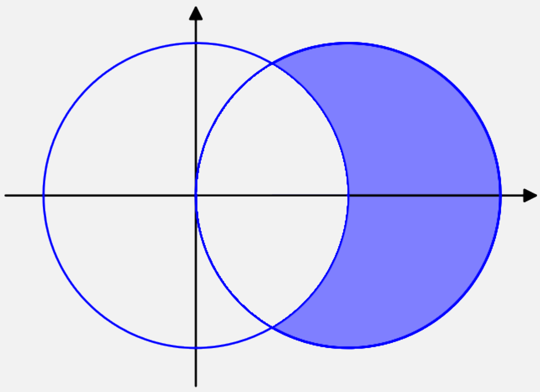
The area between two regions $r = f(\theta)$ and $r = g(\theta)$, with $f(\theta) > g(\theta)$ and $\alpha \le \theta \le \beta$, as in the figure below,



is given by

$$\frac{1}{2}\int_{\alpha}^{\beta} \left(f(\theta)^2 - g(\theta)^2\right)\, d\theta$$

Example 7.8.
Find the area of the region inside the circle $r = 2\cos(\theta)$ but outside of the circle $r = 1$.



To find our area, we first need to determine the limits of integration. We can find these by determining the θ's corresponding to the intersection points of our two circles, which is accomplished by setting the two functions equal to one another. (This is just like how you would determine where the Cartesian graphs $y = f(x)$ and $y = g(x)$ intersect by solving $f(x) = g(x)$.)

$$\begin{aligned} 2\cos(\theta) &= 1 \\ \implies \cos(\theta) &= \frac{1}{2} \\ \implies \theta &= \frac{-\pi}{3}, \frac{\pi}{3} \end{aligned}$$

Since $r = 2\cos(\theta)$ is the "outer" curve, we now simply compute

$$\begin{aligned} \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left((2\cos(\theta))^2 - 1^2\right)\, d\theta &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(4\cos^2(\theta) - 1\right)\, d\theta \\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(4 \cdot \frac{1+\cos(2\theta)}{2} - 1\right)\, d\theta \\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2 + 2\cos(2\theta) - 1)\, d\theta \\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (1 + 2\cos(2\theta))\, d\theta \\ &= \frac{1}{2}(\theta + \sin(2\theta))\Big|_{-\pi/3}^{\pi/3} \\ &= \frac{\pi}{3} + \frac{\sqrt{3}}{2} \\ &= \frac{2\pi + 3\sqrt{3}}{6}. \end{aligned}$$

Appendices

A

Detailed Review of Pre-Calculus Material

Though students in Calculus II have passed Calculus I, and so should already have a solid understanding of algebra, trigonometry, and related topics, it may sometimes be convenient to have a single source to refer to for some of this background information. This appendix is not an exhaustive review of all possible pre-calculus topics relevant to our class, but is instead focused on some of the topics which will be needed in Calculus II but which many students have forgotten the details of.

A.1 Notation

Before jumping into algebra and trig, we should discuss spend a minute to discuss notation. In mathematics in general, and in this class in particular, notation is extremely important. By *notation* we mean the standard symbols that have meaning in mathematics. Most notation you've used so frequently that you probably don't even give it a second though (e.g., using $+$ for addition or $=$ for equality), but there are a few bits of notation that students often get confused by.

Mathematical sentences and implication

The best way to think about mathematical notation is that it is the "grammar" for a mathematical expression. Just as a complete sentence in English should have a subject, a verb, and an object, so should mathematical expressions. For example, consider the equation

$$x^2 + 2 = x + 8.$$

This equation is like a "mathematical sentence" where the subject is x^2+2, the verb is "equals," and the object is $x + 8$. This sentence expresses that there is some kind of relationship between the quantities $x^2 + 2$ and $x + 8$. If we wish to find all of the solutions to this equation – all the x-values that we could plug in and satisfy the equation – then we need to manipulate this expression a little bit. Of course, the way that we will manipulate this expression in order to find the x's is to put everything on one side of the equals sign. When we subtract $x + 8$ over to the other side of the equation we have a new equation,

$$x^2 - x - 6 = 0.$$

This equation did not appear out of the blue, however; instead, it is a logical consequence of the first equation. To denote this, we often connect the first expression and the second expression with an *implication arrow* which looks like $\implies$. We do this by writing the first equation on one line, the second equation on the next line, but putting the implication arrow to the left of the second equation:

$$\begin{aligned} & x^2 + 2 = x + 8 \\ \implies & x^2 - x - 6 = 0. \end{aligned}$$

In general, when we use the implication arrow we are saying that one expression is a logical consequence of another. Put into English, the two expressions above mean "If $x^2 + 2$ is equal to $x + 8$, then it must be the case that $x^2 - x - 6$ equals 0."

As we work problems in this class you will see the implication arrow used many, many times, so it's important to understand what it means and how to use it. Again, $\implies$ means that one expression is a logical consequence of another expression.

Continuing to solve the problem we were given, we would continue to use implication arrows as follows:

$$\begin{aligned} & x^2 + 2 = x + 8 \\ \implies & x^2 - x - 6 = 0 \\ \implies & (x+2)(x-3) = 0 \\ \implies & x = -2 \text{ or } x = 3. \end{aligned}$$

Notice that we continue to string expressions together using $\implies$ saying one expression implies another which implies another which implies another. Many arguments in mathematics follow this pattern: start with some initial problem, follow by a logical consequence, follow by another logical consequence, follow by another consequence, and so on, until we arrive at our final conclusion.

Though we want to use correct mathematical notation as much as possible, simply because it is very precise, it's always okay to annotate your work with word. For example, to make it clear how the third step follows from the second, you could write "Factor" off to the side to indicate how $(x + 2)(x - 3)$ was obtained from $x^2 - x - 6$:

$$\begin{aligned} & x^2 + 2 = x + 8 \\ \implies & x^2 - x - 6 = 0 \\ \implies & (x+2)(x-3) = 0 && \text{(Factor.)} \\ \implies & x = -2 \text{ or } x = 3. \end{aligned}$$

Equality

Though we will use $\implies$ many times in this class to connect different mathematical statements together, it is not the only piece of notation we will use. In particular, it is very common for us to manipulate an expression by simply rewriting it in a different, but equivalent, way. When we do this, we use strings of equals signs. For example, in simplifying a complicated expression we may write down the original expression, then rewrite it in a slightly different (but equal) way, and connect these two expressions with an equals sign:

$$\sqrt{x^2 \cdot (x^4 + 6x^3 + 9x^2)} = \sqrt{x^2 \cdot x^2 \cdot (x^2 + 6x + 9)}$$

We may continually rewrite the expression, making it slightly simpler at each step, and putting all of these together with equals signs:

$$\sqrt{x^2 \cdot x^2 \cdot (x^2 + 6x + 9)} = \sqrt{x^4 \cdot (x+3)^2} = \sqrt{x^4}\sqrt{(x+3)^2} = x^2(x+3) = x^3+3x^2$$

When there are several expressions in such a string of equalities, we may break them up across multiple lines, just to facilitate readability.

$$\begin{aligned}\sqrt{x^2 \cdot (x^4 + 6x^3 + 9x^2)} &= \sqrt{x^2 \cdot x^2 \cdot (x^2 + 6x + 9)} \\ &= \sqrt{x^4 \cdot (x+3)^2} \\ &= \sqrt{x^4}\sqrt{(x+3)^2} \\ &= x^2(x+3) \\ &= x^3 + 3x^2\end{aligned}$$

Remark.
Implication and equality are different! We will use strings of implications and strings of equalities many times in this class, but students often confuse the two, using implication arrows where they should use equals signs and vice versa.

The rule of thumb for keeping this straight is to remember that you're constructing a sentence, and the verb you choose matters. When you want to say two things are equal, use "=," and when you want to say one thing expression implies another expression.

For example, the following **incorrect** work is often turned in by students:

$$\sqrt{x^4 \cdot (x+3)^2} \implies \sqrt{x^4}\sqrt{(x+3)^2}$$

This expression does not mean anything because the "verb" doesn't make sense. The implication arrow connects different statements by saying one is a consequence of another, but the expressions on the left and the right in the erroneous example above are not statements. It's almost like saying "If dog, then cat." That sentence does not make sense because "dog" and "cat" are individual nouns, not expressions.

A.2 Real numbers, intervals, and unions

Real numbers

A ***real number*** is a number which represents a point on the number line, and every real number can be written as an infinite decimal expansion. Sometimes we can get away with finite decimal expansions, but you can think of this as an infinite decimal expansion with infinitely-many zeros at the end. When you think of a "number," you are almost certainly thinking of a real number, but there other types of numbers.

A ***complex number*** is a number which represents a point in the plane, and every complex number can be written as $a + ib$ (this is like the point (a, b) in the plane). Because of the way multiplication is defined with complex numbers, the number i (or $0 + i1$ if you insist on writing it as $a + ib$) squares to -1: $i^2 = -1$.

We will not use complex numbers in this class, except possibly towards the very end of the semester, but I want to mention them here just so what you know that when I say something is a "real number" there are some specific types of numbers that I am not including. For our purposes in this class, you should always think of a "number" as a real number unless explicitly told otherwise.

The real numbers can be broken down into a few important families. An ***integer*** is just another word for a whole number. For example, -3, 0, 4 and $1,238$ are all integers. A positive integer is sometimes called a ***natural number***. The numbers 4, $1,238$ are natural numbers, while -3 and 0 are not. A number which can be represented as a ratio of integers, where the denominator is not zero, is called a ***rational number***. For example, $\frac{1}{2}$, 0.25 and -17 are rational numbers. A real number which can not be represented as a ratio of integers is called an ***irrational number***. This doesn't mean the number is somehow "crazy" or "bad," we just use the word *irrational* to mean the number is not a rational number, and we call fractions *rational* simply because they are ratios. Numbers like π, e, and $\sqrt{2}$ are irrational. It's usually pretty tricky to show that a given number, like π or e, is actually irrational, that there is no way whatsoever to write it as a ratio of integers.

In many (but not all!) of the examples we will do in class we will use rational numbers (or even simpler, integers) just to make the calculations a little bit easier, but we could always replace everything with irrational numbers and the math would still work out just fine.

Decimal expansions

Students often have trouble appreciating that most real numbers can not be represented by a finite decimal expansion. Every real number *can* be written as a decimal expansion, but often this requires infinitely-many digits after the decimal point. For example,

$$\pi = 3.14159...$$

where the ellipsis (the "...") indicates the numbers continue forever. If you really want the number π you have to use infinitely-many digits. Anything short of that, anything where you write only finitely-many digits, is incorrect. For example, π *is not* equal to 3.14, or 3.141, or 3.14159, or even 3.14159265358979323846. Each of these is incorrect because they're all just a little bit off. Thus if the answer to a problem is truly π, you should write the symbol π and not simply 3.14 or some other finite decimal expansion.

Some rational numbers (ratio of integers) can be represented exactly using a finite decimal expansion, such as $1/4 = 0.25$. Really this is an infinite expansion, it's just everything after the 0.25 is zeros: $1/4 = 0.25000000...$. However, not every fraction can be written with a finite decimal expansion. For example, the number $1/9$ has the infinite decimal expansion $0.1111...$ where the ones repeat forever. Thus if the answer to a problem is $1/9$, writing something like 0.1111 is actually incorrect!

In this class we often want exact answers, which means we want to use fractions and symbols as much as possible. **It is totally acceptable, and even encouraged, that you leave answers in terms of things like π, e, $\sqrt{17}$, $\sin(3/8)$, and so on instead of decimal approximations!** The one possible exception to this will be homework completed in WebWork. Some problems in WebWork will ask you for a decimal approximation, and will usually say exactly how many decimals to use. (This is just for simplicity with having WebWork grade problems.) For example, a problem on WeBWorK may have an answer of π, but explicitly ask you to give your answer to five decimals and in that situation you would want to enter 3.14159. **In all other situations, though, you should use exact answers unless explicitly told to use a decimal approximation!**

Intervals

Sometimes we will care about all of the real numbers between two given values. This is called an interval. We write $[a, b]$ to mean the collection of all real numbers x that satisfy $a \leq x \leq b$: everything between a and b, including the endpoints. If we don't want to include an endpoint, we use a parenthesis instead of a square bracket. For example, $[a, b)$ is the set of numbers x satisfying $a \leq x < b$; $(a, b]$ is the set of numbers x satisfying $a < x \leq b$; and (a, b) is the set of all numbers satisfying $a < x < b$.

Remark.
Thought most mathematical notation is standard and independent of language, intervals are one minor exception. Some European mathematicians use square brackets pointing in the other direction, where we have used parentheses. For example, a French mathematician might write $]a, b]$ and $]c, d[$ where an American mathematician would write $(a, b]$ and (c, d). This isn't really a big deal, but it is something you may see from time to time if you look at sources online or books written by a mathematician trained in Europe.

If we don't want to cap our interval off at one point, for example say we want all numbers bigger than a, we write (a, ∞) (or $[a, \infty)$ if we want to include a). Likewise, the interval containing everything less than b is $(-\infty, b)$ (or $(-\infty, b]$ if we want to include b). Notice that we will always put a parenthesis around ∞ or $-\infty$: infinity is not a real number, so we can't include it!

Sometimes we will want to represent the collection of all real numbers as an interval, and we can do this as $(-\infty, \infty)$. This is the set of all real numbers that are bigger than $-\infty$ but less than ∞, which is everything.

Unions

On odd occasion we will want to take two intervals and glue them together. We can do this by taking the *union* of two intervals, which is denoted by the symbol $\cup$. For example, $(0,1] \cup (2,3)$ consists of all the numbers which are in $(0,1]$ *or* $(2,3)$. We literally just glue the two intervals together to get a new collection of numbers.

A.3 Functions, graphs, and inverse functions

Functions

Recall that a ***function*** is just a rule for turning some real numbers into other real numbers: it's just a way of transforming numbers into other (usually different) numbers. We often specify a function by writing something like

$$f(x) = \text{how to convert } x \text{ into something else.}$$

For example, if we write

$$f(x) = \sqrt{x^2 + 7}$$

then we are saying that f is the function that takes a given value of x, squares it, adds seven, then takes the square root. For example, $f(-3) = \sqrt{(-3)^2 + 7} = \sqrt{16} = 4$.

Functions can do all sorts of weird and crazy things to transform their inputs into outputs: in fact there are some functions where it's not possible to write down a simple rule like $\sqrt{x^2 + 7}$! The only requirement that a function has to satisfy is that it takes inputs and converts them into a single output. If f is some function where $f(3) = 19$, then each and every time we plug 3 into f we have to get 19 back out.

The collection of all the values we can plug into a function is called the function's ***domain***. For this class, being able to plug a number into a function means we can evaluate the function and get a real number.

For example, the domain of the function $f(x) = \sqrt{x+2}$ is $[-2, \infty)$. Everything that's bigger than or equal to -2 can be plugged into the function and we get back a real number, but anything smaller than -2 would give us a complex number, and for this class we will avoid complex numbers *unless explicitly stated otherwise*.

As another example, the domain of the function $\frac{1}{x}$ is $(-\infty, 0) \cup (0, \infty)$: the function is defined everywhere except 0: we can't divide by 0 and get a real number. Notice that we cut out the single number 0 from the real line by gluing everything to the left of 0, $(-\infty, 0)$, and everything to the right of 0, $(0, \infty)$, with a union.

Graphs

It is often convenient to visualize a function by considering its graph, which is the set of all (x, y) points in the plane that satisfy the equation $y = f(x)$. For example, with the function $f(x) = \sqrt{x^2 + 7}$, the point $(-3, 4)$ is on the graph since $4 = f(-3)$, but the point $(2, 0)$ is not on the graph as $0 \neq f(2) = \sqrt{11}$.

The requirement that a function converts an input into a single output means that if we were to draw a vertical line in the plane, whenever that line intersects the graph it can do so only in one point. (It may miss the graph sometimes if we draw the line over a point which isn't in the domain, but if a vertical line intersects the graph of a function, it can only intersect it once.)

Inverse functions

Functions take inputs and convert them into outputs. In terms of graphs we usually use the x-coordinate to represent the input to the function and the y-coordinate to represent the output. (Although there are times when we may reverse the roles of the x- and y-coordinates.) One question we are sometimes interested in is whether we can reverse the process, taking an output from a function and determining the corresponding input.

In general this question does not have a unique answer: there may be multiple different inputs which give the same output. For example, consider the function $f(x) = x^2$. Both inputs $x = -2$ and $x = 2$ have the same output, $f(\pm 2) = 4$. Sometimes, however, there is a unique input for each output. For instance the function $f(x) = x^3$ has this property; there is exactly one input, namely $x = 2$, which has the output of 8. This isn't true simply for the output 8, however, it's true for *all* possible outputs. When a function has this special property that for each possible output there is exactly one corresponding input, we say the function is ***invertible*** or ***one-to-one***.

In terms of the graph of a function, we can easily determine if a function is one-to-one or not by seeing if it passes the ***horizontal line test***: If every horizontal line in the plane crosses the graph $y = f(x)$ *at most* once, then the function is invertible. This condition exactly says that for each possible output (this determines the horizontal line, $y = c$ for the output

c) there is at most one possible input (the x-coordinate of the intersection point of the line $y = c$ and the graph $y = f(x)$) we can plug into the function to obtain that output. That is, if $y = c$ intersects $y = f(x)$ at the point (x_0, c), then that exactly means $f(x_0) = c$. If this is the *only* place the line and the graph intersect, then that means there is only one input we can plug into the function, namely x_0, to get the output c.

A.4 Algebra

When I first took calculus, the professor stated at the start of the semester that "no one fails calculus because they can't do calculus, they fail because they can't do algebra," and I think there is a lot of truth to this statement. In particular, calculus instructors instructors do not expect their students to already know calculus and so they take the time to carefully explain calculus material, *however* they do expect students to have a solid understanding of algebra and so don't often spend much time reviewing algebra. This portion of the notes is meant to be a quick reminder of some of the algebra that will be useful in the class, but which students may have forgotten. This is not a comprehensive review of all algebra, though. If you need a more thorough review of algebra, you may want to consider the following websites:

1. The online algebra review written by James Stewart,
   `http://www.stewartcalculus.com/data/default/upfiles/AlgebraReview.pdf`
2. Paul's Online Algebra Notes,
   `http://tutorial.math.lamar.edu/Classes/Alg/Alg.aspx`
3. The Wikibook for Algebra,
   `https://en.wikibooks.org/wiki/Algebra`
4. The KhanAcademy sites for
   Algebra I, `https://www.khanacademy.org/math/algebra`
   Algebra II, `https://www.khanacademy.org/math/algebra2`

Polynomials

A ***polynomial*** is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the a_0, a_1, a_2, ..., a_{n-1} and a_n are all real numbers. If $a_n \neq 0$, then we say the polynomial has ***degree*** n. The a_0, a_1, ..., a_n are called the ***coefficients*** of the polynomial.

For example, $7x^3 + 4x^2 - 2$ is a polynomial of degree three. Here $a_0 = -2$, $a_1 = 0$, $a_2 = 4$ and $a_3 = 7$. As another example, $0x^5 - 4x^2 + x$ is a polynomial of degree 2, since x^2 is the term with the largest power, but a non-zero coefficient.

There's nothing really special about the x that appears in the polynomial above; we can just as well have polynomials in y like $-3y^3 + 2y - 4$, or z like $9z + 2$. We can even have polynomials with both x's and y's, such as $x^2 + y^2$, although in our class we will primarily be interested in polynomials of one variable.

There's nothing special about having integer coefficients, by the way: $\pi x^3 + ex^2 - \ln(13)x + \sqrt{8}$ is a perfectly legitimate polynomial.

Notice that every polynomial defines a function: if we write

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

then we have a perfectly legitimate function: a way of converting an input x into an output $f(x)$.

There is some common vocabulary that is associated with polynomials: polynomials of degree 1 are called ***linear***, polynomials of degree 2 are called ***quadratic***, polynomials of degree 3 are called ***cubic***, degree 4 polynomials are ***quartic***, degree 5 polynomials are ***quintic***, and so on. We will see polynomials of all sorts of different degrees, so just be aware that if I say something is "cubic" that means it's a polynomial of degree 3, or something that's "quadratic" is a polynomial of degree 2.

Factoring

We can multiply two polynomials together by distributing. For example, we can multiply

$$(x+3) \cdot (4x^2 - x + 1)$$

by distributing the $x + 3$ to each term of $4x^2 - x + 1$,

$$(x+3) \cdot (4x^2 - x + 1) = 4x^2(x+3) - x(x+3) + 1 \cdot (x+3),$$

then further distributing each term into its copy of $x + 3$,

$$\begin{aligned}(x+3) \cdot (4x^2 - x + 1) &= 4x^2(x+3) - x(x+3) + 1 \cdot (x+3) \\ &= 4x^3 + 12x^2 - x^2 - 3x + x + 3\end{aligned}$$

and finally combining like terms to obtain

$$(x+3)\cdot(4x^2-x+1)=4x^3+11x^2-2x+3.$$

Sometimes (for a reason we will explain in just a moment) it is convenient to go backwards, and break up a given polynomial into a product of two or more simpler polynomials.

For example, we can factor $2x^2+5x+1$ as $(2x+3)(x+1)$: we have broken $2x^2+5x+1$ up into a product of two polynomials, so we have factored it. The simpler polynomials we get when we factor are called (naturally enough) *factors* of the original polynomial.

Factoring is *extremely* useful in calculus. We will see examples through the semester which at first glance may look difficult, but become much easier after factoring.

Here's a simple example: suppose we wanted to figure out what the graph of $f(x)=\frac{x^2-4}{x-2}$ looked like *without* using a computer or graphing calculator. This doesn't look so easy at first, but becomes a lot easier if we factor x^2-4 as $(x+2)(x-2)$. We can now write

$$\frac{x^2-4}{x-2}=\frac{(x+2)(x-2)}{x-2}=x+2$$

and you might say the graph is a line with a slope of 1 and y-intercept of 2. Here factoring let's us figure that out pretty easily, and little tricks and simplifications that come by factoring like this are very common in calculus.

(There's one slight problem with the graph described above, however. The graph is indeed a line as described, but it has a hole in it at $x=2$. Keep in mind we're graphing $y=f(x)$, but 2 is not in the domain of the function: the domain is $(-\infty,2)\cup(2,\infty)$, so $(2,4)$ is actually not a point on the graph.)

It turns out that factoring and finding the roots of a polynomial go hand-in-hand because of the following fact: if $x=a$ is a root of a polynomial $f(x)$, then $x-a$ is a factor of $f(x)$. That is, if $f(x)$ is a polynomial and $f(a)=0$, then f will factor as $f(x)=(x-a)\cdot g(x)$ for some polynomial $g(x)$. Furthermore, we can find the polynomial $g(x)$ by using polynomial long division.

For example, consider the polynomial $f(x)=3x^3-6x^2+x-2$ and say we want to factor it. At first glance this doesn't seem very easy at all, but if somehow knew that $x=2$ was a factor (e.g., maybe we graphed $y=f(x)$ or had some additional information that clued us in to this fact), then we would know $f(x) = (x-2) \cdot g(x)$ for some polynomial $g(x)$. We could divide the $x-2$ over, however, to solve for $g(x)$:

$$\frac{f(x)}{x-2} = g(x).$$

This is something we can actually do with polynomial long division:

$$\begin{array}{r|l} & 3x^2 \qquad + 1 \\ x-2 & 3x^3 - 6x^2 + x - 2 \\ & -3x^3 + 6x^2 \\ \hline & \qquad\qquad x - 2 \\ & \qquad\quad -x + 2 \\ \hline & \qquad\qquad\quad 0 \end{array}$$

Thus $3x^2 + 1$ is our $g(x)$, and we can factor our original polynomial,

$$3x^3 - 6x^2 + x - 2 = (x-2)(3x^2+1)$$

If you notice that $3x^2 + 1$ is a parabola which sits above the x-axis, then you know that $3x^2 + 1$ has no (real) roots. This implies that we can't factor $3x^2 + 1$ anymore: if we could factor it, say as $(x-a)(x-b)$, then a and b would both be roots of $3x^2 + 1$, but $3x^2 + 1$ has no roots, thus we can't factor it.

When this happens, when a polynomial can not be factored any further, we say the polynomial is *irreducible*. The polynomial $3x^2 + 1$, for example, is irreducible.

The trick above where we determined $3x^2 + 1$ couldn't be irreducible because it didn't have any roots is only useful for quadratics, by the way. The polynomial $2x^4 + 5x^2 + 3$, for example, doesn't have any roots as graphing it will show you. However, it does factor as $(x^2+1)(2x^2+3)$. Each of these factors is irreducible, though, since they are both quadratics and can't have any roots (if the factors had roots, then so would the original polynomial), so we can't break this down any further.

In general it is difficult to determine whether a polynomial is irreducible or not, but despite that it is important to know the basics of factoring.

Difference of perfect squares

One of the simplest types of polynomials to factor is one of the form

$$x^2 - a^2,$$

which is often called a ***difference of perfect squares***. It is easy to verify by FOIL-ing that if you multiply $x + a$ and $x - a$, you will get back $x^2 - a^2$. That is, we can factor $x^2 - a^2$ as

$$x^2 - a^2 = (x + a)(x - a).$$

This simple trick will come up many times in the class, so you should get in the habit of immediately recognizing $x^2 - a^2$ as factoring into $(x + a)(x - a)$.

Notice that we can apply the difference of perfect squares trick even if we don't have a variable. For example, if a and b are constants, we can write $a^2 - b^2$ as $(a + b)(a - b)$. This is sometimes helpful in calculus because we may have a problem which will look difficult at first, but then greatly simplify after factoring in this way.

Completing the square and the quadratic formula

Notice that an expression of the form $(x + a)^2$ will always FOIL into $x^2 + 2ax + a^2$. This simple observation can be reversed to say that $x^2 + 2ax + a^2$ factors as $(x + a)^2$. In some problems we can use this to rewrite an expression or an equation as something more useful. For example, suppose we wanted to find all of the values of x that solved the equation

$$x^2 + 6x + 9 = 0.$$

If we recognize this has the above form (with $a = 3$), then we would see that the polynomial factors and our equation becomes

$$(x + 3)^2 = 0.$$

At this point it's clear the only solutions are $x = -3$. For a slightly more involved example, suppose we wished to solve

$$x^2 + 6x = 7.$$

Your first guess might be to write $x(x + 6) = 7$. While this equation is certainly true, it's not actually helpful. Now we have to find a value of x so that x times $x + 6$ equals seven, and that's not so obvious. If the left-hand side were actually a square, though, then we could just take a square root. So, what we will do is make the left-hand side a square by recognizing that it *almost* has the form $x^2 + 2ax + a^2$ (with $a = 3$), except we're missing the $+a^2$ term which would be 9 in our example. We're missing 9, so let's just put a 9 in. When we do that, however, we're modifying the equation and if we modify one side of the equation we have to modify the other side of the equation in the same way to still have an equality.

$$\begin{aligned} & x^2+6x=7 \\ \Longrightarrow\ & x^2+6x+9=7+9 \end{aligned}$$

We can now factor the left-hand side as $(x+3)^2$, and we may as well simplify the right-hand side to 16:

$$\begin{aligned} & x^2+6x=7 \\ \Longrightarrow\ & x^2+6x+9=7+9 \\ \Longrightarrow\ & (x+3)^2=16 \end{aligned}$$

Now we are basically home-free. Let's just take the square-root of both sides, and since the square roots of 16 are ± 4 we have

$$x+3=\pm 4.$$

Now we can just move the -3 to the other side of the equation to obtain

$$x=-3\pm 4.$$

Keep in mind this is really short-hand for two equations, $x=-3+4$ and $x=-3-4$. Of course these really just tell us the possible values of x are $x=1$ and $x=-7$, and we can very easily verify that these satisfy our original equation, $x^2+6x=7$:

$$\begin{aligned} & 1^2+6\cdot 1=1+6=7 \\ & (-7)^2+6\cdot(-7)=49-36=7. \end{aligned}$$

This process of adding something to both sides of our equation above so that we could write one side as a square is often referred to as ***completing the square***. Notice that in general if we want to complete the square using x^2+Bx, the constant B plays the role of our $2a$ from above and so when we add a^2, what we're really doing is adding half of B squared. (E.g., half of 6 is 3 and we added $3^2=9$ above.)

$$\begin{aligned} x^2+Bx &= x^2+Bx+\left(\frac{B}{2}\right)^2-\left(\frac{B}{2}\right)^2 \\ &= x^2+Bx+\frac{B^2}{4}-\frac{B^2}{4} \\ &= \left(x+\frac{B}{2}\right)^2-\frac{B^2}{4}. \end{aligned}$$

The procedure outlined above may not have been your first choice for solving the equation $x^2 + 6x = 7$: you may instead wanted to write the equation as $x^2 + 6x - 7 = 0$ and then apply the quadratic formula. That is a totally fine and valid thing to do, but it's worth pointing out that the quadratic formula is really just completing the square in disguise.

In particular, let's try to solve the quadratic

$$ax^2 + bx + c = 0$$

but by completing the square, and we'll see that we recover the familiar quadratic formula. The first thing we need to do is divide out the a that appears as the coefficient in the x^2, since our completing the square procedure requires the coefficient on the highest degree term to be a 1:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ \implies x^2 + \frac{b}{a}x + \frac{c}{a} &= 0. \end{aligned}$$

Now we can move c/a to the right-hand side to obtain

$$x^2 + \frac{b}{a}x = \frac{-c}{a}.$$

We will complete the square by adding half of the middle term's coefficient, b/a, squared to each side of the equation, obtaining

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2.$$

Now we're set up to factor the right-hand side as

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-c}{a} + \frac{b^2}{4a^2}.$$

Let's go ahead and simplify the right-hand side by getting a common denominator as follows:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Now we take the square-root of each side to obtain

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

Finally we will move the $^b/_{2a}$ from the left-hand side to the right-hand side and simplify, giving us the familiar quadratic formula:

$$\begin{aligned} x &= \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} \\ &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \end{aligned}$$

Tricks with fractions

While everyone has certainly had plenty of experience with fractions at this point in their mathematical careers, some tricks can be easy to forget if you don't use them regularly, so it's worth quickly recalling some basic facts.

The first fact is that we can multiply fractions by just multiplying their numerators and denominators separately. For example,

$$\frac{2}{3} \cdot \frac{5}{4} = \frac{2 \cdot 5}{3 \cdot 4} = \frac{10}{12}.$$

The second fact is that any number other than zero divided by itself is 1:

$$1 = \frac{2}{2} = \frac{3}{3} = \frac{-17}{-17} = \frac{\pi}{\pi}.$$

This rule *does not* apply to $\frac{0}{0}$ because division by zero is undefined.

When adding fractions, we *must* have a common denominator. For example, in order to add $^2/_3$ and $^5/_4$ we must rewrite both fractions so they have the same denominator. A very reasonable question to ask at this point is why do we need to do this: why is a common denominator necessary? You should think of the denominator as being a type of unit, such as inches or yards. You can't just add eighteen inches and two yards together together and call it twenty of something: you have to either convert your inches to yards, or your yards to inches, or both yards and inches to some common unit like feet. It's the same with fractions. If the units are "thirds" and "fourths," as in the fractions above, we have to get a common unit.

One easy way to get a common denominator, though not the only way, is to just multiply the denominators together. Since our denominators are 3 and 4 above, we'll get a common denominator of $3 \cdot 4 = 12$. We can't just replace the denominators with 12 though, we also need to change the numerators. In terms of our "eighteen inches plus two yards" idea above, it's analogous to how we have to convert eighteen inches (for example) into 1.5 feet, we can't just magically decide eighteen inches is eighteen feet because they're very different quantities!

Now the question is how do we appropriately change the numerators? To change our numerators and denominators at the same time, we'll just multiply by 1, but we'll write 1 as the denominator of the *other* fraction over itself. For example,

$$\begin{aligned} &\frac{2}{3}+\frac{5}{4} \\ =&\frac{2}{3}\cdot 1+\frac{5}{4}\cdot 1 \\ =&\frac{2}{3}\cdot\frac{4}{4}+\frac{5}{4}\cdot\frac{3}{3}. \end{aligned}$$

We can now just multiply these fractions together by multiplying numerators and multiplying denominators:

$$\begin{aligned} &\frac{2}{3}\cdot\frac{4}{4}+\frac{5}{4}\cdot\frac{3}{3} \\ =&\frac{8}{12}+\frac{15}{12}. \end{aligned}$$

Now that our fractions have a common denominator (i.e., they're in the same "units") we can just add the numerators together:

$$\frac{2}{3}+\frac{5}{4}=\frac{8}{12}+\frac{15}{12}=\frac{23}{12}.$$

Sometimes we will need to deal with "compound fractions," which are fractions whose numerator and/or denominator are themselves fractions, such as

$$\frac{\left(\frac{3}{7}\right)}{\left(\frac{5}{2}\right)}.$$

It is often convenient to rewrite compound fractions as "simple fractions" (i.e., the numerator and denominator are just integers, not fractions). This is easily accomplished in two steps. First we will multiply the fraction by 1, but written in a special way, to remove the fraction in the denominator:

$$\frac{\left(\frac{3}{7}\right)}{\left(\frac{5}{2}\right)}\cdot\frac{2}{2}.$$

Notice that we can now multiply the numerators by 2 and the denominators by 2 and in this example, this kills off the 2 in the original denominator:

$$\frac{\left(\frac{3}{7}\right)}{\left(\frac{5}{2}\right)}\cdot\frac{2}{2}=\frac{\left(\frac{6}{7}\right)}{5}.$$

Now we can get rid of the denominator in the numerator $^6/_7$ in the same way, by multiplying by 1 written as $\frac{7}{7}$:

$$\frac{\left(\frac{6}{7}\right)}{5} = \frac{\left(\frac{6}{7}\right)}{5} \cdot \frac{7}{7} = \frac{6}{35}$$

Another way of accomplishing this is to notice that when we have a compound fraction with whose denominator is 1, such as

$$\frac{1}{\left(\frac{5}{2}\right)},$$

this is equal to a fraction which is simply the denominator, but with the roles of the numerator and denominator reversed. The reason for this is again we can just multiply by 1, but with 1 written in a special way:

$$\frac{1}{\left(\frac{5}{2}\right)} = \frac{1}{\left(\frac{5}{2}\right)} \cdot \frac{\left(\frac{2}{5}\right)}{\left(\frac{2}{5}\right)} == \frac{\left(\frac{2}{5}\right)}{1} = \frac{2}{5}.$$

We could then calculate our compound fraction above by thinking of the division by $^5/_2$ as multiplication by one over $^5/_2$:

$$\frac{\left(\frac{3}{7}\right)}{\left(\frac{5}{2}\right)} = \frac{3}{7} \cdot \frac{1}{\left(\frac{5}{2}\right)} = \frac{3}{7} \cdot \frac{2}{5} = \frac{6}{35}.$$

B

Detailed Review of Calculus I

Here we collect some odds and ends from a typical Calculus I course which are necessary for Calculus II. The title of this appendix, "Detailed Review of Calculus I," is slightly misleading in that this is not a comprehensive review of all Calculus I material. Rather, this appendix fills in some of the details about topics from Calculus I that students in Calculus II will need from time to time, but may have forgotten or never fully understood.

B.1 Summations and sigma notation

A large portion of Calculus II deals with integrals and with series, both of which are essentially special types of infinite summations. To deal with summations, whether infinite or finite, in a concise way we often use "sigma notation," named for the capital Greek letter sigma, Σ, that appears.

Suppose that f is a function and we want to sum the function evaluated at several different integers. For example, we may want to evaluate

$$f(1) + f(2) + f(3) + \cdots + f(50).$$

We can use the following ***sigma notation*** to express this sum. We write a large capital Greek Σ, and below the Σ we say where we want the sum to start, above the Σ we say where we want the sum to stop, and to the right of Σ we say what we're summing up.

For example,

$$\sum_{i=1}^{50} f(i) = f(1) + f(2) + \cdots f(3)$$
$$\sum_{i=-3}^{7} f(i) = f(-3) + f(-2) + f(-1) + f(0) + \cdots + f(7)$$

This is simply a convenient way to write out certain sums.

The i that appears in such a summation is called the ***index***, and is reallly just a place-holder for numbers that start at the value indicated below the Σ and increase by one up to and including the number above the Σ. For our purposes these starting and stopping points will always be integers (whole numbers), and the index increases by one for each term in the sum.

We will typically use i as the index variable, but there's nothing magical about this choice. We could just as well use j or x or ☺; the choice of index variable is immaterial. For example, the expressions in sigma notation below all represent the exact same sum:

$$\sum_{i=1}^{5} f(i) = \sum_{j=1}^{5} f(j) = \sum_{x=1}^{5} f(x) = \sum_{☺=1}^{5} f(☺)$$
$$= f(1) + f(2) + f(3) + f(4) + f(5).$$

Sometimes we the function we're trying to sum up won't have a given name: we may simply give an expression in place of $f(i)$:

$$\sum_{i=0}^{5} i^2 = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$\sum_{i=-2}^{2} (2i-1) = (2\cdot(-2)-1)) + (2\cdot(-1)-1) + (2\cdot 0-1) + (2\cdot 1-1) + (2\cdot 2-1)$$

Remark.
A small word about notation: when using sigma notation, we often put parentheses around the expression describing the terms of the sum if these terms are made up of other terms (i.e., contain additions or subtractions). This aids in clarity. If the parentheses in the expression above were removed, for example, then that "minus one" that appears at the end would appear after the terms are added up:

$$\sum_{i=-2}^{2} 2i - 1 = 2\cdot(-2) + 2\cdot(-1) + 2\cdot 0 + 2\cdot 1 + 2\cdot 2 - 1$$

which is not the same as the earlier sum. Thus if we want each term to have a "minus one," then we need the parentheses.

Properties of sums

Let's go ahead and notice a few simple properties of sums written in this Σ-notation:

Theorem B.1.
For any two functions f and g, and for any constant k we have

$$\sum_{i=a}^{b}(f(i)+g(i)) = \sum_{i=a}^{b} f(i) + \sum_{i=a}^{b} g(i)$$
$$\sum_{i=a}^{b} kf(i) = k\sum_{i=a}^{b} f(i)$$

Proof.

$$\begin{aligned}\sum_{i=a}^{b}(f(i)+g(i)) &= f(a)+g(a)+f(a+1)+g(a+1)+\cdots+f(b)+g(b)\\ &= f(a)+f(a+1)+\cdots f(b)+g(a)+g(a+1)+\cdots g(b)\\ &= \sum_{i=a}^{b} f(i)+\sum_{i=a}^{b} g(i)\end{aligned}$$

$$\begin{aligned}\sum_{i=a}^{b} kf(i) &= kf(a)+kf(a+1)+\cdots+kf(b)\\ &= k\left(f(a)+f(a+1)+\cdots+f(b)\right)\\ &= k\sum_{i=a}^{b} f(a).\end{aligned}$$

□

Notice that the above properties are really just basic properties of sums that you learned in elementary school (you can rearrange the terms in a sum, and you can factor out any constant appearing in each term), they're just written in sigma notation.

The "constant" k in the above simply can not depend on i, but it could depend on some other quantity. For example,

$$\lim_{n\to\infty}\sum_{i=1}^{n} n^2 f(i) = \lim_{n\to\infty} n^2 \sum_{i=1}^{n} f(i).$$

Some helpful formulas

We will be dealing with sums like

$$\sum_{i=1}^{n} 1, \quad \sum_{i=1}^{n} i, \quad \text{and} \quad \sum_{i=1}^{n} i^2$$

a lot, so it would be helpful if we had some formula for calculating these sums.

Theorem B.2.

$$\sum_{i=1}^{n} 1 = n$$

Proof.

$$\sum_{i=1}^{n} 1 = \underbrace{1+1+\cdots+1}_{n \text{ times}} = n.$$

□

That is, we are simply adding up the number 1 a total of n times, and so we get n.

Theorem B.3.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof.
Let $S = 1 + 2 + \cdots + n$. Notice we can write this as $S = n + (n-1) + \cdots + 1$. Adding S to itself we have

$$2S = \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n \text{ times}} = n(n+1)$$

and so dividing by 2 gives the sum. □

The proof above is supposedly due to Carl Gauss. Carl Gauss was a German mathematician who lived in the 19th century, and is often considered to be one of the greatest mathematicians to ever live. Gauss was a prodigy and is said to have discovered the above formula when he was eight years old. The story goes that the young Gauss was being very rambunctious in school one day and so his teacher, knowing Gauss enjoyed math, decided to give Gauss some busy work to occupy him. She asked Gauss to add up the numbers $1 + 2 + 3 + \cdots + 98 + 99 + 100$, expecting this to take him all afternoon. Unfortunately for the teacher, she greatly underestimated Gauss who noted that we could write the sum $1 + 2 + 3 + \cdots + 98 + 99 + 100$ backwards, $100 + 99 + 98 + \cdots + 3 + 2 + 1$, and it would still be the same thing – call this value S. If we added these two expressions (once written forwards and once written backwards) together, we would have twice S. However, the young Gauss noticed that when we add the terms 1 and 100 we get 101; when we add the terms 2 and 99 we get 101; when we add the terms 3 and 98 we get 101; and so forth. Thus we have 101 being added to itself one-hundred times, meaning $2S = 101 \cdot 100 = 10,100$. Dividing this by 2 gives $S = 5,050$. So in just a few minutes Gauss computed what his teacher was expecting would take all afternoon.

Theorem B.4.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof.
To prove this we will use a technique called *mathematical induction*, which students usually first see in a course on Logic and Proof. That is, this proof requires a technique students usually don't learn until after they've finished their Calculus sequence. Because of this, you should not dwell on the proof or worry if it doesn't make sense to you; it's only included here for the sake of completeness.

We easily verify the formula in the case $n = 1$:

$$\sum_{i=1}^{1} i^2 = 1^1 = 1 = \frac{1 \cdot 2 \cdot 3}{6}.$$

Now suppose the formula has already been verified for all integers from 1 to $n-1$. Then the formula will also hold for n because of the following string of expressions:

$$\begin{aligned}
\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n-1} i^2 + n^2& \\
&= \frac{(n-1)n(2(n-1)+1)}{6} + n^2 \\
&= \frac{(n^2-n)(2n-1)}{6} + n^2 \\
&= \frac{2n^3-3n^2+n}{6} + n^2 \\
&= \frac{2n^3-3n^2+n+6n^2}{6} \\
&= \frac{2n^3+3n^2+n}{6} \\
&= \frac{n(2n^2+3n+1)}{6} \\
&= \frac{n(2n^2+2n+n+1)}{6} \\
&= \frac{n(2n\,[n+1]+[n+1])}{6} \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

□

Theorem B.5.

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

B.2 Linearization and differentials

The whole idea behind calculus is to take hard problems and make them simpler by approximating with something that's easier to work with.

One of the best examples of this occurs with *linearization*, which is the idea that we should approximate complicated functions with simpler ones and in particular with linear functions. This idea has had, and continues to have, a profound impact on applications of mathematics. If you've ever wondered how it is a computer or calculator is able to compute $\cos(22.091)$ or $\sqrt{16.29}$, then you may be surprised to learn that the answer relies on calculus.

Motivation

If you think about the mathematical procedures that you can "really" do, the things that you can in principle sit down and work out with pencil and paper, you may come to the realization that you only know how to do four things: add, subtract, multiply, and divide. Of course, you can do some other things like square or cube a number, but this is just multiplication applied several times. Other operations – even ones as simple as taking square roots! – are much, much harder to do by hand. Without a calculator there are probably only a small handful of numbers whose square roots you can actually calculate: things you can actually work out the answer to with just a pencil and paper.

A computer is no different. Computers are programmed by people, so if the computer is about to determine the square root of 384.193, then someone had to tell it how to do that. And computers don't posess some magical computational ability that you don't: when you get down to the nuts and bolts of it, a computer can also only add, subtract, multiply, and divide. This is meant quite literally, by the way. In terms of what the hardware of a computer is actually able to do there are special circuits that use combinations of logical operators (AND, OR, NOT) to do arithmetic with numbers represented in binary (base 2). In some sense you are actually much better at arithmetic than a computer: a computer only has a finite amount of space to store numbers, but in principle there's no actual limitation on what a person with pen and paper can do (even if there are serious practical limitations).

So, this still begs the question: if you can only add, substract, multiply, or divide, how is it that you're supposed to compute a quantity like $\tan^{-1}(\sqrt{17} + 3^{\pi})$? In terms of the functions you can build with the four arithmetic operations – i.e., the functions you can actually evaluate – all you have are rational functions: these are things built from addition, subtraction, multiplication, and division. In fact, for what we're going to consider right now, we're going to replace complicated "transcendental" functions with about the simplest type of function of all: a linear function. A linear function, by the way, is just a function whose graph is a line. So it's a function that looks something like

$$f(x) = ax + b.$$

Notice that to evaluate such a linear function we only need to be able to multiply and add.

Given a differentiable function $f(x)$ and a value a where we can compute the true values of $f(a)$ and $f'(a)$, then for "nearby" x-values $f(x)$ can be approximated by the ***linearization*** $L(x)$,

$$L(x) = f'(a)(x - a) + f(a).$$

If we're doing several calculations of nearby x-values, then we certainly don't need to re-compute the function $L(x)$ each time. For example, suppose we want to approximate $\sqrt{8.99}$ and 9.1. We can use the linearization of $f(x) = \sqrt{x}$ at $a = 9$:

$$L(x) = \frac{1}{6}(x - 9) + 3.$$

Plugging in 8.99 and 9.1 into this linearization, the only difference between the two expressions for $L(8.99)$ and $L(9.1)$ is the quantity $x - 9$:

$$\begin{aligned} L(8.99) =& \frac{1}{6}(8.99 - 9) + 3 = \frac{1}{6}(-0.01) + 3 = 2.998333... \\ L(9.1) =& \frac{1}{6}(9.1 - 9) + 3 = \frac{1}{6}(0.1) + 3 = 3.01666... \end{aligned}$$

So, once we've determined the formula for our linearization, the only thing that can change when we approximate nearby points is the factor of $x - a$. This quantity represents how much our x-value has changed from a, and so we're justified in representing this quantity as Δx.

We could now reasonably write the linearization of our function as

$$L(x) = f'(a)\Delta x + f(a).$$

Since $f(a)$ is the same for any quantity we compute with this linearization, what we really care about is how much $L(x)$ differs from $f(a)$. If we notice that $L(a) = f(a)$, we could rewrite this as follows

$$\begin{aligned} & L(x) = f'(a)\Delta x + f(a) \\ \implies & L(x) = f'(a)\Delta x + L(a) \\ \implies & L(x) - L(a) = f'(a)\Delta x. \end{aligned}$$

This quantity, $L(x) - L(a)$ is really what we're interested in: it tells us how much our approximation changes as we change the x-value. Notice too that this quantity is a change in y-values, so you might be tempted to denote this quantity by Δy. However, the convention that has been adopted is that Δy should mean the change in the true value of the function and *not* the change in the approximation. Since the Greek letter Δ is already taken, let's instead use the Latin letter d to write

$$dy = L(x) - L(a).$$

We could then write $dy = f'(a)\Delta x$, but since we're adopting the convention that we reserve Δ to mean the "true" change and d means the "approximate" change, we will write dx in place of Δx – although this is exactly the same quantity. We thus arrive at the formula $dy = f'(a)dx$

So, in the example above where $f(x) = \sqrt{x}$ and $a = 9$, we have

$$dy = \frac{1}{6}dx.$$

This quantity is called ***the differential of*** $f(x) = \sqrt{x}$ ***at*** 9.

Of course, we could have calculated this dy quantity at another place – using something other than 9 – so we should really imagine that quantity is variable, which is the x-coordinate of our original function, so let's continue to call it x.

We then define ***the differential*** of $f(x)$ to be

$$dy = f'(x)dx.$$

(The notation $df = f'(x)dx$ is also common and is also called the differential.)

There are a few things to notice about these differentials we've defined. The first is that dy and dx here area actual numeric values: they're not just symbols. In fact, dy depends on dx: dy is a function of dx with dx being an independent variable.

An interesting byproduct of our definitions is that if we divide both sides of the equation $dy = f'(x)dx$ by dx, then we have $\frac{dy}{dx} = f'(x)$. This isn't simply a coincidence: by definition, $f'(x)$ is a limit of changes in y-values over changes in x-values. The intuition behind dy and dx is that they should represent "infinitesimal changes" in x or y, and this is how people like Newton and Leibniz originally thought about derivatives. (There is a way to make "infinitesimal changes" precise, but discussing it would take us very far afield.)

The other important thing to notice is that this dy quantity is the important part of a linear approximation. We can write the linearization of a function at a as

$$L(x) = dy + f(a).$$

We can use this to help us calculate linear approximations. In the case of $\sqrt{8.99}$ and $\sqrt{9.1}$, for example, where we took $a = 9$, we have $L(x) = dy + 3 = \frac{1}{6}dx + 3$. For 8.99, $dx = 8.99 - 9 = -0.01$, and so $\sqrt{8.99} \approx \frac{-0.01}{6} + 3 = 2.998333...$, just as before. So, again, differentials are really just linearizations from another point of view.

Examples

We will first do some examples where we calculate differentials "formally," and then do some examples where we apply differentials to help us solve some approximation problems.

Example B.1.

(a) Compute dy where $y = 8x^2 + 6x$.

$$dy = \frac{d}{dx}\left(8x^2 + 6x\right)dx = (16x + 6)dx$$

(b) Compute dy where $y = \sin(\sqrt{x})\cos(x^2)$.

$$dy = \left(\frac{\sin(\sqrt{x})}{2\sqrt{x}}\cos(x^2) + 2x\sin(\sqrt{x})\cos(x^2)\right)dx$$

(c) Compute df where $f(x) = \frac{x^2+1}{x}$.

$$df = f'(x)dx = \frac{x^2 - 1}{x^2}dx.$$

Example B.2.
Use differentials to approximate $\sqrt{24.9}$, $\sqrt{25.1}$, and $\sqrt{24.975}$.

Since these values are near $\sqrt{25} = 5$, we should expect they are all approximately $dy + 5$ where dy is the differential of $\sqrt{x}$ at $x = 25$.

$$y = \sqrt{x}$$
$$\implies dy = \frac{1}{2\sqrt{x}}dx$$

When $x = 25$ we have $dy = \frac{dx}{10}$.

For $\sqrt{24.9}$, $dx = -0.1$ and so $\sqrt{24.9} \approx \frac{-0.1}{10} + 5 = 4.99$.

For $\sqrt{25.1}$, $dx = 0.1$ and so $\sqrt{25.1} \approx \frac{0.1}{10} + 5 = 5.01$.

For $\sqrt{24.975}$, $dx = -0.025$, and so $\sqrt{24.975} \approx \frac{-0.025}{10} + 5 = 5 - 0.0025 = 4.9975$.

Example B.3.
Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick on a hemispherical dome of diameter 50m.

The way to interpret this problem is that we want to know what volume of paint is needed. We basically want the volume between two spheres: one of radius 25, and one of radius 25.0005. We estimate this using differentials: we want to change in volume as we go from one sphere to the other. So our r in the problem will be 25 and $dr = 0.0005$.

The volume of a sphere of radius r is

$$V = \frac{4}{3}\pi r^3.$$

Since we have a hemisphere, we need half of this quantity,

$$V = \frac{4}{6}\pi r^3.$$

Then

$$dV = 2\pi r^2 dr.$$

When $r = 25$ and $dr = 0.0005$ we have

$$dV = 2\pi 25^2 \cdot 0.0005 = 1.964.$$

Let's notice the units here: r is in metres and squared, and dr is also in metres, so the units are cubic metres.

We need about 1.964 cubic metres of paint to paint the dome.

Example B.4.
Suppose a circular disc was measured to have a radius of 30 inches, with a margin of error of $0.2in$. What is the maximum error in the calculation of the area of the disc?

Here we use different variables than before, but we know $A = \pi r^2$. The margin of error in our original measurement corresponds to dr (how far the actual value is from 30), and the error in the area of the disc is dA. We compute

$$dA = 2\pi r dr.$$

In our particular situation $r = 30$ and $dr = 0.2$ and so we have that the error in the area of the disc is

$$dA = 2\pi \cdot 30 \cdot 0.2 \approx 37.699$$

C

Solutions to Exercises

C.1 Chapter 1, Review of Calculus I Material

1.1 We simply differentiate $\frac{2}{3}\left(x^3+x-2\right)^{3/2}+C$ to see that we get back $\sqrt{x^3+x-2}\,(3x^2+1)$.

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{2}{3}\left(x^3+x-2\right)^{3/2}+C\right)\\
=&\frac{2}{3}\frac{d}{dx}\left(x^3+x-2\right)^{3/2}+\frac{d}{dx}C\\
=&\frac{2}{3}\cdot\frac{3}{2}\left(x^3+x-2\right)^{1/2}\cdot\frac{d}{dx}\left(x^3+x-2\right)^{1/2}+0\\
=&\left(x^3+x-2\right)^{1/2}\cdot\left(3x^2+1\right)\\
=&\sqrt{x^3+x-2}\left(3x^2+1\right).
\end{aligned}$$

1.2 We differentiate our supposed antiderivative, $\frac{1}{6}(1+2x)^{3/2}-\frac{1}{2}(1+2x)^{1/2}+C$, and verify that we get back $\frac{x}{\sqrt{1+2x}}$.

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{1}{6}(1+2x)^{3/2}-\frac{1}{2}(1+2x)^{1/2}+C\right)\\
&=\frac{1}{6}\cdot\frac{3}{2}(1+2x)^{1/2}\cdot 2-\frac{1}{2}\cdot\frac{1}{2}(1+2x)^{-1/2}\cdot 2+0\\
&=\frac{1}{2}\sqrt{1+2x}-\frac{1}{2\sqrt{1+2x}}\\
&=\frac{1}{2}\sqrt{1+2x}\cdot\frac{\sqrt{1+2x}}{\sqrt{1+2x}}-\frac{1}{2\sqrt{1+2x}}\\
&=\frac{1+2x}{2\sqrt{1+2x}}-\frac{1}{2\sqrt{1+2x}}\\
&=\frac{1+2x-1}{2\sqrt{1+2x}}\\
&=\frac{x}{\sqrt{1+2x}}
\end{aligned}$$

C.2 Chapter 2, Applications

2.1 When we're computing areas we want the function we're integrating to be positive so that we don't end up computing a "negative area." The whole "top minus bottom" thing is really just to be sure our integrand is positive. Another way to do this is to integrate the absolute value, $\int_a^b |f(x) - g(x)|\,dx$. Notice that to actually compute this value – even though it's written as one integral – you basically have to do the process outlined earlier and break the integral up into pieces where you knew $f(x) > g(x)$ or $g(x) > f(x)$.

2.2 If $f(x)$ or $g(x)$ were continuous at some point c in the interval $[a, b]$ we are integrating over, then the roles of "top" and "bottom" could change at c, even though the curves don't intersect. Thus when we break the interval up into pieces where $f(x) = g(x)$, we also need to break it up at any point where either function is discontinuous.

2.3 The value $r(y_i^*) - \ell(y_i^*)$ is used because it corresponds to the width of the rectangle, and it corresponds to the width because this is the larger value minus the smaller value. If we instead have $\ell(y_i^*) - r(y_i^*)$, then we would have the negative of the width.

C.3 Chapter 3, Integration Techniques

3.4 As described just before the example, we may perform the substitution $u = 2x$, $du = 2\,dx$ to write the integral as

$$\frac{1}{2}\int \sqrt{9-u^2}\,du$$

Using the answer computed in Example 3.13, we have

$$\frac{1}{2}\int \sqrt{9-u^2}\,du = \frac{1}{2}\left(\frac{9}{2}\sin^{-1}\left(\frac{u}{3}\right) + \frac{u}{2}\sqrt{9-u^2}\right) + C.$$

Rewriting this in terms of x by writing $u = 2x$ we have

$$\int \sqrt{9-4x^2}\,dx = \frac{1}{2}\left(\frac{9}{2}\sin^{-1}\left(\frac{2x}{3}\right) + x\sqrt{9-4x^2}\right) + C.$$

D Solutions to Practice Problems

D.1 Chapter 1, Review of Calculus I Material

1.1 We will perform the substitution $u = x^3 - 1$. Notice this means $du = 3x^2\,dx$. We have an x^2 but we lack 3, so we will multiply and divide by 3 to obtain

$$\int x^2 \sin\left(x^3 - 1\right)\,dx = \frac{3}{3}\int x^2 \sin\left(x^3 - 1\right)\,dx = \frac{1}{3}\int 3x^2 \sin\left(x^3 - 1\right)\,dx$$

Now performing the substitution the integral becomes

$$\frac{1}{3}\int \sin(u)\,du = \frac{-1}{3}\cos(u) + C.$$

Rewriting our antiderivative back in terms x we have

$$\int x^2 \sin\left(x^3 - 1\right)\,dx = \frac{-1}{3}\cos\left(x^3 - 1\right) + C.$$

1.2 After performing the substitution $u = \ln(x)$, $du = \frac{1}{x}\,dx$ the integral becomes

$$\int u\,du = \frac{u^2}{2} + C$$

Rewriting in terms of x, we have

$$\int \frac{\ln(x)}{x}\,dx = \frac{\ln(x)^2}{2} + C$$

1.3 Again performing the substitution $u = \ln(x)$, $du = \frac{1}{x}\,dx$, the integral becomes

$$\int \frac{1}{u}\,du = \ln|u| + C$$

Rewriting in terms of x, we have

$$\int \frac{1}{x\ln(x)}\,dx = \ln|\ln(x)| + C$$

1.4 We perform the substitution $u = \sin(4x)$, $du = 4\cos(4x)\,dx$. We are missing a factor of 4 in the original integral, but this is easily fixed:

$$\int \cos(4x)\sin(4x)\,dx = \frac{4}{4}\int \cos(4x)\sin(4x)\,dx = \frac{1}{4}\int 4\cos(4x)\sin(4x)\,dx$$

With the above substitution, our integral thus becomes

$$\frac{1}{4}\int u\,du = \frac{u^2}{8} + C$$

Rewriting this in terms of x gives us

$$\int \cos(4x)\sin(4x)\ dx = \frac{\sin^2(4x)}{8} + C$$

1.5 First notice the integral may be written as

$$\int \frac{\sin(\tan(\theta))}{\cos^2(\theta)}\,d\theta = \int \sin(\tan(\theta))\sec^2(\theta)\,d\theta.$$

Performing the substitution $u = \tan(\theta)$, $du = \sec^2(\theta)\,d\theta$ the integral may be rewritten as

$$\int \sin(u)\,du = -\cos(u) + C.$$

Rewritten in terms of x, we thus have

$$\int \frac{\sin(\tan(\theta))}{\cos^2(\theta)}\,d\theta = -\cos(\tan(\theta)) + C.$$

1.6

$$\begin{aligned}\int \sec(x)\,dx &= \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx \\ &= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx\end{aligned}$$

If we now let $u = \sec(x) + \tan(x)$, then $du = (\sec(x)\tan(x) + \sec^2(x))\ dx$ and so our integral becomes

$$\int \frac{1}{u}\,du = \ln|u| + C$$

Rewriting this in terms of x we have

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C.$$

1.7 We will perform the substitution $u = x^2 - 3$ which means $du = 2x\,dx$. Noting $2x^3 = x^2 \cdot 2x$, we can write our original integral as

$$\int_3^9 2x^3\sqrt{x^2-3}\,dx = \int_3^9 x^2\sqrt{x^2-3}\,2x\,dx$$

Let's notice that since $u = x^2 - 3$ we have $x^2 = u + 3$, so when we perform the substitution the integrand becomes $(u+3)\sqrt{u}$. Since this is a definite integral, though, we also need to change our limits of integration. When $x = 3$, we have that $u = 3^2 - 3 = 6$; when $x = 9$ we have $u = 9^2 - 3 = 78$ and so

$$\begin{aligned}
\int_3^9 2x^3\sqrt{x^2-3}\,dx &= \int_3^9 x^2\sqrt{x^2-3}\,2x\,dx \\
&= \int_6^{78} (u+3)\sqrt{u}\,du \\
&= \int_6^{78} \left(u\sqrt{u} + 3\sqrt{u}\right)\,du \\
&= \int_6^{78} \left(u^{3/2} + 3u^{1/2}\right)\,du \\
&= \left(\frac{u^{5/2}}{5/2} + 3\frac{u^{3/2}}{3/2}\right)\Bigg|_6^{78} \\
&= \left(\frac{2u^{5/2}}{5} + \frac{6u^{3/2}}{3}\right)\Big|_6^{78} \\
&= \left(\frac{2\cdot 78^{5/2}}{5} + \frac{6\cdot 78^{3/2}}{3}\right) - \left(\frac{2\cdot 6^{5/2}}{5} + \frac{6\cdot 6^{3/2}}{3}\right)
\end{aligned}$$

1.8 Letting $u = 23 + 7x$ we have $du = 7\,dx$. Noting that $14 = 2 \cdot 7$, our original integral may be rewritten as

$$\int_{-2}^3 \frac{14}{23+7x}\,dx = \int_{-2}^3 \frac{2}{23+7x}\,7dx.$$

Now we perform the substitution, changing our limits of integration since this is a definite integral. In particular, when $x = -2$ we have $u = 23 + 7\cdot(-2) = 9$; when $x = 3$ we have $u = 23 + 7\cdot 3 = 44$ and so our integral becomes

$$\begin{aligned}
\int_{-2}^{3} \frac{14}{23+7x}\,dx &= \int_{-2}^{3} \frac{2}{23+7x}\,7dx \\
&= \int_{9}^{44} \frac{2}{u}\,du \\
&= \int_{9}^{44} 2u^{-1}\,du \\
&=2\ln|u|\Big|_{9}^{44} \\
&=2\ln(44) - 2\ln(9) \\
&= \ln(44^2) - \ln(9^2) \\
&= \ln(1936) - \ln(81) \\
&= \ln\left(\frac{1936}{81}\right).
\end{aligned}$$

A decimal approximation to this answer would be 3.1739.

1.9 Performing the substitution $u = 2x$, we have $du = 2\,dx$, or $dx = \frac{1}{2}du$. Our integral can then be rewritten as follows:

$$\int_{0}^{1} f(2x)\,dx = \frac{1}{2}\int_{0}^{2} f(u)\,du = \frac{5}{2}.$$

D.2 Chapter 2, Applications

2.1 To simplify the integral, we'll integrate this with respect to y, doing the right-hand side minus the left-hand side.

$$\begin{aligned}
\int_1^4 \left(y-(y-2)^2\right)\,dy &= \int_1^4 \left(y-\left(y^2-4y+4\right)\right)\,dy \\
&= \int_1^4 \left(-y^2+5y-4\right)\,dy \\
&= \left(\frac{-y^3}{3}+\frac{5y^2}{2}-4y\right)\Bigg|_1^4 \\
&= \frac{-4^3}{3}+\frac{5\cdot 4^2}{2}-4\cdot 4-\left(\frac{-1}{3}+\frac{5}{2}-4\right) \\
&= \frac{-64}{3}+40-16-\left(\frac{-2+15-24}{6}\right) \\
&= \frac{-64+72}{3}-\frac{-11}{6} \\
&= \frac{8}{3}+\frac{11}{6} \\
&= \frac{27}{6}=\frac{9}{2}=4.5
\end{aligned}$$

2.2

$$\begin{aligned}
\text{Area} &= \int_0^2 \left(\frac{x}{2}+1-\sin(x)\right)\,dx \\
&= \left(\frac{x^2}{4}+x+\cos(x)\right)\Bigg|_0^2 \\
&= (1+2+\cos(2))-(0+0+1) \\
&= 2+\cos(2)
\end{aligned}$$

2.3 We must determine where the curves intersect to determine the bounds of our integral and how to break the integral up into pieces where one we know which curve is on the right and which is on the left.

To see where the curves intersect we must solve the equation $-y^3+2y^2=-3y$. Let's first rewrite this equation as $y^3-2y^2-3y=0$. Factoring out a y we have $y(y^2-2y-3)=0$. Now we factor y^2-2y-3 by looking for two numbers that multiply to -3 and add to -2. Notice that -3 and 1 have these properties: $-3\cdot 1=-3$ and $-3+1=-2$. This tells us that y^2-2y-3 factors as $(y-3)(y+1)$. Thus our original equation is ultimately rewritten as $y(y-3)(y+1)=0$ which is solved by $y=0$, $y=3$

and $y = -1$. This tells us that we're ultimately going to integrate from -1 (the smallest value) to 3 (the largest value). However, the roles of which curve is on the left and which curve is on the right may change at $y = 0$.

We can of course see this from the picture, but without a picture we could plug in a y value between -1 and 0 to see which curve is on the right for $-1 \leq y \leq 0$. Plugging $y = {}^{-1}\!/_2$ into $x = -3y$, for example, gives us an output of ${}^3\!/_2$. When we plug the same $y = {}^{-1}\!/_2$ into the other expression, $x = -y^3 + 2y^2$, though, the corresponding output is $\frac{3}{8}$. Since $\frac{1}{2} > \frac{3}{8}$, this means that $x = -3y$ is the curve on the right and $x = -y^3 + 2y^2$ is the curve on the left, at least for $-1 \leq y \leq 0$.

For our other interval, $0 \leq y \leq 3$, we plug in some y-value between 0 and 3 into both of our functions. Plugging $y = 2$ into $x = -3y$ gives an output of $x = -6$, whereas it gives an output of $x = 0$ when plugged into $x = -y^3 + 2y^2$. This means that $x = -y^3 + 2y^2$ is on the right and $x = -3y$ is on the left for $0 \leq y \leq 3$.

Now we compute the area by adding the corresponding integrals together,

$$\text{Area} = \int_{-1}^{0} \left(-3y - \left(-y^3 + 2y^2\right)\right)\, dy + \int_{0}^{3} \left(-y^3 + 2y^2 - (-3y)\right)\, dy.$$

We'll simply compute these integrals separately and then add them together.

$$\begin{aligned}
\int_{-1}^{0} \left(-3y - \left(-y^3 + 2y^2\right)\right)\, dy &= \int_{-1}^{0} \left(-3y + y^3 - 2y^2\right)\, dy \\
&= \left(\frac{-3y^2}{2} + \frac{y^4}{4} - \frac{2y^3}{3}\right)\Bigg|_{-1}^{0} \\
&= 0 - \left(\frac{-3}{2} + \frac{1}{4} - \frac{-2}{3}\right) \\
&= \frac{3}{2} - \frac{1}{4} - \frac{2}{3} \\
&= \frac{18 - 3 - 8}{12} \\
&= \frac{7}{12}
\end{aligned}$$

$$\begin{aligned}
\int_0^3 \left(-y^3 + 2y^2 - (-3y)\right)\, dy &= \int_0^3 \left(-y^3 + 2y^2 + 3y\right)\, dy \\
&= \left(\frac{-y^4}{4} + \frac{2y^3}{3} + \frac{3y^2}{2}\right)\Big|_0^3 \\
&= \left(\frac{-81}{4} + \frac{64}{3} + \frac{27}{2}\right) - 0 \\
&= \frac{-243 + 256 + 162}{12} \\
&= \frac{175}{12}
\end{aligned}$$

Adding these together we have that the area is

$$\frac{7}{12} + \frac{175}{12} = \frac{185}{12}.$$

2.4 We must find where these curves intersect by solving the equation $y^2 - 2 = y$, or $y^2 - y - 2 = 0$. Notice the left-hand side factors as $(y+1)(y-2)$, and so the equation is solved by $y = -1$ and $y = 2$. Thus we will integrate the right-hand curve minus the left-hand curve over the interval $-1 \leq y \leq 2$. To determine which curve is on the right we can plug a y-value between -1 and 2 into each expression. Plugging $y = 0$ into $y^2 - 2$ yields -2, whereas plugging $y = 0$ into y yields 0, and so $x = y$ is the curve on the right and $x = y^2 - 2$ is the curve on the left.

The area is thus

$$\begin{aligned}
\text{Area} &= \int_{-1}^{2} \big(y - \big(y^2 - 2\big)\big)\ dy \\
&= \int_{-1}^{2} \big(-y^2 + y + 2\big)\ dy \\
&= \left(\frac{-y^3}{3} + \frac{y^2}{2} + 2y\right)\Bigg|_{-1}^{2} \\
&= \left(\frac{-8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) \\
&= \frac{-8 + 6 + 12}{3} - \frac{2 + 3 - 12}{6} \\
&= \frac{10}{3} - \frac{-7}{6} \\
&= \frac{20 + 7}{6} \\
&= \frac{27}{6} = \frac{9}{2}
\end{aligned}$$

2.5 We must determine where the curves intersect, which we can do by solving the equation $x^4 = 2 - x^2$. We can rewrite this as $x^4 + x^2 - 2 = 0$. Notice this is actually a quadratic "in disguise," that is, we can think of this as $(x^2)^2 + (x^2)^1 - 2 = 0$ and apply the quadratic formula. Alternatively, let $u = x^2$ and our equation becomes $u^2 + u - 2 = 0$ which the quadratic formula tells us is solved by

$$u = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-2)}}{2} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2}$$

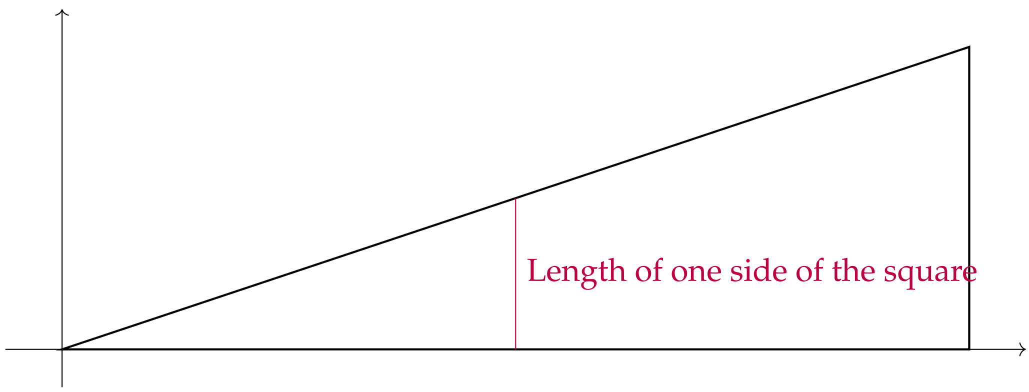
and so $u = 1$ or $u = -1$. Notice that one of these values is negative. Since $u = x^2$ and the square of a real number is positive, the negative solution is impossible and so we're left with $u = 1$. As $u = x^2$, though, we have $x^2 = 1$ so $x = \pm 1$.

To determine which of the curves is on the top we can plug in any x-value between -1 and 1. Plugging $x = 0$ into x^4 gives 0, while plugging $x = 0$ into $2 - x^2$ gives 2, and so $2 - x^2$ is on top.

The area of the enclosed region is thus

$$\begin{aligned}
\text{Area} &= \int_{-1}^{1} \left(2 - x^2 - x^4\right)\, dx \\
&= \left(2x - \frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_{-1}^{1} \\
&= \left(2 - \frac{1}{3} - \frac{1}{5}\right) - \left(-2 + \frac{1}{3} + \frac{1}{5}\right) \\
&= 2 - \frac{1}{3} - \frac{1}{5} + 2 - \frac{1}{3} - \frac{1}{5} \\
&= 4 + \frac{2}{3} + \frac{2}{5} \\
&= \frac{30 + 10 + 6}{5} \\
&= \frac{46}{5}
\end{aligned}$$

2.6 Notice that once we choose an x-value for our cross section, since the cross section is a square, we only need to find the length of one side of the square. In particular, since the cross section goes from the base of the triangle to the top of the triangle, its length is just given by the equation of the line corresponding to the top of the triangle. Since this line goes through $(0, 0)$ and $(2, 1)$, this is just $y = x/2$.



Thus the area of the cross section at a given x value is simply $(x/2)^2 = x^2/4$.

Integrating the cross sectional areas we find that the volume is

$$\begin{aligned}
\text{Volume} &= \int_0^2 \frac{x^2}{4}\,dx \\
&= \left.\frac{x^3}{12}\right|_0^2 \\
&= \frac{8}{12} \\
&= \frac{2}{3}
\end{aligned}$$

2.7 Let's first notice these curves intersect at $x = 0$ and $x = 3$, since these are the solutions to $x^2 = 3x$. Thus we will integrate from $x = 0$ to $x = 3$. Using the washer method, notice the outer radius of our washers is given by the line and the inner radius is given by the parabola. Thus the area of each washer has the form

$$\pi\left((3x)^2 - \left(x^2\right)^2\right) = \pi\left(-x^4 + 9x^2\right).$$

Now we simply integrate this quantity from $x = 0$ to $x = 3$:

$$\begin{aligned}
\text{Volume} &= \int_0^3 \pi\left(-x^4 + 9x^2\right)\,dx \\
&= \pi\left.\left(\frac{-x^5}{5} + 3x^3\right)\right|_0^3 \\
&= \pi\left(\frac{-243}{5} + 81\right) \\
&= \pi\frac{405 - 243}{5} \\
&= \frac{162}{5}\pi.
\end{aligned}$$

2.8 Since we are using the shell method and rotating around the x-axis, we need to express the heights of our shells as functions of x. In this we can do this pretty easily by just solving each of $y = x^2$ and $y = 3x$ for x to obtain $x = \sqrt{y}$ and $x = y/3$.

As we will be integrating functions of y we need to know the y-coordinates of where our curves intersect. We can do this by either solving $\sqrt{y} = y/3$ or plugging in the $x = 0$ and $x = 3$ from our solution to the previous problem to see what the corresponding y's are. We will solve $\sqrt{y} = y/3$ here. Squaring both sides this becomes $y = y^2/9$ or $9y = y^2$. We may rewrite this as $y^2 - 9y = 0$ and then factor this as $y(y-9) = 0$, so the solutions are $y = 0$ and $y = 9$.

The height of the shell associated with a given y value is the difference between the highest and lowest x-values for that y. For our region this gives us the heights

$$\sqrt{y} - \frac{y}{3}.$$

Now we can plug this into our formula for the shell method to obtain

$$\begin{aligned}
\text{Volume} &= \int_0^9 2\pi y \left(\sqrt{y} - \frac{y}{3}\right)\, dy \\
&= 2\pi \int_0^9 \left(y^{3/2} - \frac{y^2}{3}\right)\, dy \\
&= 2\pi \left(\frac{2}{5}y^{5/2} - \frac{y^3}{9}\right)\Bigg|_0^9 \\
&= 2\pi \left(\frac{2}{5}9^{5/2} - 81\right) \\
&= 2\pi \left(\frac{2}{5}243 - 81\right) \\
&= 2\pi \frac{486 - 405}{5} \\
&= 2\pi \frac{81}{5} \\
&= \frac{162}{5}\pi
\end{aligned}$$

2.9 Since we are rotating around an axis paralell to the y-axis, the heights of our shells will be functions of x. In particular, they will be the change in the functions $-x^2 + 4x + 4$ and x^2; the height of the shell associated with a point x in $[-1, 2]$ is

$$-x^2 + 4x + 4 - x^3$$

(This is because the red curve, $y = -x^2 + 4x + 4$ is on top and the blue curve, $y = x^3$, is on the bottom in the picture provided in the problem.)

The radius of the shell, however, is given by $3 - x$ since the axis of rotation $x = 3$ is to the right of the figure we are rotating.

This means the volume of the object is

$$\begin{aligned}
\text{Volume} &= \int_{-1}^{2} 2\pi(3-x)\left(-x^3-x^2+4x+4\right)\,dx \\
&= 2\pi\int_{-1}^{2}\left(-3x^3-3x^2+12x+12+x^4+x^3-4x^2-4x\right)\,dx \\
&= 2\pi\int_{-1}^{2}\left(x^4-2x^3-7x^2+8x+12\right)\,dx \\
&= 2\pi\left(\frac{x^5}{5}-\frac{x^4}{2}-\frac{7x^3}{3}+4x^2+12x\right)\Bigg|_{-1}^{2} \\
&= 2\pi\left(\frac{296}{15}-\frac{-191}{30}\right) \\
&= \frac{783}{15}\pi \\
&= \frac{261}{5}\pi
\end{aligned}$$

2.10 If we use the washer method to rotate around an axis parallel to the y-axis, then we will need to integrate a function of y. This means we need to solve each of $y = x^3$ and $y = -x^2+4x+4$ for x. The first one is of course simply $x = \sqrt[3]{y}$, and the other requies some more algebra. In particular, let's first rewrite our equation as $-y = x^2-4x-4$. Now we will complete the square to obtain $-y = x^2-4x+4-4-4$ which we may rewrite as $-y = (x-2)^2-8$, and from here we can solve $8-y = (x-2)^2$, so $x = 2\pm\sqrt{8-y}$. Since all of the x-values in our region are to the left of 2 we will need to use the negative square root and so we have $x = 2-\sqrt{8-y}$.

The outer radius of the washers will be

$$3-\left(2-\sqrt{8-y}\right) = 1+\sqrt{8-y}$$

and the inner radius will be

$$3-\sqrt[3]{y}.$$

To write our integral we need to determine where our curves intersect. In the statement of the problem we were told the x's varied from $x=-1$ to $x=2$. Plugging these x-values into the formulas for our y's, we see the y's vary from $y=(-1)^3 = 1$ to $y = 2^3 = 8$. Thus our integral for the volume using the washer method is

$$\begin{aligned}
\text{Volume} &= \int_{-1}^{8} \pi \left(\left(1+\sqrt{8-y}\right)^2 - \left(3-y^{1/3}\right)^2 \right) \, dy \\
&= \pi \int_{-1}^{8} \left(1 + 2\sqrt{8-y} + 8 - y - \left(9 - 6y^{1/3} + y^{2/3}\right)\right) \, dy \\
&= \pi \int_{-1}^{8} \left(2\sqrt{8-y} - y + 6y^{1/3} - y^{2/3}\right) \, dy
\end{aligned}$$

To evaluate this integral we will break it into two pieces,

$$2\pi \int_{-1}^{8} \sqrt{8-y}\, dy + \pi \int_{-1}^{8} \left(-y + 6y^{1/3} - y^{2/3}\right) \, dy$$

For the first of these integrals we will use the substitution $u = 8 - y$, $du = -dy$ which allows us to write the integral as

$$\begin{aligned}
-2\pi \int_{9}^{0} u^{1/2}\, du = 2\pi \int_{0}^{9} u^{1/2}\, du \\
&= \frac{4\pi}{3} u^{3/2} \Big|_0^9 \\
&= \frac{4\pi}{3} \cdot 27 \\
&= 36\pi
\end{aligned}$$

The second integral we can compute directly:

$$\begin{aligned}
&\pi \int_{-1}^{8} \left(-y + 6y^{1/3} - y^{2/3}\right) \, dy \\
=&\pi \left(\frac{-y^2}{2} + \frac{18}{4} y^{4/3} - \frac{3}{5} y^{5/3} \right) \Big|_{-1}^{8} \\
=&\pi \left(\left(-32 + 72 - \frac{96}{5}\right) - \left(\frac{-1}{2} + \frac{18}{4} + \frac{3}{5}\right) \right) \\
=&\pi \left(\frac{104}{5} - \frac{23}{5} \right) \\
=&\frac{81}{5}\pi
\end{aligned}$$

Adding these together, the volume is

$$36\pi + \frac{81}{5}\pi = \frac{180+81}{5}\pi = \frac{261}{5}\pi$$

2.11 As the force (900lb) is constant here, the work is just force times distance, and so 9000 foot-pounds of work is done.

2.12 Notice that as the crane reels the chain in, the amount of chain decreases from 30 feet to 20 feet. As it decreases the weight of the remaining chain decreases. Letting y denote the height of the block above the ground we will ultimately move the block ten feet, but because the weight of the chain is changing (since its length is changing), the force we're applying changes.

In particular, when the block is y feet above the ground, the amount of chain currently let out is $30 - y$. The weight of this portion of the chain is then $50 \cdot (30 - y) = 1500 - 50y$. This is attached to a 900 lb block, so we need to add this weight onto our force obtaining $2400 - 50y$. Now we simply integrate this force along the ten feet the block moves:

$$\begin{aligned}
\text{Work} &= \int_0^{10} (2400 - 50y)\ dy \\
&= \left(2400y - 25y^2\right)\Big|_0^{10} \\
&= 24000 - 2500 \\
&= 21500
\end{aligned}$$

Thus 21,500 foot-pounds of work is done.

2.13 We imagine that the liquid in the tank is chopped into little discs of height Δy which are distance y from the bottom of the tank. The volume of each disc is then $\pi \cdot 10^2 \Delta y$, and so the weight of that disc is $6500\pi\Delta y$ pounds. Each disc moves from its current height y to the top of the tank. Since the tank is 30 feet tall, that means the disc at height y needs to move $30 - y$ feet. The work in moving that one disc is then

$$6500\pi(30 - y)\Delta y.$$

This is just for one disc, and so we need to add up the work for each of these, and then take the limit as we use more and more (skinnier and skinnier) discs, which gives us the following integral:

$$\begin{aligned}
\text{Work} &= \int_0^{30} 6500\pi(30-y)\,dy \\
&= 6500\pi \int_0^{30} (30-y)\,dy \\
&= 6500\pi \left(30y - \frac{y^2}{2}\right)\Big|_0^{30} \\
&= 6500\pi \left(30^2 - \frac{30^2}{2}\right) \\
&= 6500\pi \frac{30^2}{2} \\
&= 2925000\pi
\end{aligned}$$

where the units are foot-pounds.

D.3 Chapter 3, Integration Techniques

3.1 We will perform integration by parts using $u = \ln(x)$, $dv = (4x^3 + 2x)\,dx$. We then have $du = \frac{1}{x}dx$ and $v = x^4 + x^2$. Using the integration by parts formula we then have

$$\begin{aligned}
\int \left(x^3 + 2x\right)\, dx &= \left(x^4 + x^2\right)\ln(x) - \int \left(x^4 + x^2\right)\frac{1}{x}\,dx \\
&= \left(x^4 + x^2\right)\ln(x) - \int \left(x^3 + x\right)\, dx \\
&= \left(x^4 + x^2\right)\ln(x) - \left(\frac{x^4}{4} + \frac{x^2}{2}\right) + C.
\end{aligned}$$

3.2 Let $u = \sin(3x)$, $dv = e^{2x}\,dx$, then $du = 3\cos(3x)$ and $v = \frac{1}{2}e^{2x}$. We then have

$$\frac{e^{2x}\sin(3x)}{2} - \frac{3}{2}\int e^{2x}\cos(3x)\,dx.$$

For the integral on the right we again do integration by parts with $u_2 = \cos(3x)$, $dv_2 = e^{2x}$, so $du_2 = -3\sin(3x)$, $v_2 = \frac{1}{2}e^{2x}$ and then have

$$\int e^{2x}\cos(3x)\,dx = \frac{e^{2x}\cos(3x)}{2} + \frac{3}{2}\int e^{2x}\sin(3x)\,dx.$$

Putting this into the above we have

$$\int e^{2x}\sin(3x) = \frac{e^{2x}\sin(3x)}{2} - \frac{3}{2}\left(\frac{e^{2x}\cos(3x)}{2} + \frac{3}{2}\int e^{2x}\sin(3x)\,dx\right)$$

or simply

$$\int e^{2x}\sin(3x) = \frac{e^{2x}\sin(3x)}{2} - \frac{3e^{2x}\cos(3x)}{4} - \frac{9}{4}\int e^{2x}\sin(3x)\,dx$$

Thus

$$\frac{13}{4}\int e^{2x}\sin(3x) = \frac{e^{2x}\sin(3x)}{2} - \frac{3e^{2x}\cos(3x)}{4}$$

and

$$\int e^{2x}\sin(3x) = \frac{2e^{2x}\sin(3x)}{13} - \frac{3e^{2x}\cos(3x)}{13} + C$$

3.3 We first write the integral as $\int \sec(x)\sec^2(x)\,dx$. Now perform integration by parts with $u = \sec(x)$, $dv = \sec^2(x)$, so $du = \sec(x)\tan(x)dx$, and $v = \tan(x)$. We then have

$$\begin{aligned}
\int \sec^3(x)\,dx &= \int \sec(x)\sec^2(x)\,dx \\
&= \sec(x)\tan(x) - \int \tan(x)\sec(x)\tan(x)\,dx \\
&= \sec(x)\tan(x) - \int \tan^2(x)\sec(x)\,dx \\
&= \sec(x)\tan(x) - \int \left(\sec^2(x) - 1\right)\sec(x)\,dx \\
&= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx
\end{aligned}$$

Moving the extra $\sec^3(x)$ to the other side this becomes

$$2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \int \sec(x)\,dx.$$

To integrate $\sec(x)$ we multiply by 1 written as $\sec(x) + \tan(x)$ over itself to obtain

$$\int \sec(x)\,dx = \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$$

Letting $w = \sec(x) + \tan(x)$ we have $dw = (\sec(x)\tan(x) + \sec^2(x))\,dx$ and the integral becomes $\int \frac{dw}{w}$ which is simply $\ln|w| + C$. Rewriting this in terms of x we find $\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$. Plugging this into the above and solving for $\int \sec^3(x)\,dx$ we find

$$\int \sec^3(x)\,dx = \frac{1}{2}\left(\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|\right) + C$$

3.4 The radius of the cylinder corresponding to x is $1-x$, and so the volume is computed using the shell method as

$$\int_{-1}^{0} 2\pi(1-x)e^{-x}\,dx = 2\pi\left(\int_{-1}^{0} e^{-x}\,dx - \int_{-1}^{0} xe^{-x}\,dx\right)$$

Now we compute these two integrals. The first one is simply

$$\int_{-1}^{0} e^{-x}\,dx = -e^{-x}\Big|_{-1}^{0} = -e^{0} + e^{1} = e - 1.$$

The second integration requires integration by parts. Using $u = x$, $dv = e^{-x}\,dx$, $du = dx$, $v = -e^{-x}$ we have

$$\begin{aligned}\int_{-1}^{0} xe^{-x}\,dx &= -xe^{-x}\Big|_{-1}^{0} + \int_{-1}^{0} e^{-x}\,dx\\ &= -xe^{-x}\Big|_{-1}^{0} + \left(-e^{-x}\right)\Big|_{-1}^{0}\\ &= \left(-xe^{-x} - e^{-x}\right)\Big|_{-1}^{0}\\ &= (0-1) - (e-e)\\ &= -1\end{aligned}$$

The volume is thus

$$2\pi\left(e - 1 - (-1)\right) = 2\pi e.$$

3.5 We will perform integration by parts with $u = \ln(\sqrt{x})$ and $dv = dx$. Then $du = \frac{1}{2\sqrt{x}\sqrt{x}}\,dx = \frac{1}{2x}$ and $v = x$. The integration by parts formula then gives us

$$\begin{aligned}\int \ln(\sqrt{x})\,dx &= x\ln(\sqrt{x}) - \int x\frac{1}{2x}\,dx\\ &= x\ln(\sqrt{x}) - \int \frac{1}{2}dx\\ &= x\ln(\sqrt{x}) - \frac{x}{2} + C\end{aligned}$$

3.6 Performing integration by parts with $u = x^2$, $dv = e^x\,dx$ we have $du = 2x\,dx$ and $v = e^x$. Thus

$$\begin{aligned}\int x^2 e^x\,dx &= x^2e^x - \int e^x 2x\,dx \\ &= x^2e^x - 2\int xe^x\,dx.\end{aligned}$$

We will compute $\int xe^x$ using integration by parts a second time with $u_2 = x$, $dv_2 = e^x\,dx$ and so $du_2 = dx$ and $v_2 = e^x$. This tells us

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

Plugging this into the $\int xe^x\,dx$ that appeared in our earlier integral we have

$$\int x^2e^x\,dx = x^2e^x - 2xe^x - 2e^x + C.$$

3.7

$$\begin{aligned}\int \sin^4(x)\,dx &= \int \left(\sin^2(x)\right)^2\,dx = \int \left(\frac{1-\cos(2x)}{2}\right)^2\,dx \\ &= \int \frac{1-2\cos(2x)+\cos^2(2x)}{4}\,dx \\ &= \int \frac{1-2\cos(2x)+\frac{1+\cos(2x)}{2}}{4}\,dx \\ &= \int \left(\frac{1}{4} - \frac{\cos(2x)}{2} + \frac{1}{8} + \frac{\cos(4x)}{8}\right)\,dx \\ &= \int \left(\frac{3}{8} - \frac{\cos(2x)}{2} + \frac{\cos(4x)}{8}\right)\,dx \\ &= \frac{3}{8}x - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C\end{aligned}$$

3.8 Writing $\tan(x)$ as $\frac{\sin(x)}{\cos(x)}$ the integral becomes

$$\int \frac{\sin^2(x)}{\cos^2(x)}\sin(x)\,dx.$$

Now we can rewrite $\sin^2(x) = 1 - \cos^2(x)$ to obtain

$$\int \frac{1-\cos^2(x)}{\cos^2(x)}\sin(x)\,dx = \int \frac{\sin(x)}{\cos^2(x)}\,dx - \int \sin(x)\,dx.$$

The first integral we can compute with the substitution $u = \cos(x)$, $du = -\sin(x)\,dx$ to obtain

$$-\int \frac{du}{u^2} = \frac{1}{u} + C$$

which in terms of x tells us

$$\int \frac{\sin(x)}{\cos^2(x)}\,dx = \frac{1}{\cos(x)} + C = \sec(x) + C.$$

(Alternatively, we might observe $\frac{\sin(x)}{\cos^2(x)} = \tan(x)\sec(x)$.)

The second integral is simply $\int \sin(x)\,dx = -\cos(x) + C$.

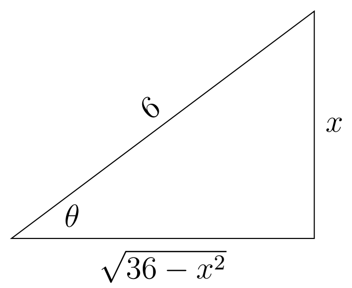
Putting these together we have

$$\int \tan^2(x)\sin(x)\,dx = \sec(x) - \cos(x) + C.$$

3.9 As our integrand involves the form $a^2 - x^2$, we will perform the substitution $x = a\sin(\theta)$, which in this case means $x = 6\sin(\theta)$, $dx = 6\cos(\theta)\,d\theta$. Our integral then becomes

$$\int \frac{6\sin(\theta)}{\sqrt{36 - 36\sin^2(\theta)}} 6\cos(\theta)\,d\theta = 6\int \frac{\sin(\theta)}{\cos(\theta)}\cos(\theta)\,d\theta$$

Of course, this becomes just $6\int \sin(\theta)\,d\theta$ which is simply $-6\cos(\theta)+C$. To write this in terms of x we consider the right triangle whose side opposite of θ has length x and whose hypotenuse has length 6. By the Pythagorean theorem, the length of the adjacent side is $\sqrt{36 - x^2}$.



Noting $\cos(\theta) = \frac{\sqrt{36-x^2}}{6}$, we have

$$\int \frac{x}{\sqrt{36 - x^2}}\,dx = -\sqrt{36 - x^2} + C$$

3.10 Since this integrand involves an expression of the form $x^2 + a^2$, where $a = \sqrt{2}$, we will perform the substitution $x = \sqrt{2}\tan(\theta)$, $dx = \sqrt{2}\sec^2(\theta)\,d\theta$. Our integral then becomes

$$\int \frac{2^{5/2}\tan^5(\theta)}{\sqrt{2\tan(\theta)+2}}\sqrt{2}\sec^2(\theta)\,d\theta = \frac{2^3}{\sqrt{2}}\int \frac{\tan^5(\theta)}{\sec(\theta)}\sec^2(\theta)\,d\theta = 2^{5/2}\int \tan^5(\theta)\sec(\theta)\,d\theta$$

To integrate this we will rewrite $\tan(\theta)$ and $\sec(\theta)$ in terms of sines and cosines to obtain

$$2^{5/2}\int \frac{\sin^5(\theta)}{\cos^5(\theta)}\cdot\cos(\theta)\,d\theta$$

We may further rewrite this as

$$\begin{aligned}
&2^{5/2}\int \frac{\sin^5(\theta)}{\cos^4(\theta)}\,d\theta\\
=&2^{5/2}\int \frac{\sin^4(\theta)}{\cos^4(\theta)}\sin(\theta)\,d\theta\\
=&2^{5/2}\int \frac{\left[\sin^2(\theta)\right]^2}{\cos^4(\theta)}\sin(\theta)\,d\theta\\
=&2^{5/2}\int \frac{\left(1-\cos^2(\theta)\right)^2}{\cos^4(\theta)}\sin(\theta)\,d\theta
\end{aligned}$$

Now we can perform the substitution $u = \cos(\theta)$, $du = -\sin(\theta)\,d\theta$ and the integral becomes

$$-\int \frac{(1-u^2)^2}{u^4}\,du = -\int \frac{1-2u^2+u^4}{u^4}\,du = -\int \left(u^{-4} - 2u^{-2} + 1\right)\,du$$

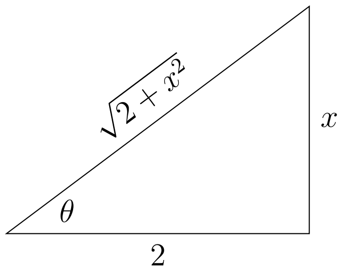
Thus the antiderivative in terms of u is

$$\frac{1}{3u^3} - \frac{1}{u} + u + C$$

which in terms of θ is

$$\frac{1}{\cos^3(\theta)} - \frac{1}{\cos(\theta)} + \cos(\theta) + C$$

Finally, to put this back in terms of x we recall that $x = \sqrt{2}\tan(\theta)$. Considering the right triangle where the side opposite the angle θ has length x and the side adjacent to θ has length $\sqrt{2}$, the hypotenuse must have side $\sqrt{2+x^2}$:



Thus $\cos(\theta) = \sqrt{\frac{2}{2+x^2}}$, we have that the antiderivative is

$$\frac{1}{3\left(\frac{2}{2+x^2}\right)^{3/2}} - \sqrt{\frac{2+x^2}{2}} + \sqrt{\frac{2}{2+x^2}} + C$$

3.11 We will perform the substitution $x = \sec(\theta)$, $dx = \sec(\theta)\tan(\theta)$ and the integral then becomes

$$\int \frac{\sec(\theta)}{\sqrt{\sec^2(\theta)-1}} \sec(\theta)\tan(\theta)\, d\theta = \int \frac{\sec(\theta)}{\tan(\theta)} \sec(\theta)\tan(\theta)\, d\theta = \int \sec^2(\theta)\, d\theta$$

Of course, the antiderivative of $\sec^2(\theta)$ is simply $\tan(\theta) + C$. To rewrite this in terms of x we consider the right triangle where the side adjacent to θ has length 1 and the hypotenuse has length x. The opposite side then has length $\sqrt{x^2-1}$. As tangent is opposite over hypotenuse we have that our antiderivative is simply

$$\int \frac{x}{\sqrt{x^2-1}}\, dx = \sqrt{x^2-1} + C.$$

3.12 To write this as a sum of simpler fractions we must first factor the denominator, which factors as $(x+1)(x-1)$. Thus we will write the fraction as

$$\frac{5x-1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}.$$

To determine the values of A and B we add the fractions on the right to obtain

$$\begin{aligned}
&\frac{A}{x+1}+\frac{B}{x-1}\\
&=\frac{A}{x+1}\cdot\frac{x-1}{x-1}+\frac{B}{x-1}\cdot\frac{x+1}{x+1}\\
&=\frac{Ax-A+Bx+B}{x^2-1}\\
&=\frac{(A+B)x+(-A+B)}{x^2-1}
\end{aligned}$$

Equating coefficients with the numerator of the original fraction gives us the following system of equations:

$$\begin{aligned}
A+B&=5\\
-A+B&=-1
\end{aligned}$$

Adding the equations together tells us $2B=4$, so $B=2$. Once we know $B=2$, we easily see $A=3$ and thus

$$\begin{aligned}
\int\frac{5x-1}{x^2-1}\,dx&=\int\left(\frac{3}{x+1}+\frac{2}{x-1}\right)\,dx\\
&=3\ln|x+1|+2\ln|x-1|+C
\end{aligned}$$

3.13 We first factor the denominator as $(x+4)(x-3)$. We thus want to write our fraction as

$$\frac{-35}{x^2+x-12}=\frac{A}{x+4}+\frac{B}{x-3}$$

We now write out the right-hand side as

$$\begin{aligned}
&\frac{A}{x+4}+\frac{B}{x-3}\\
=&\frac{A(x-3)}{x+4}+\frac{B(x+4)}{x-3}\\
=&\frac{Ax-3A+Bx+4B}{x^2+x-12}\\
=&\frac{(A+B)x+(-3A+4B)}{x^2+x-12}
\end{aligned}$$

Comparing this to our earlir fraction we have

$$\begin{aligned}
A+B&=0\\
-3A+4B&=-35
\end{aligned}$$

The first equation tells us $A=-B$. Plugging this into the second equation, the second equation $3B+4B=-35$, so $7B=-35$, thus $B=-5$ and $A=5$. We now have

$$\begin{aligned}
\int\frac{-35}{x^2+x-12}\,dx&=\int\left(\frac{5}{x+4}-\frac{5}{x-3}\right)dx\\
&=5\ln|x+4|-5\ln|x-3|+C
\end{aligned}$$

3.14 As we have a repeated root, we will write the fraction as

$$\frac{4x^2+12x-4}{(x+2)^2(x-4)}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{x-4}.$$

Adding the terms on the right-hand side together gives us

$$\frac{A(x+2)(x-4)+B(x-4)+C(x+2)^2}{(x+2)^2(x-4)}$$

which we may write as

$$\frac{Ax^2-2Ax-8A+Bx-4B+Cx^2+4Cx+4C}{(x+2)^2(x-4)}.$$

After combining like-terms this becomes

$$\frac{(A+C)x^2+(-2A+B+4C)x+(-8A-4B+4C)}{(x+2)^2(x-4)}.$$

Comparing this to our original fraction we have

$$\begin{aligned} A+C&=4\\ -2A+B+4C&=12\\ -8A-4B+4C&=-4 \end{aligned}$$

Adding twice the first row to the second, the system becomes

$$\begin{aligned} A+C&=4\\ B+6C&=20\\ -8A-4B+4C&=-4 \end{aligned}$$

We now add eight times the first row to the third to obtain

$$\begin{aligned} A+C&=4\\ B+6C&=20\\ -4B+12C&=28 \end{aligned}$$

Now we add four times the second row to the third and we have

$$\begin{aligned} A+C&=4\\ B+6C&=20\\ 36C&=108 \end{aligned}$$

From this we can determine $C=3$, $B=2$, and $A=1$.

We now compute our integral as

$$\int \frac{4x^2+12x-4}{(x+2)^2(x-4)}\,dx = \int \left(\frac{1}{x+2}+\frac{2}{(x+2)^2}+\frac{3}{x-4}\right)\,dx.$$

We can integrate this term-by-term, where in each term we would perform a u-substitution with u being the denominator of the fraction in that term. This would give us the final antiderivative

$$\ln|x+2|-\frac{2}{x+2}+3\ln|x-4|+C.$$

3.15 The denominator simply factors as $x(x^2+1)$. Since there is an irreducible quadratic, we will write our fraction as

$$\frac{A}{x}+\frac{Bx+C}{x^2+1}.$$

Adding these together gives us

$$\frac{Ax^2+A+Bx^2+Cx}{x^2+1}=\frac{(A+B)x^2+Cx+A}{x^2+1}.$$

Comparing this to our original fraction, the corresponding system of equations is

$$\begin{aligned}A+B&=8\\C&=-1\\A&=3\end{aligned}$$

This system is of course solved by $A=3$, $B=5$, and $C=-1$ and so our integral becomes

$$\int\frac{8x^2-x+3}{x^3+x}\,dx=\int\left(\frac{3}{x}+\frac{5x-1}{x^2+1}\right)\,dx$$

The first term integrates to $3\ln|x|+C$. For the second term we break the integral up as

$$\int\frac{5x}{x^2+1}\,dx-\int\frac{1}{x^2+1}\,dx.$$

The first integral is now solved by using the substitution $u=x^2+1$, $du=2x\,dx$. This turns the integral into

$$\frac{1}{2}\int\frac{5}{u}\,du=5\ln|u|+C$$

which in terms of x tells us

$$\int\frac{5x}{x^2+1}\,dx=\frac{5}{2}\ln|x^2+1|+C$$

The last remaining integral is solved by performing the trig substitution $x=\tan(\theta)$, $dx=\sec^2(\theta)\,dx$. The integral will then become $\int 1\,d\theta=\theta+C$ which in terms of x is just $\tan^{-1}(x)+C$.

Combining all the integrals above together, we have

$$\int\frac{8x^2-x+3}{x^3+x}\,dx=3\ln|x|+\frac{5}{2}\ln|x^2+1|+\tan^{-1}(x)+C$$

3.16 We first write the integral as

$$\lim_{b\to\infty}\int_{4/\pi}^{b}\frac{\sec^2\left(1/x\right)}{x^2}\,dx$$

Now we can perform the substitution $u=\frac{1}{x}$, $du=\frac{-dx}{x^2}$, so the integral becomes

$$\begin{aligned}\lim_{b\to\infty}-\int_{\pi/4}^{1/b}\sec^2(u)\,du &= \lim_{b\to\infty}-\tan(u)\Big|_{\pi/4}^{1/b}\\ &= \lim_{b\to\infty}-\tan\left(\frac{1}{b}\right)+\tan\left(\frac{\pi}{4}\right)\\ &= 1\end{aligned}$$

3.17 Notice that $\ln(x)$ is undefined at $x=0$, so we must consider the limit

$$\lim_{b\to 0^+}\int_b^1 x\ln(x)\,dx.$$

Performing integration by parts with $u=\ln(x)$, $dv=x\,dx$ we have $du=\frac{1}{x}dx$, $v=\frac{x^2}{2}$ we may write the integral as

$$\begin{aligned}&\lim_{b\to 0^+}\left(\frac{x^2\ln(x)}{2}\Big|_b^1-\int_b^1\frac{x}{2}\,dx\right)\\ &=\lim_{b\to 0^+}\frac{-b^2\ln(b)}{2}-\frac{x^2}{4}\Big|_b^1\\ &=\lim_{b\to 0^+}\frac{b^2-2b^2\ln(b)}{4}-\frac{1}{4}\\ &=\lim_{b\to 0^+}\frac{-2b^2\ln(b)}{4}-\frac{1}{4}\end{aligned}$$

Now, to take the limit of the first time we will want to use l'Hôpital's rule, but first we write that limit as

$$\lim_{b\to 0^+}\frac{-2\ln(b)}{4b^{-2}}$$

Then l'Hôpital's rule tells us this is equal to

$$\lim_{b\to 0^+}\frac{-2\cdot\frac{1}{b}}{-8b^{-3}}=\lim_{b\to 0^+}\frac{b^3}{4b}=\lim_{b\to 0^+}\frac{b^2}{4}=0.$$

Thus

$$\int_0^1 x\ln(x)\,dx=-\frac{1}{4}$$

3.18 Since the integrand is undefined at ± 2, we write our integral as the limit

$$\lim_{b \to 2} \int_{-b}^{b} \frac{dy}{\sqrt{4-y^2}}.$$

Now we perform the trig substitution $y = 2\sin\theta$, then $dy = 2\cos\theta d\theta$ and our antiderivative is computed as

$$\int \frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}} d\theta = \int d\theta = \theta + C$$

Writing this in terms of y we have $\theta = \sin^{-1}\left(\frac{y}{2}\right)$ and so our integral becomes

$$\begin{aligned}
\int_{-2}^{2} \frac{dy}{\sqrt{4-y^2}} &= \lim_{b \to 2} \int_{-b}^{b} \frac{dy}{\sqrt{4-y^2}} \\
&= \lim_{b \to 2} \sin^{-1}\left(\frac{y}{2}\right)\Bigg|_{-b}^{b} \\
&= \lim_{b \to 2} \left(\sin^{-1}\left(\frac{b}{2}\right) - \sin^{-1}\left(\frac{-b}{2}\right) \right) \\
&= \frac{\pi}{2} - \frac{-\pi}{2} \\
&= \pi
\end{aligned}$$