Ordinary Differential Equations

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Introduction to the Course

Being a student is easy. Learning requires actual work.

WILLIAM CRAWFORD

Welcome to Math 320, the Ordinary Differential Equations course at Western Carolina University. In this course we will introduce the basic theory and computational methods associated with solving special types of equations which involve derivatives. As we will see throughout the course, differential equations have numerous applications within mathematics, engineering, and the sciences. Unlike many other types of equations, however, there isn't one simple technique or algorithm which applies to all differential equations. In fact there are some equations which we (meaning mankind) do not yet know how to solve and we are forced to resort to numerical approximations of the solutions. The types of equations we can solve are broken up into various "families," each of which has its own solution techniques and associated theory. A large part of our class will be devoted to learning about these various families, where they arise in applications, and how to solve equations from those families. As we will see, these families are not mutually exclusive and certain equations belong to multiple families and can be solved in multiple different ways.

We will also see computers can be used to help us solve (or at least approximate solutions to) various differential equations. In using a computer we have to make a choice of what software to use, and here there are several possibilities: Maple, Mathematica, Matlab, Octave, and many other pieces of software could be used for our class. For simplicity we will use a piece of software called Sage, not because it is inherently "better" than the other possibilities, but because we can jump into using it minimal prerequisites. In particular, we will run Sage in our web browsers and don't need to worry about installing any additional software on your own computer. (You could even use Sage like this on a phone or tablet.)

One caution about the course is in order: we have *a lot* of material to cover, and relatively little time to cover it. As a consequence our course will move quickly, generally spending only one day (sometimes two days) on a given topic. This means that it can be very easy to become overwhelmed and very difficult to catch up if you start to fall behind. For that reason it is important that students understand at the beginning of the semester that they will need to work hard in this course to keep pace with the material and need to get into the habit of studying for this course on a regular basis (ideally daily).

There is no denying that this course is difficult, and students will need to work hard to do well in this course. I firmly believe, however, that all students are capable of succeeding in this course *if they are willing to study regularly, start on assignments early, and take the course seriously from the very beginning*. You should be reading these lecture notes and the corresponding sections of the textbook *before* coming to class, and come to class with questions. No question is too simple or basic, and you should feel free to ask questions anytime you have them. You can ask questions during class time, or before or after class, or during office hours, or through email. I will always do my best to try to give you a complete answer to your question that you can understand. I also encourage you to work outside of class with other students. Sometimes simply bouncing an idea off of someone else can help you see how to start on a problem, and explaining a concept to someone else can help solidify your own understanding. *You are strongly encouraged to work on out-of-class assignments with other people!*

The lecture notes

The notes you are reading are in their fourth incarnation, having evolved from the handwritten examples I used when I first taught a version of this course as a postdoc at Indiana University, Bloomington. This is the second time I have made these notes available to students, and as a consequence the notes are likely “rough around the edges” in some places and likely contain typos and mistakes (though hopefully those are all minor). If you see something in the notes you think is a mistake, it may very well be, and it would be greatly appreciated if you would email me (`cjohnson@wcu.edu`) to let me know about any mistakes. While these notes are my primary resource for the examples I use in the lecture videos, they should not be a substitute for the textbook. Besides the fact that your textbook has fewer mistakes than these notes (probably not mistake-free, but relatively few and minor mistakes) since it was professionally edited, the textbook also has lots of exercises and practice problems, which these notes do not. I hope these notes are helpful to you, but you should not use them as your only source of study material.

Chris Johnson
Fall 2022

1 Introduction

In order to put his system into mathematical form at all, Newton had to devise the concept of differential quotients and propound the laws of motion in the form of total differential equations – perhaps the greatest advance in thought that a single individual was ever privileged to make.

ALBERT EINSTEIN
The World As I See It

1.1 What are differential equations and why should I care?

This course is a first introduction to the field of "differential equations," which is both a very old and developed, but also a very contemporary and active, area of mathematics. Our basic goal in this course is to solve equations which involve derivatives, study the solutions to these equations, and see how these types of equations can be used to develop mathematical models.

There are many reasons why we may be interested in differential equations, and we will give just a few sample applications to illustrate where differential equations arise.

Remark.
You don't need to have a complete understanding of the examples below, they only serve to illustrate that knowledge of differential equations is useful in many different disciplines.

- The basic laws of physics are often stated as differential equations. For example, Newton's second law that force is equal to mass times acceleration, $F = ma$, is really a differential equation since acceleration is the derivative of velocity:

$$F = ma = m\frac{dv}{dt}.$$

A less trivial example concerns the motion of a wave. It can be shown that the height of a one-dimensional wave (e.g., the kind of wave that occurs by whipping one end of a rope up and down) satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(t, x)$ represents the height of the rope at a position x units after t seconds.

- The size of populations of two species, predators are prey, may be modeled using a system of differential equations,

$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = \delta xy - \gamma y$$

where x represents the size of population of predator, y is the size of prey population, and the Greek letters are parameters that describe how the populations interact (e.g., how abundance or scarcity of prey influences the size of the predators' population).

- Rates of chemical reactions are modeled by differential equations in chemistry. For example, nitrogen dioxide, NO_2, decomposes into nitric oxide, NO, and dioxygen, O_2, according to

$$2NO_2 \to 2NO + O_2.$$

The rate at which the concentration of NO_2 changes is proportional to the square of the current concentration,

$$\frac{d[NO_2]}{dt} = -k[NO_2]^2.$$

- Mathematicians interested in complex analysis often focus on special types of functions referred to as *holomorphic* (also called *conformal* or *complex analytic*) and a complex-valued function of the complex plane will be holomorphic precisely when its real and imaginary parts satisfy the following system of partial differential equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- Electrical engineers use differential equations to model voltage and current in circuits. For example, the charge $q(t)$ of a capacitor in an RLC circuit at time t can be shown to satisfy the differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

  where L, R, and C represent the inductance, resistance, and capacitance of the elements of the circuit.

There are many, many other applications of differential equations, not just to the hard sciences, but also to social sciences, economics, and other areas of mathematics. We won't jump into all of the applications right away, but it's good to know that some interesting applications exist.

In this course we will start at the beginning and work our way up to more interesting material. And though our focus will be on the conceptual and theoretical underpinnings of the theory of differential equations, we will make repeated detours into the applications to help us keep in mind the utility of all theory we're building up.

But what is a differential equation?

Let's begin by first being a little bit more careful about defining exactly what a "differential equation" is. A ***differential equation*** is simply an equation which involves a derivative. For example, the equation

$$\frac{dy}{dx} = x^2 + 3x$$

is a particularly simple kind of differential equation. Often we will abbreviate differential equations as ***diff. eq.*** or simply ***DE***.

A ***solution*** to a differential equation is a function that satisfies the equation. In the case of our differential equation above,

$$\frac{dy}{dx} = x^2 + 3x,$$

a solution would be a function of x, which we'll denote y, whose derivative is $x^2 + 3x$. By integrating we see that for any constant C the function

$$y(x) = \frac{x^3}{3} + 3x^2 + C$$

is such a solution. In the simplest possible situations we can solve the differential equation (i.e., compute the solution) by simply integrating.

Example 1.1.
Find a function $f(x)$ so that $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = x\sin(x^2).$$

We want to find a function whose derivative is $x\sin(x^2)$. That is, we want to find the antiderivative of $x\sin(x^2)$:

$$f(x) = \int x\sin(x^2)\,dx.$$

Performing the substitution $u = x^2$, $du = 2x\,dx$, we may compute the antiderivative as

$$\begin{aligned}\int x\sin(x^2)\,dx &= \frac{1}{2}\int \sin(u)\,du \\ &= \frac{-1}{2}\cos(u) + C \\ &= \frac{-1}{2}\cos(x^2) + C.\end{aligned}$$

So, for any choice of the constant C, the function

$$y = \frac{-1}{2}\cos(x^2) + C$$

satisfies the differential equation $\frac{dy}{dx} = x\sin(x^2)$.

This is only the simplest possible scenario, however. In general the derivative $\frac{dy}{dx}$ may also depend on y. For example, we may have a differential equation such as

$$\frac{dy}{dx} = xy^2.$$

That is, a solution to this differential equation is a function whose derivative is equal to x times the function squared. Can a function with such a property even exist? If it does exist, how can we go about finding it? In general these are hard questions to answer, although let's notice that if we have a putative solution to a differential equation, we can always easily check to see if our function satisfies the differential equation or not.

Example 1.2.
Check that for any value of C the function

$$y = \frac{-2}{x^2 + C} = -2(x^2 + C)^{-1}$$

satisfies the differential equation

$$\frac{dy}{dx} = xy^2.$$

We simply compute $\frac{dy}{dx}$ by differentiating the function above, and see if we can rewrite it as xy^2. Notice that the derivative of $y = -2(x^2 + C)^{-1}$ can easily be computed as

$$\frac{dy}{dx} = -2 \cdot (-1) \cdot (x^2 + C)^{-2} \cdot 2x.$$

We can simplify this a little bit to write it as

$$4x(x^2 + C)^2 = \frac{4x}{(x^2 + C)^2}.$$

Let's rewrite this a little bit more by factoring an x off to the side and noticing $4 = (-2)^2$:

$$\frac{4x}{(x^2 + C)^2} = x \cdot \frac{(-2)^2}{(x^2 + C)^2} = x \cdot \left(\frac{-2}{x + C}\right)^2.$$

Notice, though, the factor being square is exactly the original function. Writing this as y we see our expression is xy^2, and so $\frac{dy}{dx} = xy^2$ and the differential equation is satisfied.

How do you solve a differential equation?

It's easy to see if a supposed solution really is a solution or not, but how do we go about actually finding the solution to begin with? This is usually rather difficult, so we'll begin by restricting ourselves to situations where we can build up some theory and techniques for solving these equations.

Sometimes it's helpful to have a graphical representation of solutions to a differential equation. Even if we can't analytically solve the equation, these graphical representations can provide some valuable insights. One way to visualize solutions to a differential equation is with a "slope field," as we'll now describe.

Given a differential equation, let's say $\frac{dy}{dx} = y - x$ just to have a concrete example in mind, any solution $y = f(x)$ gives us a graph in the plane. Can we figure out what such a graph looks like without first knowing what $f(x)$ is? If we somehow knew a particular point (a, b) was on the graph, then that would mean we would also know the slope of the line tangent to $y = f(x)$ at that point (a, b). In particular, it must be $b - a$ since our equation tells us $f'(x) = y - x$ for each point (x, y) on the graph.

For example, if we somehow knew the curve passed through the point $(1, 3)$, then the slope of the tangent line would have to be 2 at that point; if the graph passed through the $(-2, 4)$, then the slope would have to be 6 at that point; and so on.

The association of a slope to each point (x, y) in the plane gives us a ***slope field***, and we visualize a slope field by drawing a small line segment with the corresponding slope at each point in the plane (really we can only draw these line segments for some finite collection of point). The slope field determined by $y - x$ is visualized by Figure 1.1.

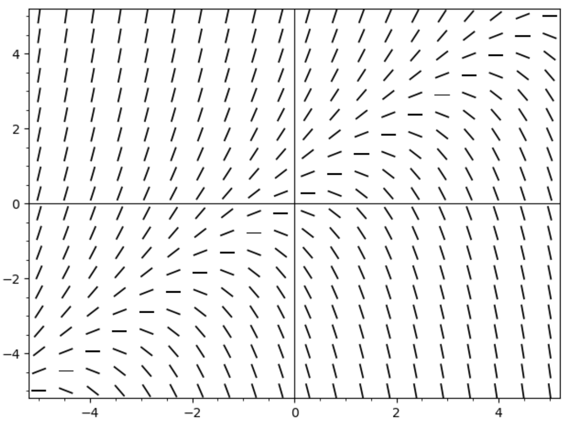


Figure 1.1: The slope field determined by $y - x$.

Making pictures of slope fields is extremely tedious to do by hand, but very easy to do with a computer, and so this is a good time to introduce the mathematical software Sage which can draw these kinds of pictures for us.

Sage

Sage is a free, open source collection of mathematical software that can be used to do lots of interesting things. One of the nice things about Sage is that you can run it in your browser without having to download and install any additional software, simply by visiting the website `http://sagecell.sagemath.org`, entering commands in the textbox, and then hitting `Shift-Enter` on your keyboard.

In this course we will introduce Sage gradually as we need it. For now we just want to see how to plot a slope field in Sage. To do this you would simply enter the commands

```
x, y = var('x, y')
plot_slope_field(y - x, (x, -5, 5), (y, -5, 5))
```

into the textbox that appears on `http://sagecell.sagemath.org` and holding down the `Shift` key on your keyboard while hitting `Enter`. (Or you can click on the 'Evaluate' button that appears on the webpage.) You should then see something similar to Figure 1.2.

The first command entered into Sage above, the `x, y = var('x,y')`, tells Sage that `x` and `y` are mathematical variables. The second line tells Sage to plot the slope field determined by `y - x` in a window where the x-values range from -5 to 5, and the y-values range from -5 to 5.

What does a slope field tell us?

Even without solving a differential the differential equation $\frac{dy}{dx} = y - x$, the slope field of Figure 1.1 tells us a few things about the solutions of the equation, whatever they happen to be. For example, the slope field tells us that if $y = f(x)$ is a solution to $\frac{dy}{dx} = y - x$, then we can see that $\lim_{x\to\infty} f(x)$ diverges to either positive or negative infinity, and these two cases are separated by whether $y = f(x)$ is above or below the line $y = x + 1$.

In general if a differential equation has one solution then it has infinitely-many solutions (this basically corresponds to the infinitely-many choices of the "$+C$" that appears when performing an integration). In terms of slope fields, these correspond to the different curves $y = f(x)$ which are tangent to the line segments of the slope field at every point on the curve. That is, the slope fields tell us what graphs of solutions to a differential

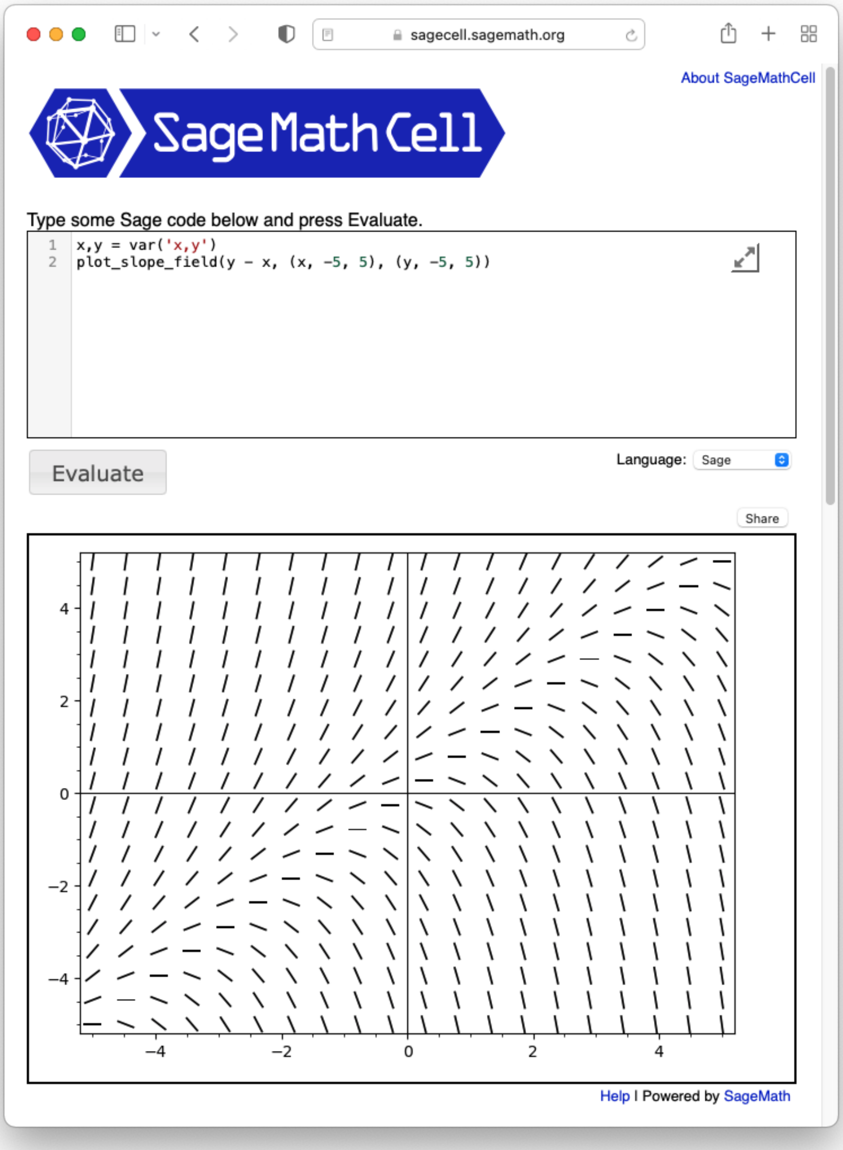


Figure 1.2: Plotting the slope field for $y - x$ in Sage.

equation may look like. In Figure 1.3, for example, we have three different curves which are plotted and which stay tangent to the slope field at every point. That is, each of those curves represents the graph of some solution of the underlying differential equation.

The curves which stay tangent to the slope field at each point of the

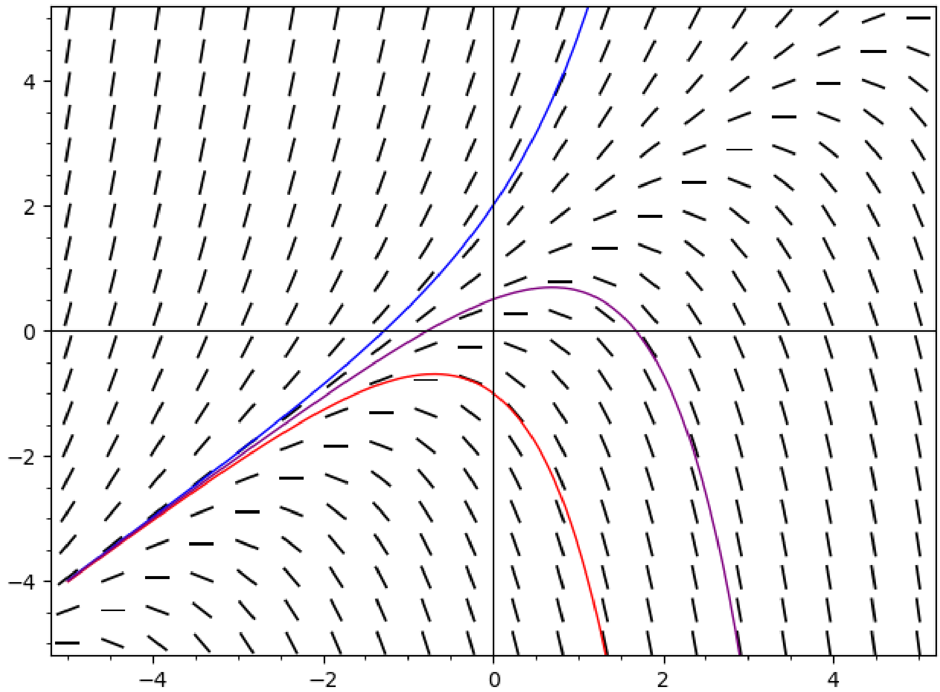


Figure 1.3: Some integral curves of the slope field which corresponds to the differential equation $\frac{dy}{dx} = y - x$.

curve (i.e., the graphs of solutions of the differential equation) are called the ***integral curves*** of the slope field.

Though there are infinitely-many solutions to a differential equation, for a given application we may only care about one particular solution. We can single out a particular solution by giving an extra piece of information called an ***initial condition***. A differential equation together with an initial conditional is called an ***initial value problem*** or ***IVP***. You have actually already solved some initial value problems before in calculus: any time you found an antiderivative and a particular choice of $+C$, you solved an initial value problem.

Example 1.3.
Find a function $y = f(x)$ which solves the differential equation

$$\frac{dy}{dx} = x\ln(x)$$

and which also satisfies $f(1) = 1/2$.

We of course just integrate $x \ln(x)$. Performing integration by parts with

$$u = \ln(x) \qquad dv = x\,dx$$
$$du = \frac{1}{x}\,dx \qquad v = \frac{x^2}{2}$$

we have

$$\begin{aligned}\int x\ln(x)\,dx &= \frac{x^2\ln(x)}{2} - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx \\ &= \frac{x^2\ln(x)}{2} - \frac{1}{2}\int x\,dx \\ &= \frac{x^2\ln(x)}{2} - \frac{x^2}{4} + C.\end{aligned}$$

So, for any of the infinitely-many choices of C, the function

$$y = f(x) = \frac{x^2\ln(x)}{2} - \frac{x^2}{4} + C$$

satisfies $\frac{dy}{dx} = x\ln(x)$. Only one choice of C will give us a function that satisfies $f(1) = 1/2$, though, and we can determine that C with some simple algebra:

$$\begin{aligned} f(1) &= \frac{1}{2} \\ \implies \frac{1^2\cdot\ln(1)}{2} - \frac{1^2}{4} + C &= \frac{1}{2} \\ \implies \frac{-1}{4} + C &= \frac{1}{2} \\ \implies C &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.\end{aligned}$$

Thus the function

$$f(x) = \frac{x^2\ln(x)}{2} - \frac{x^2}{4} + \frac{3}{4}$$

solves our initial value problem.

Mathematical modeling

One of the main applications of differential equations comes from mathematical modeling. That is, the discipline of developing the mathematical systems that model the situations we may care about in problems that arise in physics, engineering, or other fields.

It's a bit hard to imagine when first learning about this topic, but sometimes it can be difficult to "directly" find a function which models some phenomenon we are interested in, but we may be able to more easily write down a differential equation which represents how that quantity changes. One simple example of this is given below where use the fact that evaporation occurs at the surface of a liquid to determine how the volume of the liquid changes.

Example 1.4.
Suppose a spherical raindrop evaporates at a rate proportional to its surface area. Find a differential equation whose solution is the volume of the raindrop as a function of time.

Ultimately what we'd like is a function $V(t)$ which gives the volume of the raindrop as a function of time. It's not immediately clear what such a function would be, but we do know something about the function's rate of change. Letting S denote the surface are of the rain drop, we are told

$$\frac{dV}{dt} = -kS.$$

That is, as the raindrop evaporates its volume decreases, so its derivative is negative. This rate of change is proportional to the surface area of the rain drop since evaporation occurs at the surface of a liquid. (The constant k depends on various factors that influence the rate at which the liquid evaporates, such as the temperature and humidity of the surrounding air.)

Even though this differential equation models the situation we're interested in, it isn't something we can very easily solve right now. For one thing, let's notice that the left-hand side of the equation is in terms of V and t, not S. We can fix this, though, by writing S in terms of V.

If the radius of our raindrop was r, then let's notice that

$$S = 4\pi r^2$$
$$V = \frac{4}{3}\pi r^3.$$

We can now perform a little bit of algebra to obtain the following:

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ \implies \frac{3V}{4\pi} &= r^3 \\ \implies \left(\frac{3V}{4\pi}\right)^{1/3} &= r \\ \implies 4\pi\left(\frac{3V}{4\pi}\right)^{2/3} &= 4\pi r^2 = S. \end{aligned}$$

Performing just a touch of of arithmetic to simplify this we have

$$S = (6V)^{2/3}\pi^{1/3},$$

and we can use this to rewrite our earlier differential equation as

$$\frac{dV}{dt} = -k6^{2/3}\pi^{1/3}V^{2/3}.$$

This is a differential equation that we will learn how to solve later, but for now let's notice that it's easy to verify that the differential equation is solved by

$$V = \left(\frac{-k6^{2/3}\pi^{1/3}}{3}t + C\right)^3.$$

To see that this really does solve our differential equation, we just compute the derivative $\frac{dV}{dt}$ of the function V above and perform some simple algebra to see that it can be written as $-k6^{2/3}\pi^{1/3}V^{2/3}$:

$$\begin{aligned}\frac{dV}{dt} &= 3\left(\frac{-k6^{2/3}\pi^{1/3}}{3}t + C\right)^2 \cdot \frac{-k6^{2/3}\pi^{1/3}}{3}\\ &= -k6^{2/3}\pi^{1/3} \cdot \left(\frac{-k6^{2/3}\pi^{1/3}}{3}t + C\right)^2\\ &= -k6^{2/3}\pi^{1/3} \cdot \left[\left(\frac{-k6^{2/3}\pi^{1/3}}{3}t + C\right)^3\right]^{2/3}\\ &= -k6^{2/3}\pi^{1/3}V^{2/3}.\end{aligned}$$

Case study: free fall with air resistance

Let's now put everything we've discussed together in one "case study," and let's also see how to solve the differential equation that we'll derive in that example.

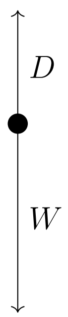
Suppose an object is dropped from a height near the surface of the Earth. Assuming drag due to air resistance is proportional to the object's velocity, find the velocity of the object t seconds after being dropped.

How are we going to get a differential equation out of this? Let's first recall that Newton's second law tells us

$$F = ma = m\frac{dv}{dt}.$$

So if we know what the force F is, we will have a differential equation.

There are actually two different forces which are acting on our object: the weight W of the object is pulling the object down, but the drag D is resisting the direction of motion.



Thus the total force on our object is the sum of these two forces: $F = W + D$.

If the object has mass m measured in kilograms, then its weight will be $W = -9.8m$. (This -9.8 is coming from the acceleration due to gravity and is measured in kilograms per second squared.)

The drag D is proportional to the velocity v, but is in the opposite direction of the motion. This means D has the form $D = -\gamma v$ for some positive constant γ that depends on the shape of our object, the density of the surrounding air, and other various parameters.

Putting this all together we have $F = ma = m\frac{dv}{dt}$ from Newton, but we also have $F = w + D = -9.8m - \gamma v$. These both equal our force, though, and so they equal one another, and this gives us our differential equation,

$$\begin{aligned} m\frac{dv}{dt} &= -9.8m - \gamma v \\ \implies \frac{dv}{dt} &= -9.8 - \frac{\gamma}{m}v. \end{aligned}$$

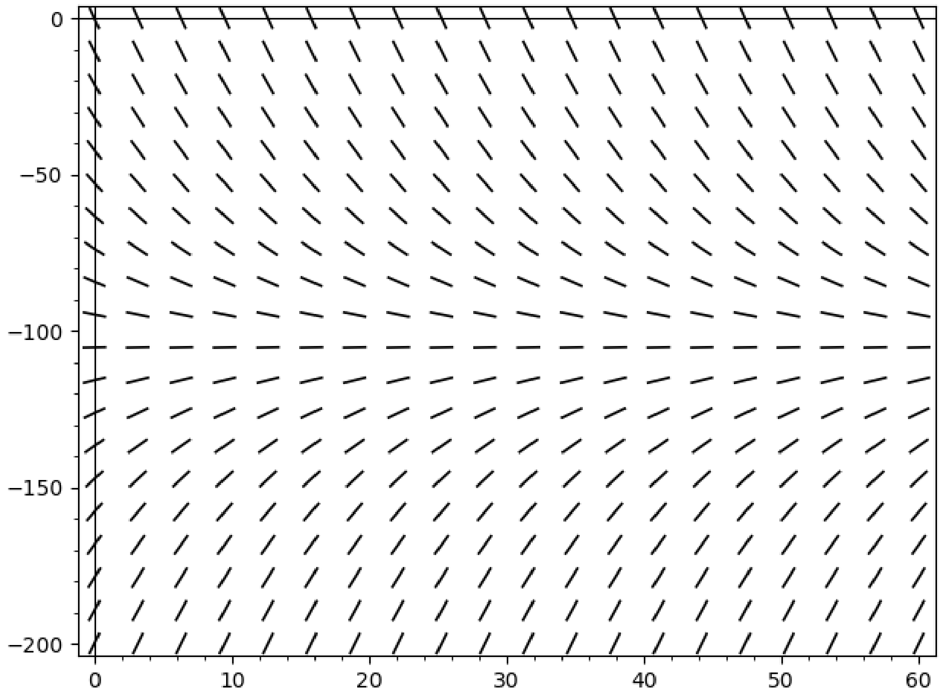
For example, if we had a spherical object near sea level, from experimental evidence we know that γ is about 0.47. If that object had a mass of five kilograms, then our differential equation would become

$$\frac{dv}{dt} = -9.8 - \frac{0.47}{5}v = -9.8 - 0.094v.$$

We will analytically solve this differential equation n a moment, but first let's consider the slope field in the (t, v) plane corresponding to our differential equation, $\frac{dv}{dt} = -9.8 - 0.094v$. We can plot this slope field in Sage using

```
t,v = var('t,v')
plot_slope_field(-9.8 - 0.094 * v, (t, 0, 60), (v, -200, 0))
```

This produces the following figure:



Of the infinitely-many possible solutions to our differential equations (i.e., the infinitely-many integral curves which are tangent to this slope field), the one that we care about is the one which passes through the origin. We care about this particular solution because the object is *dropped*, meaning its initial velocity at time $t = 0$ is $v = 0$.

From the slope field we see that our velocity is going to level off somewhere around $-105\frac{\mathrm{m}}{\mathrm{s}}$. This corresponds to the "terminal velocity," where the drag D balances out the weight W and the net force is zero; once the force is zero, Newton's second law tells us there is no acceleration, and so the object continues to move at the same velocity.

Now the question becomes whether or not we can find an exact formula for the velocity of our falling object; i.e., can we solve the differential equation

$$\frac{dv}{dt} = -9.8 - \frac{\gamma}{m}v.$$

(The tricks we are about to use may feel a little bit out of the blue, but don't worry about that too much right now. Later on we'll see more precise, formulaic procedures for solving differential equations.)

Let's first just do some algebra to rewrite our differential equation by factoring $-\frac{\gamma}{m}$ out of the right-hand side, leaving us with

$$\frac{dv}{dt} = \frac{-\gamma}{m}\left(\frac{9.8m}{\gamma} + v\right).$$

Now we can divide both sides of the equation by $\frac{9.8m}{\gamma} + v$ to obtain

$$\frac{\frac{dv}{dt}}{\frac{9.8m}{\gamma} + v} = \frac{-\gamma}{m},$$

or simply

$$\frac{1}{\frac{9.8m}{\gamma} + v} \cdot \frac{dv}{dt} = \frac{-\gamma}{m}.$$

Our goal is to find v and we have an equation involving its derivative, so it seems like integrating both sides of the equation would be a reasonable thing to do. Since both sides of the equation above are equal (that's the definition of an equation, after all), their integrals must be equal, and so we have the following equation:

$$\int \frac{1}{\frac{9.8m}{\gamma} + v} \cdot \frac{dv}{dt}\, dt = \int \frac{-\gamma}{m}\, dt.$$

The right-hand side of this equation is easy to integrate:

$$\int \frac{-\gamma}{m}\, dt = \frac{-\gamma}{m}t + C.$$

The left-hand side is just a tiny bit more involved since we need to perform a u-substitution. Taking $u = \frac{9.8m}{\gamma} + v$, we would have $du = \frac{dv}{dt}\, dt$. Our integral then becomes

$$\int \frac{1}{u}\, du = \ln|u| + C.$$

Rewriting u in terms of v this becomes

$$\int \frac{1}{\frac{9.8m}{\gamma} + v} \frac{dv}{dt}\, dt = \ln\left|\frac{9.8m}{\gamma} + v\right| + C.$$

Now there's a slightly subtle point we need to be careful of right here. We have performed two integrations and each one gave us a "$+C$." These *are not* necessarily the same C, however! The "$+C$" is just a placeholder, like a variable, for any constant we might like to attach to our antiderivative, and there's nothing that magically forces these "$+C$" values to be the same. So, really, the technically more correct thing we should have done was to use different letters for these different constants. For example, we could have said something like

$$\int \frac{-\gamma}{m}\,dt = \frac{-\gamma}{m}t + C_1$$
$$\int \frac{1}{\frac{9.8m}{\gamma}+v}\,\frac{dv}{dt}\,dt = \ln\left|\frac{9.8m}{\gamma}+v\right| + C_2$$

to emphasize that these constants were different.

Now, after integrating both sides of our differential equation, we are left with the equation

$$\ln\left|\frac{9.8m}{\gamma}+v\right| + C_2 = \frac{-\gamma}{m}t + C_1.$$

Now there's a little trick that makes life a just a tiny bit easier, but it does take some getting used to. The "$+C$" values, C_1 and C_2 above, we just went out of our way to emphasize were different are just constants, whatever they happen to be for the problem at hand. We could thus move both constants to the same side of the equation, such as

$$\ln\left|\frac{9.8m}{\gamma}+v\right| = \frac{-\gamma}{m}t + C_1 - C_2$$

and then just write C for this combination of constants $C_1 - C_2$. This would give us the (very slightly) simpler equation

$$\ln\left|\frac{9.8m}{\gamma}+v\right| = \frac{-\gamma}{m}t + C.$$

This kind of manipulation where we combine different constants from different integrations into one constant might seem a little strange at first, so don't worry too much about it right now: we'll see lots and lots of other examples of this kind of manipulation later.

Anyway, now we have integrated both sides of our initial differential equation and we are left with a new equation,

$$\ln\left|\frac{9.8m}{\gamma}+v\right| = \frac{-\gamma}{m}t + C.$$

Keeping in mind our goal is to find v, let's now just try to do the algebra to solve for v in this expression. First we will exponentiate both sides of the equation to obtain

$$\left|\frac{9.8m}{\gamma} + v\right| = e^{\frac{-\gamma}{m}t+C}.$$

(Our goal here is to get v by itself on the left-hand side of the equation, so we're trying to get rid of the other "stuff" around the v. Since we had a natural log on the left-hand side, we need to remove it, which we can do by raising e to everything on both sides of the equation since $e^{\ln(x)} = x$.)

Let's now notice the right-hand side could be written as

$$e^{\frac{-\gamma}{m}t+C} = e^{\frac{-\gamma}{m}t}e^{C}.$$

Now, again, C is just some constant. Thus e^C is just some constant as well. Our convention going forward is that whenever we have an expression that's "just some constant," we'll replace it by C. That is, we will write e^C as simply C. (Again, this probably seems weird. The idea is just that if we have an arbitrary constant we'll always just call it C; so, technically, these are all different C's that we're writing down.) Our equation is now

$$\left|\frac{9.8m}{\gamma} + v\right| = Ce^{\frac{-\gamma}{m}t}$$

We're a little bit closer to getting v by itself on the left-hand side, but there's still more to do. In particular, we need to get rid of the absolute values. Here we'll use one other really minor trick. Let's just observe that if $|x| = 2$, then the only options are $x = 2$ or $x = -2$, which we might write simply as $x = \pm 2$. In general, we can drop absolute values on one side of an equation, but when we do so we pick up a plus/minus on the other side. Using this, our equation above becomes

$$\frac{9.8m}{\gamma} + v = \pm Ce^{\frac{-\gamma}{m}t}.$$

Once again, C is just some constant, so $\pm C$ is also just some constant, and we'll replace $\pm C$ by simply C to obtain

$$\frac{9.8m}{\gamma} + v = Ce^{\frac{-\gamma}{m}t}.$$

Finally, we can easily solve for v:

$$v = Ce^{\frac{-\gamma}{m}t} - \frac{9.8m}{\gamma}.$$

Let's go ahead and verify that this really does solve our original differential equation of $\frac{dv}{dt} = -9.8 - \frac{\gamma}{m}v$ by just differentiating and rewriting as necessary.

$$
\begin{aligned}
v &= Ce^{\frac{-\gamma}{m}t} - \frac{9.8m}{\gamma} \\
\implies \frac{dv}{dt} &= \frac{-\gamma}{m}Ce^{\frac{-\gamma}{m}t} \\
&= \frac{-\gamma}{m}Ce^{\frac{-\gamma}{m}t} + 9.8 - 9.8 \\
&= \frac{-\gamma}{m}\left(Ce^{\frac{-\gamma}{m}t} - \frac{9.8m}{\gamma}\right) - 9.8 \\
&= \frac{-\gamma}{m}v - 9.8 \\
&= 9.8 - \frac{\gamma}{m}v
\end{aligned}
$$

Thus we *do* in fact have a solution to our differential equation.

The last thing that remains is to find the solution to our initial value problem. I.e., the choice of C which will give us $v(0) = 0$. This is just some more algebra, however. As we have determined

$$v(t) = Ce^{\frac{-\gamma}{m}t} - \frac{9.8m}{\gamma}$$

we know that

$$v(0) = Ce^{\frac{-\gamma}{m}\cdot 0} - \frac{9.8m}{\gamma} = C - \frac{9.8m}{\gamma}$$

However, this is supposed to equal 0, so we have the equation

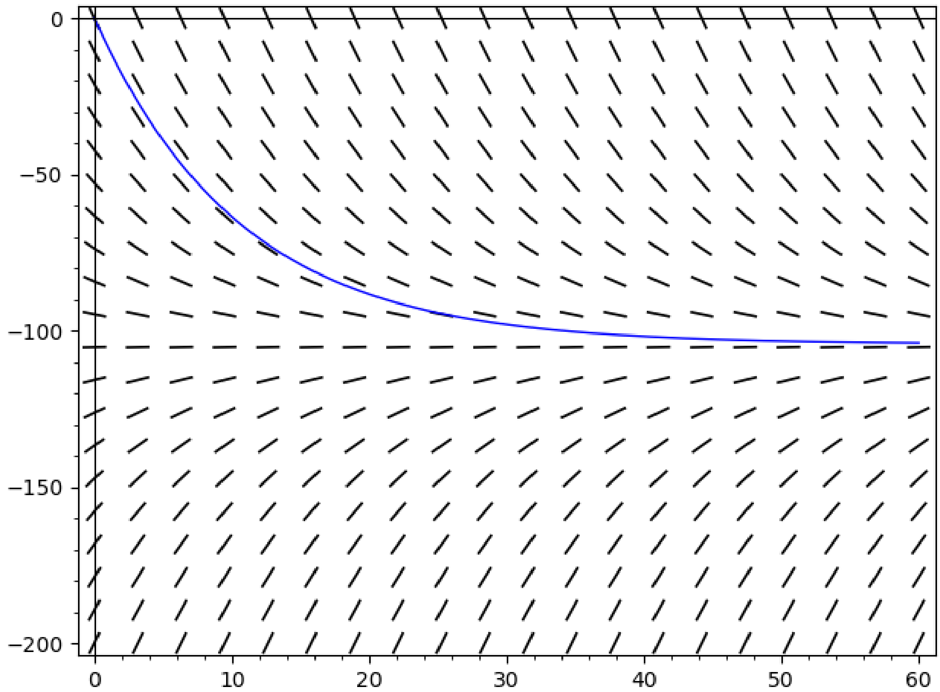
$$C - \frac{9.8m}{\gamma} = 0,$$

which is of course solved by $C = \frac{9.8m}{\gamma}$.

After all of that, we now know that if an object of mass m is dropped near the surface of the Earth, and if the drag due to air resistance is proportional to the object's velocity, then the velocity t seconds after being dropped is

$$v(t) = \frac{9.8m}{\gamma}e^{\frac{-\gamma}{m}t} - \frac{9.8m}{\gamma} = \frac{9.8m}{\gamma}\left(e^{\frac{-\gamma}{m}t} - 1\right).$$

To verify that this gives us the answer we desire, we can plot the graph of this function together with the slope field in Sage as follows. Using $m = 5$ and $\gamma = 0.47$ as before, we could plot the following:



Notice that, as we expected, the curve approaches the limiting terminal velocity as t increases.

The image above was plotted in Sage using the following commands:

```
t,v = var('t,v')
m = 5
gamma = 0.47
img = plot_slope_field(-9.8 - gamma / m * v, (t, 0, 60), (v, -200, 0))
img += plot(9.8 * m / gamma * (e^(-gamma / m * t) - 1), (t, 0, 60))
img.show()
```

You don't need to worry about the details of all of the commands above for right now, but they are summarized below in case you are curious. (You can very safely avoid reading the next paragraph if you want to, though.)

In addition to creating our slope field with `plot_slope_field`, we've done a few other things here. In order to simplify our commands to come we introduced variables `m` and `gamma` to contain our $m = 5$ and $\gamma = 0.47$ values. This is a convenient thing to do because it allows us to see the "meaning" of some of the numbers that appear in our equation. This also allows us to very easily modify our commands if we were to change the problem by changing, for example, the mass. Instead of immediately plotting our slope field, we saved the plot as a variable `img` and then added the plot of our integral curve computed above to this using `+=`. In order to plot graphs of functions we use the command `plot` which takes two arguments: the function to graph, and then a range for the independent variable. Finally, we show the image on the screen by using the `show()` method on our `img` variable.

> **Remark.**
> It's worth pointing out at this point that Sage is basically just Python with some extra packages for doing some mathematical calculations, graphing, solving equations, etc. So, if you know Python you already know the basics of Sage. If you don't know Python, then you basically get to learn Python – which is a useful skill! – as a consequence of learning Sage.

1.2 Manipulating "$+C$" and a few more examples

What's going on with the "$+C$" values?

At various points during our process of solving the differential equation at the end of the last section, we performed manipulations of the arbitrary constant "$+C$" from our integration, but left this as "$+C$." It's worth taking a moment to explain what's going on here.

When we find an antiderivative, we always get a "$+C$" which is a placeholder for *any* constant, because the derivative of any constant is always zero. For instance,

$$\int x\,dx = \frac{x^2}{2} + C$$

and the C that appears could be 0 or 15 or $-\sqrt{17}$, or anything else: the C is completely arbitrary. If we modify C by adding 3, or by dividing by 2, or taking the sine of C, we still have an arbitrary constant. In general, we may need to do several of these manipulations over the course of solving a problem. Since writing things like

$$\frac{3C+2}{-7} - 5$$

gets old very quickly, and since this is still just an arbitrary constant, we often just write "$+C$" instead of these more complicated expressions.

Similarly, if we have multiple arbitrary constant – different "$+C$" values from different integration steps – we often combine them together as one single C.

Examples of solving some simple differential equations

Let's now generalize the process we used to solve the differential equation that appeared at the end of the last section.

Suppose that we have a differential equation which is written as

$$\frac{dy}{dx} = my + b.$$

That is, the right-hand side is a linear function of the unknown y. If we can get all of the y's and the $\frac{dy}{dx}$'s together on one side of the equation, then we might be able to integrate that side of the equation using a u-substitution. Mimicking what we did to solve the differential equation in the last section, let's factor the m out from the right-hand side of the equation to obtain

$$\frac{dy}{dx} = m\left(y + \frac{b}{m}\right).$$

Now we can divide both sides of the equation by $y + \frac{b}{m}$, giving us

$$\frac{1}{y+\frac{b}{m}}\frac{dy}{dx} = m.$$

Since these functions are equal, their antiderivatives are equal, and so

$$\int \frac{1}{y+\frac{b}{m}}\frac{dy}{dx}\,dx = \int m\,dx.$$

Of course, the right-hand side is simply $mx + C$. We can integrate the left-hand side by using the following u-substitution:

$$u = y + \frac{b}{m}$$
$$du = \frac{dy}{dx}\,dx$$

We can thus rewrite the left-hand side of our equation above as

$$\int \frac{1}{u}\,du = \ln|u| + C.$$

Since u is really $y + \frac{b}{m}$, we thus have

$$\int \frac{1}{y + \frac{b}{m}} \frac{dy}{dx}\,dx = \ln\left|y + \frac{b}{m}\right| + C.$$

Hence after integrating both sides of our differential equation we now have

$$\ln\left|y + \frac{b}{m}\right| = mx + C \tag{1.1}$$

where we have gone ahead and combined the arbitrary "$+C$" from each integration into a single "$+C$" which we wrote on the right-hand side.

Our differential equation has now reduced to an algebra problem: solving for y in Equation 1.1. We can solve this algebra problem using the same sequence of tricks we saw at the end of our example at the end of the last section: we will exponentiate to get rid of the natural log, then drop the absolute values for plus/minus, and finally get y by itself on one side of the equation. Along the way we will replace expressions involving the arbitrary constant C with a single C. This leads us to the following sequence of algebra:

$$\begin{aligned}
\ln\left|y+\frac{b}{m}\right| &= mx+C \\
\implies e^{\ln\left|y+\frac{b}{m}\right|} &= e^{mx+C} \\
\implies \left|y+\frac{b}{m}\right| &= e^{C}e^{mx} \\
\implies \left|y+\frac{b}{m}\right| &= Ce^{mx} \\
\implies y+\frac{b}{m} &= \pm Ce^{mx} \\
\implies y+\frac{b}{m} &= Ce^{mx} \\
\implies y &= Ce^{mx}-\frac{b}{m}
\end{aligned}$$

That is, we claim the solution to the differential equation

$$\frac{dy}{dx} = my + b$$

is given by

$$y = Ce^{mx} - \frac{b}{m}.$$

It's relatively easy for us to verify this really is the correct solution by differentiating, and so we will go ahead and double-check that our solution is correct:

$$\begin{aligned}
y &= Ce^{mx} - \frac{b}{m} \\
\implies \frac{dy}{dx} &= \frac{d}{dx}\left(Ce^{mx} - \frac{b}{m}\right) \\
&= mCe^{mx}
\end{aligned}$$

Right now this doesn't quite look like what we want: we want to see that $\frac{dy}{dx}$ is equal to $my+b$, and what we have instead is mCe^{mx}. This does not necessarily mean our answer is incorrect, it just means we need to do a little bit more work to rewrite our derivative:

$$\begin{aligned}\frac{dy}{dx} &= mCe^{mx} \\ &= mCe^{mx} - b + b \\ &= m\left(Ce^{mx} - \frac{b}{m}\right) + b \\ &= my + b.\end{aligned}$$

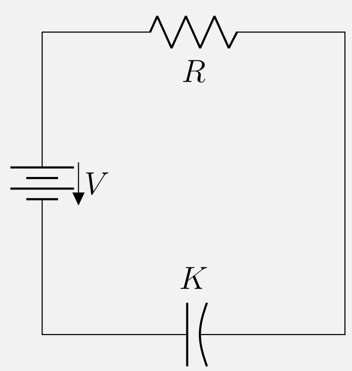
Thus we have verified our putative solution of $y = Ce^{mx} - \frac{b}{m}$ really does solve our differential equation $\frac{dy}{dx} = my + b$.

We can go a step further and determine what value of C solves the initial condition $y(x_0) = y_0$. (I.e., what value of C will additionall guarantee that our solution has the output of y_0 when the input is x_0; equivalently, the graph of our solution goes through the point (x_0, y_0).)

$$\begin{aligned} y(x_0) &= y_0 \\ \implies Ce^{mx_0} - \frac{b}{m} &= y_0 \\ \implies Ce^{mx_0} &= y_0 + \frac{b}{m} \\ \implies C &= \frac{y_0 + \frac{b}{m}}{e^{mx_0}}\end{aligned}$$

Let's now use our formula for the solution we've developed to solve two particular differential equations.

Example 1.5.
Consider a circuit containing a capacitor of capacitance K, a resistor of resistance R, and a battery of voltage V.



By Kirchhoff's laws,

$$R\frac{dQ}{dt} + \frac{Q}{K} = V$$

where $Q(t)$ is the charge of the capacitor at time t. Notice that this is a differential equation of the form described above. In particular, by performing a minor amount of algebra to put $\frac{dQ}{dt}$ by itself on the left-hand side, the equation becomes

$$\frac{dQ}{dt} = \frac{-1}{KR}Q + \frac{V}{R}.$$

Though the letters are different, this is exactly the kind of thing we had above. That is, this *is* in the form $\frac{dy}{dx} = my + b$ where $y = Q$, $x = t$, $m = \frac{-1}{KR}$, and $b = \frac{V}{R}$. Thus our differential equation is solved by

$$\begin{aligned} Q &= Ce^{\frac{-t}{KR}} - \frac{V/R}{-1/KR} \\ &= Ce^{\frac{-t}{KR}} + KV \end{aligned}$$

If we had the initial condition that $Q(0) = 0$ (i.e., the capacitor initially has zero charge), then we can compute that the constant C equals

$$C = \frac{0 + \frac{V/R}{-1/KR}}{e^{\frac{-1}{KR}\cdot 0}} = -KV.$$

Hence

$$\begin{aligned} Q(t) &= -KVe^{\frac{-t}{KR}} + KV \\ &= KV\left(1 - e^{\frac{-t}{KR}}\right). \end{aligned}$$

Example 1.6.

Carbon-15 has a half-life of about 2.5 seconds. Given an initial sample of 1000 grams of Carbon-15, find a function $M(t)$ for the mass of Carbon-15 remaining after t seconds.

Notice that the rate of change of the mass is proportional to the mass. I.e., we have the differential equation

$$\frac{dM}{dt} = -kM$$

where k depends on how quickly the mass decays. Since this is a linear differential equation (i.e., it has the form $\frac{dy}{dx} = my + b$ where here we have $y = M$, $x = t$, $m = -k$ and $b = 0$), we can solve the equation and we see the solution has the form

$$M = Ce^{-kt}.$$

Now, since we know that the half life is 2.5 seconds and we start with 1000 grams, we have the following two pieces of information:

$$\begin{aligned} M(0) &= 1000 \\ M(2.5) &= 500. \end{aligned}$$

That is,

$$\begin{aligned} Ce^{-k \cdot 0} &= 1000 \\ Ce^{-2.5k} &= 500 \end{aligned}$$

The first equation tells us that $C = 1000$. We can plug this into the second equation to solve for k:

$$\begin{aligned} 1000e^{-2.5k} &= 500 \\ \implies e^{-2.5k} &= \frac{1}{2} \\ \implies -2.5k &= \ln\left(\frac{1}{2}\right) = -\ln(2) \\ \implies k &= \frac{\ln(2)}{2.5} \approx 0.2773. \end{aligned}$$

Thus the mass of Carbon-15 at time t, measured in grams, is

$$M = 1000e^{\frac{-\ln(2)}{2.5}t} \approx 1000e^{-0.2773t}.$$

1.3 Some differential equations vocabulary

Let's end our introductory discussion of differential equations by giving some basic terminology that we will use throughout the semester.

Differential equations are usually divided into two families, "ordinary differential equations" and "partial differential equations." An ***ordinary differential equation*** is a differential equation where the unknown function has only one variable and so the derivatives involved are the "ordinary" derivative that you learned about in your first semester of calculus. A ***partial differential equation*** involves functions of multiple variables and the derivatives specified are the partial derivatives of the various variables, as you would have learned about in a third semester of calculus. In this course we will exclusively consider ordinary differential equations and will save partial differential equations for another course.

When solving a differential equation, we will often be concerned with how many derivatives are specified in the equation. For example, does the differential equation involve only first derivatives, or does it involve second derivatives as well? The ***order*** of a differential equaton is simply the highest order of a derivative that appears. For example, the equation

$$3\frac{dy}{dx} + y = x^2$$

is a first order differential equation since it only involves the first dirst derivative of y. However, the equation

$$7x\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} = y$$

is a second order differential equation because it involves the second derivative of y; and something like

$$\frac{d^3y}{dx^3} = y^2$$

is a third order differential equation as it involves a third derivative.

In general, an n-th order differential equation (i.e., a differential equation involving the n-th derivative of the function, and possibly lower order derivatives, but nothing higher) can always be written as

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0.$$

That is, we simply move everything to the left-hand side of the equation, and are left with an expression involving x, y, and the derivatives of y up to the n-th derivative.

For example, the first-order equation

$$3\frac{dy}{dx} + y = x^2$$

could be rewritten as

$$-x^2 + y + 3\frac{dy}{dx} = 0.$$

In this case our function F would be $F(a,b,c) = -a^2 + b + 3c$ where we plug in x for a, y for b, and $\frac{dy}{dx}$ for c.

The second-order equation

$$7x\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} = y$$

could be rewritten as

$$-y + 7x\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} = 0$$

and our function F would be $F(a,b,c,d) = -b + 7ad - 2bc$.

The first set of differential equations we will study will be the "linear" differential equations, which is might not be quite what you'd expect based on the name. In general, we will say that a differential equation is ***linear*** if it is linear in the unknown (e.g., the y) and its derivatives (such as $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$). A slightly strange and subtle point is that we *do not* require this equationt o be linear in the independent variable (the x).

This basically means that linear differential equations will be of the form

$$f_0(x)y + f_1(x)\frac{dy}{dx} + f_2(x)\frac{d^2y}{dx^2} + \ldots + f_n(x)\frac{d^ny}{dx^n}$$

where we do not assume the $f_i(x)$ functions are linear! As strange as it sounds, the short version of this is that linear differential equations can involve non-linear functions of x – they just can't involve non-linear functions of y or its derivatives.

For example, the differential equations

$$\frac{dy}{dx} = x^2y - 3x \quad \text{and} \quad x^2\frac{dy}{dx} + e^x y = x$$

are both linear, even though they contain non-linear functions of x. However differential equations such as

$$\frac{dy}{dx} = y^2 \quad \text{and} \quad y\frac{dy}{dx} = x$$

are not linear. The differential equation $\frac{dy}{dx} = y^2$ is not linear because of the y^2, and $y\frac{dy}{dx} = x$ is not linear because of the product of y and $\frac{dy}{dx}$.

Remark.
The language is strange and takes some getting used to, so don't worry too much about it right now if this seems weird. It will gradually become easier to determine which equations are linear and which are nonlinear as we do more examples.

1.4 Practice problems

Problem 1.1. For each differential equation described below, use Sage to plot the associated slope field. Using that slope field, describe what happens to the solution y of the differential equation as x goes to infinity.

(a) $\dfrac{dy}{dx} = -y$

(b) $\dfrac{dy}{dx} = -y + 3$

(c) $\dfrac{dy}{dx} = -2y + 3$

(d) $\dfrac{dy}{dx} = y$

(e) $\dfrac{dy}{dx} = y + 3$

Problem 1.2. Based on your answers to the problems in Problem 1, determine the values of m and b such that solutions to the differential equation $\frac{dy}{dx} = my + b$ have the described behavior. Use Sage to plot the slope field for your differential equation to verify if its solutions have the behavior that is described.

(a) All solutions approach the line $y = 2$ as x goes to infinity.

(b) All solutions approach the line $y = \frac{3}{4}$ as x goes to infinity.

(c) All solutions diverge away from the line $y = \frac{-2}{5}$ as x goes to infinity.

Problem 1.3. Suppose an object in free fall near the surface of the Earth experiences drag that is proportional to the object's velocity. Assume the drag coefficient will be approximately 0.47.

(a) Determine a differential equation whose solution models the velocity of the object t seconds after it begins free fall, assuming the object's mass is 15 kg and the drag coefficient is 0.47.

(b) Use Sage to plot the slope field corresponding to the differential equation from part (a), and use this slope field to estimate the object's terminal velocity.

Problem 1.4. Determine if each differential equation below is linear or not, and determine its order.

(a) $\dfrac{dy}{dx} = x^2y + x$

(b) $e^x\dfrac{d^2y}{dx} + y\dfrac{dy}{dx} = xy$

(c) $\dfrac{d^2y}{dx} + y\dfrac{dy}{dx} = \dfrac{d^2y}{dx} - x^3\dfrac{dy}{dx}$

(d) $\sin(x)\dfrac{d^3y}{dx} + x^2\dfrac{d^2y}{dx} = y$

(e) $\dfrac{d^2y}{dx^2} = y$

Problem 1.5. Solve each of the differential equations below.

(a) $\dfrac{dy}{dx} = -y + 5$

(b) $\dfrac{dy}{dx} = 2y - 3$

(c) $3y - \dfrac{dy}{dx} = 4y + 1$

2 First Order Ordinary Differential Equations

Part of the charm in solving a differential equation is in the feeling that we are getting something for nothing. So little information appears to go into the solution that there is a sense of surprise over the extensive results that are derived.

GEORGE ROBERT STIBITZ

2.1 Integrating Factors

In this section we introduce the "method of integrating factors" which will allow us to solve any first order linear ordinary differential equation. (Or, more precisely, gives us a method of converting the differential equation into an integration problem. Whether we can actually compute the integral is another story.)

To get started, let's work our way through a particular example, and then we will try to generalize the process involved.

Consider the differential equation

$$x^2\frac{dy}{dx} + 2xy = x^3.$$

It may not be immediately obvious at first glance, but the left-hand side of this equation can actually be written much more simply using the product rule. In particular, since y is some unknown function of x, we may as well call it $f(x)$ for the moment and so $\frac{dy}{dx}$ is $f'(x)$. The left-hand side of the equation above could then be written as

$$x^2f'(x) + 2xf(x).$$

After a moment's thought, you should be able to convince yourself that this expression, $x^2f'(x) + 2xf(x)$, is equal to the derivative of $x^2f(x)$ by the product rule. That is, the left-hand side of our differential equation above is simply $\frac{d}{dx}x^2f(x)$. Keeping in mind that y *is* $f(x)$, we can just as easily call this $\frac{d}{dx}x^2y$ and so our differential equation becomes

$$\frac{d}{dx}x^2y = x^3.$$

Remark.
In problems such as this where we have a derivative $\frac{dy}{dx}$ that appears in our differential equation, we must have that y is a function of x; $y = f(x)$ for some currently unknown $f(x)$. We usually won't bother to explicitly point this out each time, but if you are ever confused by some manipulation that appears in working through a differential equation, it may be helpful to replace that y's that appear by $f(x)$ and $\frac{dy}{dx}$ by $f'(x)$: this can help you realize where derivative rules such as the product rule are being used.

Since we have written our differential equation as

$$\frac{d}{dx}x^2y = x^3,$$

we now see that integrating both sides of the equation becomes very simple. In particular, integrating both sides gives us the new equation

$$\int \frac{d}{dx}x^2y\,dx = \int x^3\,dx.$$

The right-hand side is of course very easy to integrate, but the left-hand side is easy to integrate as well. In particular, on the left-hand side we're basically trying to compute the antiderivative of the derivative of x^2y. Of course these operations of "compute the antiderivative of the derivative" cancel each other out, and so the left-hand side becomes simply x^2y. That is, we now have the equation

$$x^2y = \frac{x^4}{4} + C.$$

Our ultimate goal is to solve for y, but at this point that is very simple and we have

$$y = \frac{x^2}{4} + \frac{C}{x^2}.$$

That is, we are claiming that $y = x^2/4 + Cx^{-2}$ solve our initial differential equation, $x^2\frac{dy}{dx} + 2xy = x^3$. Let's now verify our y is a solution to the differential equation.

Differentiating $y = x^2/4 + Cx^{-2}$ tells us

$$\frac{dy}{dx} = \frac{2x}{4} - 2Cx^{-3} = \frac{x}{2} - \frac{2C}{x^3}.$$

Now we will plug this expression for $\frac{dy}{dx}$ and our expression for x into the left-hand side of our differential equation, and see if we can simplify it down to the right-hand side:

$$\begin{aligned}
&x^2\frac{dy}{dx} + 2xy \\
=&x^2\left(\frac{x}{2} - \frac{2C}{x^3}\right) + 2x\left(\frac{x^2}{4} + \frac{C}{x^2}\right) \\
=&\frac{x^3}{2} - \frac{2C}{x} + \frac{2x^2}{4} + \frac{2C}{x} \\
=&\frac{x^3}{2} + \frac{x^3}{2} \\
=&x^3
\end{aligned}$$

and so the differential equation is satisfied.

Let's work through one more similar example before describing the process in general. Consider the differential equation

$$\sin(x)\frac{dy}{dx} + y\cos(x) = 0.$$

Using the product rule, we may rewrite the left-hand side of this equation to obtain the following:

$$\frac{d}{dx}y\sin(x) = 0.$$

Now we can integrate both sides of the equation to obtain

$$\begin{aligned}
&\int \frac{d}{dx}y\sin(x)\,dx = \int 0\,dx \\
\implies &y\sin(x) = C.
\end{aligned}$$

And of course we can easily solve this for y:

$$y = \frac{C}{\sin(x)} = C\csc(x).$$

Exercise 2.1.
Verify that $y = C\csc(x)$ solves the differential equation

$$\sin(x)\frac{dy}{dx} + y\cos(x) = 0.$$

Notice that

$$\frac{dy}{dx} = -\csc(x)\cot(x).$$

Now we simply plug our expressions for y and $\frac{dy}{dx}$ into the original differential equation and see if this becomes zero or not.

$$\begin{aligned}
&\sin(x)\frac{dy}{dx} + y\cos(x)\\
=&\sin(x)\left(-\csc(x)\cot(x)\right) + C\csc(x)\cos(x)\\
=&-C\sin(x)\csc(x)\cot(x) + C\csc(x)\cos(x)\\
=&-C\sin(x)\,\frac{1}{\sin(x)}\,\frac{\cos(x)}{\sin(x)} + C\,\frac{1}{\sin(x)}\,\cos(x)\\
=&-C\,\frac{\cos(x)}{\sin(x)} + C\,\frac{\cos(x)}{\sin(x)}\\
=&0.
\end{aligned}$$

You may notice in the two examples described above that we got "lucky" in that the left-hand side of the equation happened to be a product rule. What would happen if we were not so lucky? For example, consider the differential equation

$$\frac{dy}{dx} + 3y = x.$$

Here it's not so clear that the left-hand side can be written as a product rule: in fact, the left-hand side *can not* be written as a product rule. But perhaps we can fix that if we're clever.

Notice that that for any function $g(x)$, multiplying both sides of our earlier differential equation by $g(x)$ would give us the equation

$$g(x)\frac{dy}{dx} + 3g(x)y = xg(x).$$

Can we make a clever choice of $g(x)$ so that the left-hand side of this new differential equation is a product rule? That is, can we choose $g(x)$ so that

$$\frac{d}{dx}yg(x) = g(x)\frac{dy}{dx} + 3g(x)y?$$

The product rule tells us that $\frac{d}{dx}yg(x)$ *must* equal $g(x)\frac{dy}{dx} + yg'(x)$. So, if we were to have $\frac{d}{dx}yg(x) = g(x)\frac{dy}{dx} + 3g(x)y$, we would need that $g'(x) = 3g(x)$. There *is* a choice of $g(x)$ that would have this property, namely $g(x) = e^{3x}$.

Just to summarize what we've said thus far, if we start off with the equation

$$\frac{dy}{dx} + 3y = x$$

and multiply both sides of the equation by e^{3x}, we have a new differential equation,

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{3x}x.$$

Even though the left-hand side of our original equation can not be rewritten as a product rule, the left-hand side of this new differential equation *can* be. This gives us

$$\frac{d}{dx}e^{3x}y = e^{3x}x.$$

Integrating both sides of the equation would give us a new equation,

$$\int \frac{d}{dx}e^{3x}y\,dx = \int e^{3x}x\,dx.$$

The right-hand side of this equation is trivial: it is simply $e^{3x}y$. To compute the right-hand side of the equation we need to use integration by parts with

$$\begin{aligned} u &= x & dv &= e^{3x} \\ du &= dx & v &= \frac{1}{3}e^{3x}. \end{aligned}$$

This gives us

$$\begin{aligned} \int xe^{3x}\,dx &= \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx \\ &= \frac{1}{3}xe^{3x} - \frac{1}{9}\int e^{3x} + C. \end{aligned}$$

Thus, after integrating, our differential equation has become

$$e^{3x}y = \frac{1}{3}xe^{3x} - \frac{1}{9}\int e^{3x} + C$$

which we can easily solve for y:

$$y = \frac{x}{3} - \frac{1}{9} + Ce^{-3x}.$$

Let's now verify this truly does solve our initial differential equation:

$$\begin{aligned}\frac{dy}{dx} + 3y &= \frac{1}{3} - 3Ce^{-3x} + 3\left(\frac{x}{3} - \frac{1}{9} + Ce^{-3x}\right)\\ &= x.\end{aligned}$$

We may generalize the procedure of the last example to solve any linear first order ordinary differential equation. To see this, let's first notice that a linear first order ODE may be written in the form

$$f(x) + g(x)y + h(x)\frac{dy}{dx} = 0.$$

Using simple arithmetic we may write this as

$$\frac{dy}{dx} + \frac{g(x)}{h(x)}y = -f(x).$$

Now for notation convenience, let us write $p(x)$ for $\frac{g(x)}{h(x)}$ and $q(x) = -f(x)$ so the equation above becomes

$$\frac{dy}{dx} + p(x)y = q(x).$$

We would like it if the left-hand side could be rewritten as a product rule, but there's no reason this should necessarily be the case. However, as in the last example above, we may be able to multiply through by some currently unknown function which will allow us to turn the left-hand side into a product rule. Calling this unknown function $\mu(x)$, this leads us to the differential equation

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x).$$

We would like to choose $\mu(x)$ so that the left-hand side of the equation becomes the $\frac{d}{dx}\mu(x)y$. That is, we want $\mu(x)$ to have the property that

$$\frac{d}{dx}\mu(x)y = \mu(x)\frac{dy}{dx} + \mu(x)p(x)y.$$

However, the product rule tells us that we must have

$$\frac{d}{dx}\mu(x)y = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y.$$

So, what we really require is that $\frac{d\mu}{dx}$ equal $\mu(x)p(x)$. Notice that finding such a $\mu(x)$ is really an integration problem:

$$\begin{aligned} &\frac{d\mu}{dx} = \mu(x)p(x) \\ \implies &\frac{1}{\mu(x)}\frac{d\mu}{dx} = p(x) \\ \implies &\int \frac{1}{\mu(x)}\frac{d\mu}{dx}\,dx = \int p(x)\,dx. \end{aligned}$$

Now letting $u = \mu(x)$ so $du = \frac{d\mu}{dx}\,dx$, we have

$$\begin{aligned} &\int \frac{1}{u}\,du = \int p(x)\,dx \\ \implies &\ln|u| + C = \int p(x)\,dx \\ \implies &e^{\ln|u|+C} = e^{\int p(x)\,dx} \\ \implies &Ce^{\ln|u|} = e^{\int p(x)\,dx} \\ \implies &|u| = Ce^{\int p(x)\,dx} \\ \implies &u = Ce^{\int p(x)\,dx}. \end{aligned}$$

Since $u = \mu(x)$, we have that for any choice C the function $\mu(x) = Ce^{\int p(x)\,dx}$ has our desired derivative, $\frac{d\mu}{dx} = \mu(x)p(x)$. As any choice of C suffices, we will always just choose C to be 1 to simplify our notation slightly.

Exercise 2.2.
Verify that $\mu(x) = e^{\int p(x)\,dx}$ satisfies the equation

$$\frac{d\mu}{dx} = \mu(x)p(x).$$

$$
\begin{aligned}
\mu(x) &= e^{\int p(x)\,dx} \\
\implies \frac{d\mu}{dx} &= e^{\int p(x)\,dx} \frac{d}{dx} \int p(x)\,dx \qquad \text{(By the chain rule)} \\
&= e^{\int p(x)\,dx} p(x) \\
&= \mu(x)p(x).
\end{aligned}
$$

To summarize what we have said thus far, if we are given a first order linear ODE of the form

$$\frac{dy}{dx} + p(x)y = q(x),$$

then we can multiply both sides of the equation by $e^{\int p(x)\,dx}$ to obtain

$$e^{\int p(x)\,dx} \frac{dy}{dx} + e^{\int p(x)\,dx} p(x)y = e^{\int p(x)\,dx} q(x)$$

The left-hand side of this equation may be rewritten using the product rule to turn the equation into

$$\frac{d}{dx} e^{\int p(x)\,dx} y = e^{\int p(x)\,dx} q(x).$$

Integrating both sides of the equation gives us

$$e^{\int p(x)\,dx} y = \int e^{\int p(x)\,dx} q(x)\,dx,$$

and solving for y tells us

$$y = e^{-\int p(x)\,dx} \int e^{\int p(x)\,dx} q(x)\,dx.$$

Remark.
Though this looks very ugly written in this general form, the actual quantities that will appear in problems won't be so bad.

The function $\mu(x) = e^{\int p(x)\,dx}$ that we multiplied through above is called an ***integrating factor***, and this procedure for solving linear first order differential equations is called ***the method of integrating factors***.

Let's end our discussion of integrating factors by seeing some concrete examples.

Example 2.1.
Solve the differential equation

$$\frac{dy}{dx} + xy = 2x.$$

In this example we have $p(x) = x$ and $q(x) = \sin(x)$, and so we will multiply through by the integrating factor

$$\mu(x) = e^{\int x\,dx} = e^{x^2/2}$$

to obtain

$$e^{x^2/2}\frac{dy}{dx} + xe^{x^2/2}y = 2xe^{x^2/2}$$

which we may write as

$$\frac{d}{dx}e^{x^2/2}y = 2xe^{x^2/2}.$$

Integrating both sides of the equation gives us

$$\int \frac{d}{dx}e^{x^2/2}y\,dx = \int 2xe^{x^2/2}\,dx$$
$$\implies e^{x^2/2}y = \int 2xe^{x^2/2}\,dx.$$

To perform the integration on the right-hand side we will use the substitution

$$u = \frac{x^2}{2} \qquad\qquad du = x\,dx$$

to obtain

$$\int 2xe^{x^2/2}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{x^2/2} + C.$$

That is, integrating both sides of the differential equation gives us

$$e^{x^2/2}y = 2e^{x^2/2} + C$$

which we may easily solve for y:

$$y = \frac{2e^{x^2/2} + C}{e^{x^2/2}} = 2 + Ce^{-x^2/2}.$$

Let's verify this really solves our differential equation:

$$\begin{aligned}&\frac{dy}{dx} + xy\\ =&\frac{d}{dx}\left(2 + Ce^{-x^2/2}\right) + x\left(2 + Ce^{-x^2/2}\right)\\ =&-Cxe^{-x^2/2} + 2x + 2Cxe^{-x^2/2}\\ =&2x.\end{aligned}$$

Example 2.2.
Solve the differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = x^3.$$

We will multiply through by our integrating factor which is

$$\begin{aligned}\mu(x) &= e^{\int \frac{2}{x}\,dx}\\ &= e^{2\int \frac{dx}{x}}\\ &= e^{2\ln|x|}\\ &= e^{\ln|x|^2}\\ &= e^{\ln(x^2)}\\ &= x^2.\end{aligned}$$

and so our differential equation becomes

$$x^2 \frac{dy}{dx} + x^2 \frac{2}{x} y = x^3 x^2$$

or simply

$$x^2 \frac{dy}{dx} + 2xy = x^5.$$

We may write this as

$$\frac{d}{dx} x^2 y = x^5$$

and integrating both sides gives us

$$\int \frac{d}{dx} x^2 y \, dx = \int x^5 \, dx$$
$$\implies x^2 y = \frac{x^6}{6} + C$$
$$\implies y = \frac{1}{x^2} \left(\frac{x^6}{6} + C \right) = \frac{x^4}{6} + \frac{C}{x^2} = \frac{x^6 + C}{6x^2}.$$

Exercise 2.3.
Verify that $y = \frac{x^6+C}{6x^2}$ solves the differential equation

$$\frac{dy}{dx} + \frac{2}{x} y = x^3.$$

$$\begin{aligned}
&\frac{dy}{dx} + \frac{2}{x}y \\
=&\frac{d}{dx}\left(\frac{x^6 + C}{6x^2}\right) + \frac{2}{x}\left(\frac{x^6 + C}{6x^2}\right) \\
=&\frac{6x^2 \cdot 6x^5 - 12x(x^6 + C)}{(6x^2)^2} + \frac{2x^6 + 2C}{6x^3} \\
=&\frac{36x^7 - 12x^7 - 12Cx}{36x^4} + \frac{2x^6 + 2C}{6x^3} \\
=&\frac{24x^7 - 12Cx}{36x^4} + \frac{12x^7 + 12Cx}{36x^4} \\
=&\frac{36x^7}{36x^4} \\
=&x^3.
\end{aligned}$$

2.2 Separable equations

At this point we are able to solve any first order linear ordinary differential equation – at least up to performing some integration coming from the integrating factor. But how can we deal with non-linear equations? Integrating factors are completely useless for a differential equation such as

$$y\,\frac{dy}{dx} = x.$$

In general *there is no procedure that works for all non-linear differential equations*. However, there are different procedures for different "families" of non-linear differential equations. The simplest family consists of the "separable differential equations."

A differential equation is called ***separable*** if it can be written in the form

$$p(y)\frac{dy}{dx} = q(x).$$

In the equation above, for example, we have $p(y) = y$ and $q(x) = x$. That is, for a separable differential equation we can "separate" the y-part of the equation on one side from the x-part of the equation on the other side. Before describing the general procedure for separable differential equations, let's see if we can solve the separable equation mentioned above.

Starting from

$$y\,\frac{dy}{dx} = x,$$

let's keep in mind that y is really some function of x, so we could replace the y's that appear with $f(x)$ and replace $\frac{dy}{dx}$ with $f'(x)$. The equation above would then become

$$f(x) \cdot f'(x) = x.$$

Since these functions of x (the left-hand side is the function $f(x) \cdot f'(x)$ and the right-hand side is simply the function x), their antiderivatives must be equal, leading us to the equation

$$\int f(x) \cdot f'(x)\,dx = \int x\,dx.$$

Of course, the right-hand side is simply $\frac{x^2}{2} + C$, but what about the left-hand side? Notice we could actually integrate the left-hand side by applying the substitution

$$u = f(x), \quad du = f'(x)\,dx.$$

This allows us to to replace $\int f(x)f'(x)\,dx$ with

$$\int u\,du = \frac{u^2}{2} + C$$

which we can rewrite in terms of $f(x)$ to obtain

$$\int f(x)f'(x)\,dx = \frac{f(x)^2}{2} + C.$$

(Notice we can easily double-check that $\frac{f(x)^2}{2} + C$ really is the antiderivative of $f(x)f'(x)$.)

After performing this integration, our earlier differential equation becomes

$$\frac{f(x)^2}{2} = \frac{x^2}{2} + C$$

which we can now attempt to solve for $f(x)$:

$$\frac{f(x)^2}{2} = \frac{x^2}{2} + C \implies \qquad f(x)^2 = x^2 + C$$
$$\implies f(x) = \pm\sqrt{x^2 + C}.$$

That is, we claim that $y = \pm\sqrt{x^2 + C}$ solves the differential equation $y\frac{dy}{dx} = x$. Notice first that we're really claiming there are two solutions: $y = \sqrt{x^2 + C}$ is one solution and $y = -\sqrt{x^2 + C}$ is the other solution. We will only verify the positive square root solution here and leave the verification of the negative square root solution as an exercise.

$$\begin{aligned}
y &= \sqrt{x^2 + C} = (x^2 + C)^{1/2} \\
\implies y \cdot \frac{dy}{dx} &= (x^2 + C)^{1/2} \cdot \frac{d}{dx}(x^2 + C)^{1/2} \\
&= (x^2 + C)^{1/2} \cdot \frac{1}{2}(x^2 + C)^{-1/2} \cdot 2x \\
&= \frac{\sqrt{x^2 + C}}{2\sqrt{x^2 + C}} \cdot 2x \\
&= x.
\end{aligned}$$

We can "streamline" the procedure above a little bit by noticing if $y = f(x)$, then the differential dy is given by $dy = f'(x)\,dx = \frac{dy}{dx}\,dx$. That is, the integral that we had above,

$$\int f(x)\,f'(x)\,dx$$

can be written more succinctly as

$$\int y\,dy.$$

Our procedure for solving the differential equation would then become

$$\begin{aligned}
y\frac{dy}{dx} &= x \\
\implies \int y\frac{dy}{dx}\,dx &= \int x\,dx \\
\implies \int y\,dy &= \int x\,dx.
\end{aligned}$$

Once written like this, we can integrate each side and solve for y, giving us

$$\begin{aligned}
&\int y\,dy = \int x\,dx \\
\implies &\frac{y^2}{2} = \frac{x^2}{2} + C \\
\implies &y^2 = x^2 + C \\
\implies &y = \pm\sqrt{x^2 + C}.
\end{aligned}$$

Notice this gives us the same solution we had above.

Let's see another example of solving a separable differential equation in this "streamlined" form. Consider the equation

$$3y^2\frac{dy}{dx} = x\cos(x^2).$$

Integrating both sides of the equation with respect to x yields

$$\int 3y^2\frac{dy}{dx}\,dx = \int x\cos(x^2)\,dx.$$

On the left-hand side, let's again notice $dy = \frac{dy}{dx}dx$ and so our equation becomes

$$\int 3y^2 dy = \int x\cos(x^2)\,dx.$$

Performing the described integration on each side leads us to

$$y^3 = \frac{1}{2}\sin(x^2) + C$$

(The substitution $u = x^2$, $du = 2x\,dx$ was used on the right-hand side.) We can now solve easily solve this for y:

$$y = \left(\frac{1}{2}\sin(x^2) + C\right)^{1/3}.$$

Let's quickly verify this really solves our differential equation:

$$\begin{aligned}
&3y^2\frac{dy}{dx}\\
=&3\left(\frac{1}{2}\sin(x^2)+C\right)^{2/3}\cdot\frac{d}{dx}\left(\frac{1}{2}\sin(x^2)+C\right)^{1/3}\\
=&3\left(\frac{1}{2}\sin(x^2+C)\right)^{2/3}\cdot\frac{1}{3}\left(\frac{1}{2}\sin(x^2)+C\right)^{-2/3}\cdot\frac{1}{2}\cdot\cos(x^2)\cdot 2x\\
=&x\cos(x^2).
\end{aligned}$$

In general, when we have a separable equation

$$p(y)\frac{dy}{dx} = q(x)$$

and integrate both sides with respect to x giving us

$$\int p(y)\frac{dy}{dx}\,dx = \int q(x)\,dx$$

we can use the fact that y is some (currently unknown) function of x to replace $\frac{dy}{dx}\,dx$ with the differential dy. This turns our equation into

$$\int p(y)\,dy = \int q(x)\,dx$$

and we then simply integrate the left-hand side of the equation with respect to y, integrate the right-hand side with respect to x, then do the algebra to solve for y.

Remark.
The procedure just described is the "short cut" version of what's happening. What's really going on is that we're performing a u-substitution. In particular, since y is some function of x – say $y = f(x)$ – our equation can really be written as

$$\int p(f(x))f'(x)\,dx = \int q(x)\,dx.$$

Performing the substitution $u = f(x)$, $du = f'(x)\,dx$ the right-hand side becomes

$$\int p(u)\,du.$$

If we momentarily let $P(u)$ denote the antiderivative of $p(u)$, our integration would then give us $P(u)$, but since u is $f(x)$ we'd have $P(f(x))$ and the equation becomes

$$P(f(x)) = \int q(x)\,dx.$$

Solving for $f(x)$ then really means we are applying the inverse function P^{-1} to both sides of the equation to obtain

$$f(x) = P^{-1}\left(\int q(x)\,dx\right).$$

Notice this procedure can be tricky: computing inverses can be hard, and not every function is invertible! For now we won't worry too much about this and will resign ourselves to problems where these inverses are "easy," but we will eventually have to deal with this issue when we talk about explicit and implicit solutions to differential equations later.

Our "streamlined" version of the solution above is really just using the variable y instead of u in our substitution: Given

$$\int p(f(x))f'(x)\,dx = \int q(x)\,dx.$$

we substitute $y = f(x)$, $dy = f'(x)\,dx$ to obtain

$$\int p(y)\,dy = \int q(x)\,dx.$$

And the "solve for y" we had mentioned above is really applying the inverse of the antiderivative of the integrand on the left-hand side.

Let's consider a few more examples of solving separable equations.

Example 2.3.
Solve the differential equation

$$\frac{1}{1+y}\frac{dy}{dx} = \ln(x).$$

Integrating both sides of the equation gives us

$$\int \frac{1}{1+y}\frac{dy}{dx}\,dx = \int \ln(x)\,dx,$$

which we may rewrite as

$$\int \frac{1}{1+y}\,dy = \int \ln(x)\,dx.$$

To integrate the left-hand side we'll perform the substitution $u = 1+y$, $du = dy$ so the left-hand side becomes

$$\int \frac{1}{u}\,du = \ln|u| + C$$

and so we know $\int \frac{dy}{1+y} = \ln|1+y| + C$.

For the right-hand side of our earlier equation we will need to use integration by parts with

$$\begin{aligned} u &= \ln(x) & dv &= dx \\ du &= \frac{1}{x} & v &= x \end{aligned}$$

we then have

$$\begin{aligned} \int \ln(x)\,dx &= x\ln(x) - \int x \cdot \frac{1}{x}\,dx \\ &= x\ln(x) - \int dx \\ &= x\ln(x) - x + C. \end{aligned}$$

Thus

$$\ln|1+y| = x\ln(x) - x + C.$$

Now we just have to solve for y. Exponentiating both sides gives us

$$|1+y| = e^{x\ln(x)-x+C} = Ce^{x\ln(x)-x}.$$

We can drop the absolute values on the left-hand side by picking pu $\pm$ on the right-hand side, giving us

$$1+y = \pm Ce^{x\ln(x)-x} = Ce^{x\ln(x)-x}$$

and so $y = -1 + Ce^{x\ln(x)-x}$.

Exercise 2.4.
Verify that $y = -1 + Ce^{x\ln(x)-x}$ solves the differential equation described in Example 2.3.

Example 2.4.
Solve the following initial value problem:

$$\frac{dy}{dx} = \frac{3x^2 + 4x - 4}{2y - 4}$$

where $y(1) = -1$.

First we rewrite our equation as

$$(2y - 4)\frac{dy}{dx} = 3x^2 + 4x - 4.$$

Now we integrate both sides to obtain

$$\int (2y-4)dy = \int (3x^2 + 4x - 4)\,dx$$
$$\Longrightarrow y^2 - 4y = x^3 + 2x^2 - 4x + C$$

We now need to solve for y. We can do this by first completing the square which will give us

$$y^2 - 4y = x^3 + 2x^2 - 4x + C$$
$$\Longrightarrow y^2 - 4y + 4 = x^3 + 2x^2 - 4x + C + 4$$
$$\Longrightarrow (y-2)^2 = x^3 + 2x^2 - 4x + C$$
$$\Longrightarrow y - 2 = \pm\sqrt{x^3 + 2x^2 - 4x + C} \quad \Longrightarrow \quad y = 2 \pm \sqrt{x^3 + 2x^2 - 4x + C}.$$

Notice that in the third step above the "$+4$" which we had attached to the right-hand side in the previous equation was incorporated into the "$+C$".

At this point we have two different functions which solve our differential equation, since we have a $\pm$ that appears because of the square root. Let's notice, though, that we also have the initial condition $y(1) = -1$. This means we *must* use the negative square root in order to acheive the negative value. Now to determine our choice of C we simply need to do the algebra. Our initial condition may be written as

$$
\begin{aligned}
& 2-\sqrt{1^3+2\cdot 1^2-4\cdot 1+C}=-1 \\
\Longrightarrow\ & 2-\sqrt{1+2-4+C}=-1 \\
\Longrightarrow\ & 3=\sqrt{C-1} \\
\Longrightarrow\ & 9=C-1 \\
\Longrightarrow\ & C=10.
\end{aligned}
$$

Thus we claim our initial value problem is solved by

$$y=2-\sqrt{x^3+2x^2-4x+10}.$$

We will verify this truly does solve our IVP:

$$
\begin{aligned}
y=2-\sqrt{x^3+2x^2-4x+10} \implies \frac{dy}{dx} &= \frac{d}{dx}\left(2-\sqrt{x^3+2x^2-4x+10}\right) \\
&= \frac{-1}{2\sqrt{x^3+2x^2-4x+10}}\cdot\left(3x^2+4x-4\right) \\
&= \frac{-(3x^2+4x-4)}{2\sqrt{x^3+2x^2-4x+10}-4+4} \\
&= \frac{-(3x^2+4x-4)}{-2(-\sqrt{x^3+2x^2-4x+10}+2)+4} \\
&= \frac{-(3x^2+4x-4)}{-2(2-\sqrt{x^3+2x^2-4x+10})+4} \\
&= \frac{-(3x^2+4x-4)}{-2y+4} \\
&= \frac{3x^2+4x-4}{2y-4}
\end{aligned}
$$

And so the differential equation is solved, and the initial condition is very easy to check:

$$2-\sqrt{1^3+2\cdot 1^2-4\cdot 1+10}=2-\sqrt{9}=2-3=-1.$$

Our next example will require us to use partial fractions, so let's first spend some time quickly reviewing how partial fractions work.

Recall that when you add two fractions together, you have to get a common denominator. One way of doing this is to multiply eac term by the denominator of the other term over itself. For example, consider

$$\begin{aligned}
&\frac{7}{x-1}+\frac{2}{x+3}\\
=&\frac{7}{x-1}\cdot 1+\frac{2}{x+3}\cdot 1\\
=&\frac{7}{x-1}\cdot\frac{x+3}{x+3}+\frac{2}{x+3}\cdot\frac{x-1}{x-1}\\
=&\frac{7(x+3)}{(x-1)(x+3)}+\frac{2(x-1)}{(x+3)(x-1)}\\
=&\frac{7x+21}{x^2+2x-3}+\frac{2x-2}{x^2+2x-3}\\
=&\frac{9x+19}{x^2+2x-3}.
\end{aligned}$$

"Partial fractions" is just doing this process in reverse. That is, if we started with $\frac{9x+19}{x^2+2x-3}$, then our goal would be to break it up into the sum of simpler fractions $\frac{7}{x-1}+\frac{2}{x+3}$. The point of doing this is to try to make integrating that function easier:

$$\begin{aligned}
\int \frac{9x+19}{x^2+2x-3}\,dx &= \int\left(\frac{7}{x-1}+\frac{2}{x+3}\right)dx\\
&= 7\int\frac{dx}{x-1}+2\int\frac{dx}{x+3}\\
&= 7\ln|x-1|+2\ln|x+3|+C.
\end{aligned}$$

Of course, this example is "cheating" a little bit since we started off with the sum of simpler fractions and then added them together to get the more complicated fraction. So, the question now becomes how do we perform this reverse procedure in general. For example, how could we take the fraction

$$\frac{x+5}{x^2+5x+6}$$

and write it as a sum of simpler fractions? Let's first notice that if we can factor the denominator, its factors should tell us the denominators of the simpler fractions that appear in our sum. This is simply because when we add the fractions with those denominators together we'll multiply denominators, and this will give us back the denominator of the original fraction we started with. In the case of the fraction above, the denominator factors as

$$x^2+5x+6=(x+2)(x+3).$$

The claim is that this means we should be able to write our original fraction as the sum of "something" over $x + 2$ plus "something else" over $x + 3$.

$$\frac{x+5}{x^2+5x+6} = \frac{x+5}{(x+2)(x+3)} = \frac{?}{x+2} + \frac{?}{x+3}.$$

Now we need to determine what those "somethings" are. Let's treat this as an algebra problem where there are two unknowns (the numerators of our simpler fractions) that we need to solve for; let's just call those numerstors a and b to obtain

$$\frac{x+5}{x^2+5x+6} = \frac{a}{x+1} + \frac{b}{x+3}.$$

Now let's just see what happens when we add $\frac{a}{x+1}$ and $\frac{b}{x+3}$ together, and compare that to the original fraction we started with. This would lead us to the following bit of arithmetic:

$$\begin{aligned} &\frac{a}{x+1} + \frac{b}{x+3} \\ =&\frac{a}{x+1} \cdot \frac{x+3}{x+3} + \frac{b}{x+3} \cdot \frac{x+1}{x+1} \\ =&\frac{ax+3a}{x^2+5x+6} + \frac{bx+b}{x^2+5x+6} \\ =&\frac{(a+b)x+3a+b}{x^2+5x+6}. \end{aligned}$$

But we want this fraction to equal our earlier fraction:

$$\frac{x+5}{x^2+5x+6} = \frac{(a+b)x+3a+b}{x^2+5x+6}.$$

That is, we want $a+b$ to be equal to 1, and $3a+b$ to be equal to 5, and this gives us a system of equations:

$$\begin{aligned} a+b &= 1 \\ 3a+b &= 5 \end{aligned}$$

All we have to do now is solve this system of equations. Subtracting the first equation from the second gives us

$$\begin{aligned} &3a+b-(a+b) = 5-1 \\ \implies &2a = 4 \\ \implies &a = 4. \end{aligned}$$

Now that we know $a = 4$ we can plug this into $a + b = 1$ to obtain $b = -3$. Thus we have the partial fraction decomposition we were after

$$\frac{x+5}{x^2+5x+6} = \frac{4}{x+1} + \frac{-3}{x+3}$$

We will need partial fractions to solve the differential equation in the next example.

Example 2.5.
Solve

$$2y\frac{dy}{dx} = \frac{3x+1}{x^2+3x+2}.$$

Integrating both sides of the equation gives us

$$\begin{aligned} \int 2y\,dy &= \int \frac{3x+1}{x^2+3x+2}\,dx \\ \implies y^2 &= \int \frac{3x+1}{x^2+3x+2}\,dx. \end{aligned}$$

To compute the integral on the right we will need the partial fraction decomposition, and the first step is to factor the denominator. It only takes a moment's thought to realize the factorization is

$$x^2 + 3x + 2 = (x+1)(x+2).$$

Thus we will write our fraction as a sum of two simpler fractions with these denominators, treating the numerators as unknowns we can solve for.

$$\begin{aligned} \frac{3x+1}{x^2+3x+2} &= \frac{a}{x+1} + \frac{b}{x+2} \\ &= \frac{a(x+2)+b(x+1)}{(x+1)(x+2)} \\ &= \frac{(a+b)x+2a+b}{x^2+3x+2} \end{aligned}$$

Thus we are lead to the system of equations

$$\begin{aligned} a + b &= 3 \\ 2a + b &= 1. \end{aligned}$$

Subtracting the first equation from the second to get rid of the b's tells us

$$\begin{aligned} 2a + b - (a+b) &= 1 - 3 \\ \implies a &= -2. \end{aligned}$$

The first equation then becomes $-2 + b = 3$ and so $b = 5$. That is,

$$\frac{3x+1}{x^2+3x+2} = \frac{-2}{x+1} + \frac{5}{x+2}.$$

The right-hand side of our integral equation earlier is now much simpler:

$$\begin{aligned} \int \frac{3x+1}{x^2+3x+2}\,dx &= \int \left(\frac{-2}{x+1} + \frac{5}{x+2}\right) dx \\ &= -2\ln|x+1| + 5\ln|x+2| + C. \end{aligned}$$

Hence we have

$$\begin{aligned} y^2 &= -2\ln|x+1| + 5\ln|x+2| + C \\ \implies y &= \pm\sqrt{-2\ln|x+1| + 5\ln|x+2| + C}. \end{aligned}$$

It is easy to check this solves our differential equation:

$$\begin{aligned} &2y\frac{dy}{dx} \\ =&2\left(\pm\sqrt{-2\ln|x+1| + 5\ln|x+2| + C}\right) \cdot \frac{1}{\pm 2\sqrt{-2\ln|x+1| + 5\ln|x+2| + C}} \cdot \\ &\left(\frac{-2}{x+1} + \frac{5}{x+2}\right) \\ =&\frac{3x+1}{x^2+3x+2} \end{aligned}$$

In the examples we've seen so far, we've basically always been able to easily, algebraically solve for y. This does not always happen, however, as the next example illustrates.

Example 2.6.
Consider the following initial value problem:

$$\frac{dy}{dx} = \frac{1+3x^2}{3y^2-6y} \qquad y(0) = 1.$$

Since this equation is separable, we have

$$\int \left(3y^2 - 6y\right)\, dy = \int \left(1+3x^2\right)\, dx$$
$$\Longrightarrow y^3 - 3y^2 = x + x^3 + C$$

Let's go ahead and find the C which will satisfy our initial condition. If $y = 1$ when $x = 0$ (i.e., $y(0) = 1$), then the equation above becomes $1 - 3 = C$, and so $C = -2$ and our initial value problem is solved by

$$y^3 - 3y^2 = x + x^3 - 2.$$

In Example 2.6, notice that we can not solve for y. That is, we may not be able to do the algebra to get a single y by itself on one side of the equation; $y^3 - 3y^2 = x + x^3 - 2$ for example has this property. The reason for this is that *if we could* solve for y in this expression, writing $y = f(x)$, that would mean that y is a function of x. However, if we were to graph the set of (x, y)-points which satisfy $y^3 - 3y^2 = x + x^3 - 2$, we see something as in Figure 2.1. Notice that this *is not* the graph of a function, however. Thus we should not expect that we could solve this expression for y.

Notice, though, that even when we have a curve which fails the vertical line test and so is not the graph of a function, such as in Figure 2.1, we *can* imagine breaking the graph up into pieces where each piece does satisfy the vertical line test, such as the red, blue, and purple portions of Figure 2.2.

That is, there are three functions whose graphs together give us these three pieces. Call the red portion of the curve the graph of $f_1(x)$, the blue portion the graph of $f_2(x)$, and the purple portion $f_3(x)$. We see that the piece we really care about is $f_2(x)$ since this is the only one that can satisfy our initial condition, as it is the only function whose domain contains 0 which is the x-coordinate of our initial condition, $y(0) = 1$.

We will often have to leave the solution to a differential equation in an ***implicit form***, such as in Example 2.6. That is, instead of writing y as a

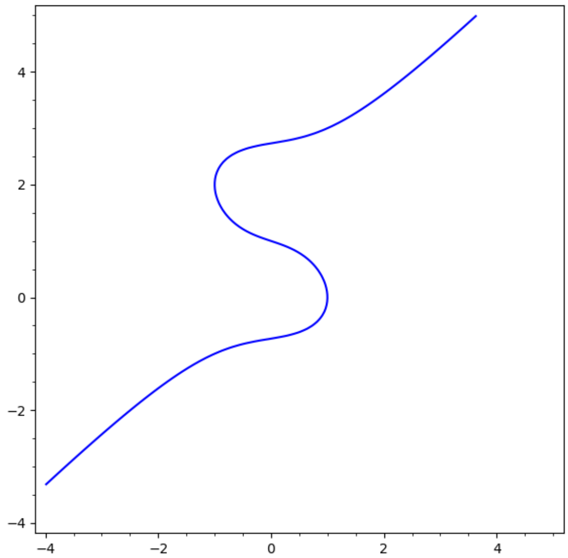


Figure 2.1: The (x, y) points satisfying $y^3 - 3y^2 = x + x^3 - 2$.

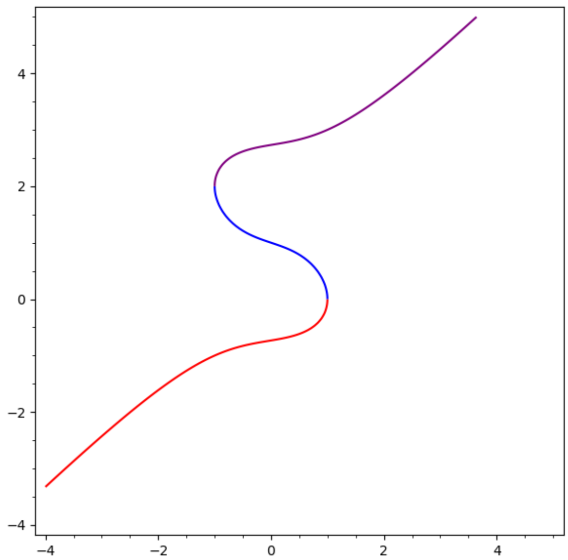


Figure 2.2: We can break a up a curve failing the vertical line test into individual pieces which pass the vertical line test, and so are graphs of functions.

function of x, we give an equation representing a relationship the x and y must satisfy. (If we can explicitly write y as a function of x, we say the solution is ***explicit***.) There may be several such functions satisfying this relationship (in the case of Example 2.6, we see from Figure 2.2 that there are three functions satisfying the relation $y^3 - 3y^2 = x + x^3 - 2$), so we must give an initial condition or the domain of the function we care about in order to single out one particular function.

In the case of the solution to the initial value problem of Example 2.6, we see the domain of the function we care about (the $f_2(x)$ whose graph was in blue in Figure 2.2) is $(-1, 1)$. In general, the domain of the function defined implicitly like this is an open interval (a, b), where we may possibly have $a = -\infty$ and $b = \infty$. If a and b are finite, however, the tangent lines to the graph will become vertical as x approaches a from the right or b from the left.

Remark.
The above discussion is a bit hand-wavy, but can be made precise using an important theorem called *The Implicit Function Theorem*. The implicit function theorem is a bit technical to state precisely, so we're glossing over the details here. If you want to see all the nitty-gritty details of the implicit function theorem, though, perhaps the best place to look is in the book *Vector calculus, linear algebra, and differential forms* by Hubbard and Hubbard.

2.3 Modeling using first-order differential equations

We now turn our attention away from solution techniques of first-order differential equations to discuss one of the main applications of differential equations: mathematical modeling.

Many real-world problems are too complex to analyze directly, and so it is often desirable to have a simpler model of the problem which can be studied. Though mathematical models not necessarily always involve differential equations, diff. eq.'s are often one of the primary tools used in modeling.

Every problem is different, and there isn't one set "algorithm" for coming up with a model, so we will consider a few special examples. First, though, let's note some general ideas which may be helpful to consider in constructing a model. Some general guidelines for constructing mathematical models are the following:

- Identify independent and dependent variables.
- Choose convenient units of measurement.
- Determine any underlying principles related to the problem at hand (e.g., Newton's laws of motion).
- Express the underlying principle mathematically in terms of the variables and units above.
- Check that the resulting units make sense.

After determining the model, we wish to analyze it. For us, this usually means solving the resultant differential equation, or perhaps estimating the behavior of a solution from a slope field if the differential equation can not be solved analytically. Finally, we may wish to compare estimates made from our model to observed data. If the observations agree with what our model predicted, then we may feel confident the model is correct. If our predictions do not agree with the observations, then we should revise the model.

Example 2.7.
Suppose a five-hundred gallon reservoir of water contains a fertilizer mixed into the water at a concentration of half-a-pound of fertilizer per gallon of water. If clean water is pumped into the tank at a rate of ten gallons per minute, and fertilized water is pumped out of the tank at a rate of ten gallons per minute as well, what is the concentration of the fertilizer in the water after thirty minutes?

Let's let C denote the concentration of the fertilizer in the water, measured in pounds per gallon. Notice this is a function of time. Let's let t denote the time, measured in minutes, after clean water is pumped in and fertilized water is pumped out. So, our goal will be to find $C(30)$ given that $C(0) = 1/2$.

Assuming the fertilizer is well-mixed into the water, how is the concentration changing? Note that the concentration is decreasing: we're losing ten gallons of fertilized water with concentration C every minute. This means the rate of change in the amount of fertilizer is

$$-10\frac{\text{gal}}{\text{min}} \cdot C\frac{\text{lb}}{\text{gal}}.$$

The concentration is obtained by dividing this by the total number of gallons, giving us

$$\frac{-10\frac{\text{gal}}{\text{min}} \cdot C\frac{\text{lb}}{\text{gal}}}{500\text{gal}} = \frac{-10C}{500}\frac{\left({}^{\text{lb}}/_{\text{gal}}\right)}{\text{min}}.$$

That is,

$$\frac{dC}{dt} = \frac{-C}{50}.$$

This is a differential equation we can solve in a few different ways. For the sake of recalling some of the solution techniques we've seen thus far, let's mention three different ways to solve this differential equation. In what follows we'll use k to represent the arbitrary constant of integration, since we're using C to represent the concentration of fertilizer in this problem.

Recognizing the form $\frac{dy}{dx} = my + b$ Using our general formula we developed for solving differential equations of this form we see

$$\begin{aligned} \frac{dC}{dt} &= \frac{-C}{50} \\ \implies C &= ke^{-t/50} \end{aligned}$$

Using an integrating factor Treating this as a linear differential equation we can use an integrating factor to obtain the following:

$$\begin{aligned} \frac{dC}{dt} &= \frac{-C}{50} \\ \implies \frac{dC}{dt} + \frac{1}{50}C &= 0 \\ \implies e^{t/50}\frac{dC}{dt} + e^{t/50} \cdot \frac{1}{50}C &= 0 \\ \implies \frac{d}{dt}e^{t/50}C &= 0 \\ \implies e^{t/50}C &= k \\ \implies C &= ke^{-t/50} \end{aligned}$$

As a separable equation We can also think of this differential equation as a separable equation: it's just that the right-hand side is a constant function. We then have

$$\begin{aligned}
&\frac{dC}{dt} = \frac{-C}{50} \\
\implies &\frac{50}{C}\frac{dC}{dt} = -1 \\
\implies &\int \frac{50}{C} dC = \int (-1)\, dt \\
\implies &50 \ln |C| = -t + k \\
\implies &\ln |C| = \frac{-t}{50} + k \\
\implies &|C| = e^{-t/50+k} = e^{-t/50}e^{k} = ke^{-t/50} \\
\implies &C = \pm ke^{-t/50} = ke^{-t/50}
\end{aligned}$$

Regardless of which solution technique we choose, we see that the concentration of fertilizer at time t is given by $C = ke^{-t/50}$ for some constant k. To solve for k we simply use our initial condition of $C(0) = 1/2$ to obtain

$$\frac{1}{2} = C(0) = ke^{-0/50} = k,$$

and so the concentration of fertilizer is

$$C(t) = \frac{1}{2}e^{-t/50}.$$

After thirty minutes, the concentration will thus be

$$C(30) = \frac{1}{2}e^{-30/50} \approx 0.2744 \frac{\text{lb}}{\text{gal}}.$$

One thing to point out about the solution to modeling problem above is that as t goes to infinity, $C(t)$ goes to zero, though for each finite value of t there is still some small amount of fertilizer which remains.

For the next example we want to see how radiocarbon dating can be used to determine how old a piece of organic material may be. Before jumping into the differential equation, though, let's first spend just a minute discussing the idea behind radiocarbon dating.

Carbon-14 is a radioactive isotope of carbon with a half-life of 5730 years, and is naturally produced in the atmosphere when cosmic rays from the sun interact with Carbon-12. This has been going on since the beginning of the Earth, and the ratio of Carbon-14 to Carbon-12 is known and is considered to be relatively stable.

When plants breathe in carbon dioxide, they take in both Carbon-12 and Carbon-14 indiscriminately, and so the proportion of ${}^{14}C$ to ${}^{12}C$ is the same in the plant as it is in the atmosphere. Similarly when animals eat plants, or other animals that have eaten plants, they take in both Carbon-14 and Carbon-12.

While a plant or animal is alive, then, its proportion of Carbon-14 to Carbon-12 is about the same as the proportion in the atmosphere, which can be measured. When the organism dies, however, it stops taking in carbon. The Carbon-12 it took in while alive is stable, but the Carbon-14 is radioactive and so it decays. This means that the proportion of Carbon-14 to Carbon-12 changes over time. By measuring the amount of Carbon-14 and Carbon-12 in a sample of organic material, we can thus determine what this ratio is and deduce how long ago the corresponding organism died, as the next example illustrates.

Example 2.8.
Suppose a sample of wooly mammoth fur recovered from frozen arctic tundra is sent to a lab which determines that only 30% of the fur's original Carbon-14 remains. Given that Carbon-14 has a half-life of 5730 years, how old is the sample of fur?

Letting $M(t)$ denote the mass of Carbon-14 in the sample t years after the mammoth's death, we know from a previous problem about exponential decay (see Example 1.6 on page 26) that the mass satisfies the differential equation

$$\frac{dM}{dt} = -rM$$

since the rate of change of the mass of Carbon-14 is proportional to the current amount of Carbon-14. Since the half-life is 5730 years, we can determine that the constant of proportionality r is

$$r = \frac{\ln(2)}{5730}.$$

That is,

$$M(t) = M_0 e^{-t\,\frac{\ln(2)}{5730}}$$

where M_0 is the initial mass of Carbon-14 in the sample of mammoth fur. If the sample is T years old, then we would have

$$M_0 e^{-T\frac{\ln(2)}{5730}} = 0.3M_0$$

since only 30% of the Carbon-14 remains. From this we can simply solve for the age T:

$$\begin{aligned} & M_0 e^{-T\frac{\ln(2)}{5730}} = 0.3M_0 \\ \implies & e^{-T\frac{\ln(2)}{5730}} = 0.3 \\ \implies & -T\frac{\ln(2)}{5730} = \ln(0.3) \\ \implies & T = -5730\frac{\ln(0.3)}{\ln(2)} \approx 9952.81 \end{aligned}$$

And so the fur sample is a little less than 10,000 years old.

We'll finish up our examples of modeling applications with one more involved example about compute escape velocity. It should be noted this example is longer and more involved than the previous examples, and you shouldn't worry too much about trying to internalize every detail of the example. It's mainly included as a fun example of something interesting we can calculate based on what we've done thus far.

Example 2.9.
Suppose a rocket of mass m is launched straight up from the surface of the Earth at sea level. Recalling that the force of gravity satisfies an inverse square law (the force of gravity between two objects is inversely proportional to the square of the distance between the objects), what initial velocity v_0 would be required to reach a maximum height of ξ above the Earth? What velocity insures the object never falls back down to Earth? (This last quantity is sometimes called *escape velocity*.)

Here the only force acting on the rocket is its weight (i.e., gravity between the rocket and the Earth). At sea level this is $-mg$ where m is the mass measured in kilograms and g is the familiar acceleration due to gravity near the surface of the Earth, $9.8\frac{m}{s^2}$. This weight changes, however, as we get higher and higher above the Earth. (I.e., we approach "weightlessness" the higher the rocket gets.) In general, the weight is inversely proportional to the square of the distance between the height of the rocket above the Earth and the center of the planet. If the radius of the Earth is R and our height above the surface is x, the total distance between the rocket and the center of the Earth is $R + x$, and so the weight of the rocket has the form

$$W(x) = \frac{-k}{(R+x)^2}$$

where k is some constant of proportionality we can compute.

When $x = 0$ (the rocket is at sea level on the surface of the planet), we know $W(0) = -mg$, and so we have

$$\begin{aligned} W(0) &= \frac{-k}{(R+0)^2} = -mg \\ \implies -k &= -mgR^2. \end{aligned}$$

Thus the rocket's weight, once it's x meters above the Earth, is

$$W(x) = \frac{-mgR^2}{(R+x)^2}.$$

This weight is a force, and Newton's second tells us force is mass times acceleration (the derivative of velocity). Equating that with our expression for the weight above, we have the following differential equation:

$$m\frac{dv}{dt} = \frac{-mgR^2}{(R+x)^2}.$$

Since m is a factor on both sides of the equation we can cancel it out and write the differential equation as simply

$$\frac{dv}{dt} = \frac{-gR^2}{(R+x)^2}.$$

Notice that the problem we're considering concerns velocity and position, not time. That is, we would like to remove the t from the equation above and be left with v and x. Since our rocket is launched straight up with some initial velocity, we can interpret the velocity as a function of the rocket's position. Since the rocket's position is a function of time since launch, this means we can think of the velocity as $v(x(t))$. Differentiating this gives us

$$\frac{dv}{dt} = \frac{d}{dt}v(x(t)) = v'(x(t))x'(t) = \frac{dv}{dx} \cdot \frac{dx}{dt}.$$

Notice, though, that $\frac{dx}{dt}$ *is* the velocity v, and so we have

$$\frac{dv}{dt} = v\frac{dv}{dx}.$$

Plugging this into our earlier equation we have

$$\begin{aligned} \frac{dv}{dt} &= \frac{-gR^2}{(R+x)^2} \\ \implies v\frac{dv}{dx} &= \frac{-gR^2}{(R+x)^2} \end{aligned}$$

which is a separable differential equation. Integrating both sides gives us

$$\int v\,dv = -gR^2 \int \frac{1}{(R+x)^2}\,dx.$$

Of course, the left-hand side is simply $\frac{v^2}{2}$. For the right-hand side we can do the substitution $u = R + x$, $du = dx$ so the integral becomes

$$-gR^2 \int u^{-2}\,du = gR^2u^{-1} + C$$

which in terms of x tells us

$$-gR^2 \int \frac{1}{(R+x)^2}\,dx = \frac{gR^2}{R+x} + C.$$

So, after integrating both sides of our differential equation we have

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + C.$$

Now, if we have an initial velocity $v_0 > 0$ at position $x = 0$ (i.e., when the rocket first launches from the ground), then our equation becomes

$$\frac{v_0^2}{2} = \frac{gR^2}{R+0} + C = gR + C$$

and so $C = {v_0^2}/{2} - gR$. Plugging this into the above and solving for v gives us

$$\begin{aligned}\frac{v^2}{2} &= \frac{gR^2}{R+x} + \frac{v_0^2}{2} - gR \\ \implies v &= \pm\sqrt{\frac{2gR}{R+x} + v_0^2 - 2gR}.\end{aligned}$$

This tells us that if our rocket has an initial velocity of v_0 at take-off, then at the moment the rocket is x meters above the ground its velocity is given by

$$v = \pm\sqrt{\frac{2gR}{R+x} + v_0^2 - 2gR}.$$

(The physical interpretation of the plus/minus sign is that the rocket passes through a position twice: once going up, and once coming down. The speed of the rocket at each point will be the same, but the direction flips.)

The maximum altitude the rocket reaches, let's call that ξ, occurs when the velocity is zero (i.e., at the apex of the rocket's trajectory). Plugging in $x = \xi$ and $v = 0$, we can solve for ξ to see what this maximum altitude is:

$$\begin{aligned}0 &= \frac{2gR^2}{R+\xi} + v_0^2 - 2gR \\ \implies 2gR - v_0^2 &= \frac{2gR^2}{R+\xi} \\ \implies R + \xi &= \frac{2gR^2}{2gR - v_0^2} \\ \implies \xi &= \frac{2gR^2}{2gR - v_0^2} - R = \frac{v_0^2 R}{2gR - v_0^2}.\end{aligned}$$

This tells us the maximum height we will achieve with an initial velocity of v_0. We could instead have solved for v_0 to determine the initial velocity required to reach a maximum altitude of ξ:

$$\begin{aligned} 0 &= \frac{2gR^2}{R+\xi} + v_0^2 - 2gR \\ \implies v_0^2 &= 2gR - \frac{2gR^2}{R+\xi} \\ &= \frac{2gR(R+\xi) - 2gR^2}{R+\xi} \\ &= \frac{2gR\xi}{R+\xi}. \end{aligned}$$

Thus if the initial velocity is $v_0 \geq \sqrt{\frac{2gR\xi}{R+\xi}}$, the rocket will reach a height of at least ξ.

Now, to determine the escape velocity, we want to see what will happen as the rocket gets higher and higher and higher before coming back down. I.e., we want to consider what happens as ξ goes to infinity:

$$\begin{aligned} \lim_{\xi\to\infty} \sqrt{\frac{2gR\xi}{R+\xi}} &= \sqrt{\lim_{\xi\to\infty} \frac{2gR\xi}{R+\xi}} \\ &= \sqrt{\lim_{\xi\to\infty} \frac{2gR\xi}{R+\xi} \cdot \frac{1/\xi}{1/\xi}} \\ &= \sqrt{\lim_{\xi\to\infty} \frac{2gR}{R/\xi + 1}} \\ &= \sqrt{2gR}. \end{aligned}$$

That is, if we want our rocket to go down and never come back up, we need its initial velocity to be at least $\sqrt{2gR}$ where g is acceleration due to gravity at the surface of the Earth and R is the radius of the Earth. These two values can easily be looked up:

$$g = 9.8\frac{\mathrm{m}}{\mathrm{s}^2}, \text{ and } R = 6.371 \times 10^6\mathrm{m}.$$

The escape velocity is thus

$$\begin{aligned}
&\sqrt{2 \cdot 9.8 \frac{\text{m}}{\text{s}^2} \cdot 6.371 \times 10^6 \text{m}} \\
\approx &\sqrt{124.8716 \times 10^6 \frac{\text{m}^2}{\text{s}^2}} \\
\approx &11.175 \times 10^3 \frac{\text{m}}{\text{s}} \\
= &11.175 \frac{\text{km}}{\text{s}}.
\end{aligned}$$

And so a rocket needs to have an initial velocity of a little more than eleven kilometers (about 7 miles) per second to escape the Earth and never come back down.

Remark.
It's worth noting that in the example above we were assuming that no force other than the gravity of the Earth was affecting the rocket. This means, in particular, the rocket does not continue to burn fuel to accelerate in our example. Rockets in the real world don't work like this, of course. (If they did, you'd basically have one giant explosion when the rocket first took off, and then let all of that initial velocity do all of the work.) In reality, rockets continually burn fuel, and many space-bound rockets burn fuel in stages, dumping booster rockets as they get higher into the air to make the rocket lighter so that less fuel is needed. This fact will be familiar to anyone who has played Kerbal Space Program.

2.4 Existence and uniqueness of solutions

We now discuss some theoretical issues that we have side-stepped regarding the existence and uniqueness of solutions of initial value problems. Along the way we'll have to make a short excursion into multivariable calculus in order to define partial derivatives, which are needed to precisely state some of our theorems.

We begin, though, by stating a theorem which we have essentially already seen how to prove, just to set the stage for what's to come.

Theorem 2.1.
Suppose p and q are continuous functions defined in an open interval (a, b), and let x_0 be any value in (a, b). Then for any first-order linear differential equation

$$\frac{dy}{dx} + p(x)y = q(x)$$

any number x_0 in (a, b), and any real number y_0 there exists a unique function $f(x)$ defined on (a, b) such that $y = f(x)$ solves the differential equation above as well as the initial condition $f(x_0) = y_0$.

Proof.
We have already seen that a solution exists and can be computed using an integrating factor. For uniqueness of the solution, suppose there were two different solutions, say $f(x)$ and $g(x)$. We would then have

$$f'(x) + p(x)f(x) = q(x), \text{ and}$$
$$g'(x) + p(x)g(x) = q(x).$$

Since the left-hand sides of both equations equal $q(x)$, they must equal one another and so we have

$$f'(x) + p(x)f(x) = g'(x) + p(x)g(x).$$

Now let $\mu(x)$ be the integrating factor, $\mu(x) = e^{\int p(x)\,dx}$. Multiplying both sides of the equation by the integrating factor gives us

$$\mu(x)f'(x) + p(x)\mu(x)f(x) = \mu(x)g'(x) + p(x)\mu(x)g(x)$$

which we may rewrite as

$$\frac{d}{dx}\mu(x)f(x) = \frac{d}{dx}\mu(x)g(x).$$

Integrating both sides gives us that

$$\mu(x)f(x) = \mu(x)g(x) + C$$

for all x and some choice of C. Notice, though, since both $f(x)$ and $g(x)$ solve our initial value problem we must have

$$\mu(x_0)f(x_0) = \mu(x_0)g(x_0) + C$$

which becomes

$$\mu(x_0)y_0 = \mu(x_0)y_0 + C$$

and so we must have $C = 0$. Thus $\mu(x)f(x) = \mu(x)g(x)$ and since $\mu(x)$ is never equal to zero (since it's defined as $e^{\int p(x)\,dx}$), we can divide through by $\mu(x)$ and we have that $f(x) = g(x)$. □

So, we can always solve linear, first order initial value problems, and we can do this explicitly. We have seen that for non-linear differential equations we can sometimes solve the equations implicitly (e.g., this occurs in the case of separable equations), but we may not be able to find an explicit solution. Still, the question remains whether solutions are even guaranteed to exist or not. Is there a theorem analogous to Theorem 2.1 for non-linear equations? To answer this we need to know a little bit about partial derivatives.

Recall that for a function of one variable $f(x)$, the derivative of f at x_0 is defined as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

For a function of two variables, $f(x, y)$, the ***partial derivative of f at (x_0, y_0) with respect to x*** is defined similarly to the derivative of a function of one variable, except we leave the y-value in the function alone. This quantity we are about to describe is often denoted as $f_x(x_0, y_0)$ or

$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$,

$$\begin{aligned} f_x(x_0,y_0) &= \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \\ &= \lim_{x\to x_0} \frac{f(x,y_0)-f(x_0,y_0)}{x-x_0} \\ &= \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}. \end{aligned}$$

The partial derivative with respect to y is defined similarly:

$$\begin{aligned} f_y(x_0,y_0) &= \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \\ &= \lim_{y\to y_0} \frac{f(x_0,y)-f(x_0,y_0)}{x-x_0} \\ &= \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}. \end{aligned}$$

We can compute these partial derivatives using our "usual" calculus rules, but we keep the other variable constant. That is, when differentiating with respect to x, we treat y as a constant; when differentiating with respect to y, we treat x as a constant. For example, if $f(x,y) = xy^2 + x^3\sin(y)$, then

$$\begin{aligned} \frac{\partial f}{\partial x} &= y^2 + 3x^2\sin(y) \\ \frac{\partial f}{\partial y} &= 2xy + x^3\cos(y) \end{aligned}$$

With partial derivatives at our disposal, we can now state a general theorem concerning the existence and uniqueness of solutions of initial value problems.

Theorem 2.2.
Suppose $g(x,y)$ is a function of two variables defined on the open rectangle $(a,b)\times(c,d)$ in the plane, and suppose also that both $g(x,y)$ and its partial derivative with respect to y, $\frac{\partial g}{\partial y}$, are continuous on this rectangle. Then for any $(x_0,y_0)\in(a,b)\times(c,d)$ there exists a function $f(x)$ defined on an open

interval $(x_0 - \delta, x_0 + \delta)$ *solving the first-order differential equation*

$$\frac{dy}{dx} = g(x, y)$$

and satisfying the initial condition $f(x_0) = y_0$.

The proof of this theorem is considerably more complicated than the proof of Theorem 2.1 above. The idea behind the proof is to show that any putative solution to the differential equation must satisfy a certain type of integral equation, and then construct a sequence of functions solving that integral equation in such a way that their limit also solves the differential equation. This is a rather technical process, so we won't say any more about the details of the proof in this class, though we will certainly rely on Theorem 2.2.

Remark.
Details of the proof can easily be looked up online, however, if you're curious; just Google for *proof of the Picard-Lindelöf theorem*.

Though the theorem (and its proof) do not describe an effective way of computing the solution of a general first-order initial value problem, they do at least promise us the existence of such a solution. The theorem tells us that for some interval around our x_0-coordinate in the initial condition, a solution to the initial value problem will exist as long as g and $\frac{\partial g}{\partial y}$ are both continuous "near" the point (x_0, y_0). Exactly what that interval where a solution is defined is depends heavily on the differential equation and its initial condition. For non-linear differential equations there's no simple formula for even determining the size of this interval in general.

For linear differential equations, on the other hand, we see that a solution to

$$\frac{dy}{dx} + p(x)y = q(x),\ y(x_0) = y_0$$

exists in largest interval around x_0 where both p and q are continuous.

Example 2.10.
For what interval does the initial value problem

$$(x-3)\frac{dy}{dx} + \ln(x)y = 2x,\ y(1) = 2$$

have a solution?

Writing this as

$$\frac{dy}{dx} + \frac{\ln(x)}{x-3}y = 2x$$

we see we need the interval around 1 where $\frac{\ln(x)}{x-3}$ are both continuous. Since $2x$ is a polynomial it is continuous everywhere, whereas the function $\frac{\ln(x)}{x-3}$ is continuous in $(0,3) \cup (3,\infty)$. Hence the initial value problem will be solved in the interval $(0,3)$. Notice we did not need to actually solve the differential equation in order to determine where the solution would be defined.

2.5 Autonomous equations, the logistic equation, and equilibria

Autonomous equations

We now consider another family of first-order differential equations that arise in many applications: the "autonomous" equations. We say a differential equation of the form

$$\frac{dy}{dx} = p(y)$$

where p is a function only of y, not of x, an ***autonomous differential equation***. For example, a differential equation such as

$$\frac{dy}{dx} = ry$$

which appeared when we discussed half-life and radioactive decay is autonomous. Using one of the techniques discussed earlier, such as how to solve separable equations, we see that the solution to $\frac{dy}{dx} = ry$ is given by

$$y = y_0 e^{rt}$$

where y_0 is the value of $y(0)$. This equation, $y = y_0 e^{rt}$ models "exponential growth" or "exponential decay," depending on whether r is positive or negative.

One application of such equations occurs in population dynamics. For example, many populations (animals, plants, fungi, bacteria, ...) experience exponential growth under ideal conditions.

As an example, consider the number of yeast cells that are in a batch of fermenting beer. These number of yeast cells grows at a rate that's proportional to the current number of yeast cells. E.g., if the yeast cells reproduce asexually by "budding" (where one yeast cell creates a "bud" that becomes another yeast cell), then how quickly the number of yeast cells grows depends on how many yeast cells we currently have, and this is exactly the kind of situation described above.

To be particular, let's suppose that after pitching yeast into our "wort" (the unfermented sugary liquid that will eventually become beer) we have 100,000 yeast cells. If our strain of yeast is known to triple every day, how many yeast cells will there be after t days?

Letting y denote the number of yeast cells in our fermenting beer, we are told in the statement of the problem that $y(0) = 100,000$ and since the cells triple each day, we have $y(1) = 300,000$. Now, since the growth rate of the number of yeast cells is proportional to the current number of cells, the population size y satisfies an equation of the form

$$\frac{dy}{dt} = ry.$$

This equation is solved by

$$y = 100,000e^{rt}.$$

Since we know that $y(1) = 300,000$, we can perform some simple algebra to determine r:

$$\begin{aligned} &y(1) = 300,000 \\ \Longrightarrow\ &100,000e^{r} = 300,000 \\ \Longrightarrow\ &e^{r} = 3 \\ \Longrightarrow\ &r = \ln(3). \end{aligned}$$

Thus the total number of yeast cells after t days is

$$y = 1000,000e^{\ln(3)t} = 100,000 \cdot 3^{t}.$$

After 36 hours (1.5 days), for example, the number of yeast cells is

$$y(1.5) = 100,000 \cdot 3^{1.5} \approx 519,615.$$

Of course, in the above we were assuming our yeast cells could continue to reproduce at this rate forever, which isn't realistic. Eventually the yeast will devour all of the sugar in our wort, and without sugar to metabolize the yeast will not be able to continue reproducing. (On this plus side this means fermentation is over and our beer is ready!)

More generally, as the amount of unconsummed sugar in our fermenting beer starts to dwindle, less yeast will be able to find sugar to consume and fewer yeast cells will be able to reproduce. That is, the rate at which the yeast reproduces will slow down over time.

The logistic equation

We can modify our earlier differential equation to account for this. In particular, we may ask that the constant of proportionality – the r in $\frac{dy}{dt} = ry$ – be a function of y, and so our earlier equation becomes replaced by an equation of the form

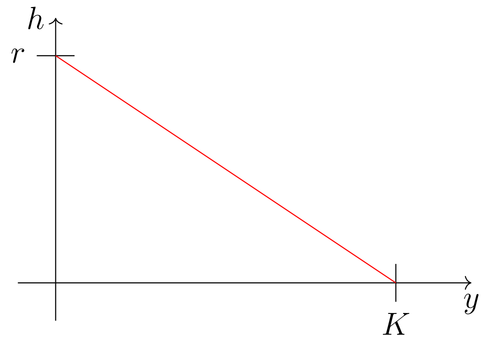
$$\frac{dy}{dx} = h(y) \cdot y.$$

In our situation we may want our function $h(y)$ to satisfy a few reasonable-sounding properties:

- When the population of yeast is as small as possible there are maximum resources available, and so we may ask that $h(0) = r$ for some constant r.
- For some maximum number of yeast cells, let's call this maximum number K, the population size can't grown any more and we may want $h(K) = 0$.
- As long as we have not yet reached our maximum possible size, the number of yeast cells should continue to grow, even if the growth is slow. That is, we may want $h(y) > 0$ for $0 < y < K$.

Let's notice that the three conditions above are basically just saying that that the rate of growth of our yeast cells is the fastest when there are very few yeast cells (as there are more resources available for the cells to consume), and then the rate of growth decreases as the population size grows, and finally stops once we reach some maximum population size.

There are several possible functions for what the $h(y)$ could be, but perhaps the simplest possibility is the function whose graph is a line segment through $(0, r)$ and $(K, 0)$.



The slope of this line is simply

$$\frac{0-r}{K-0} = \frac{-r}{K}$$

and so the function $h(y)$ is

$$\begin{aligned} h(y) &= \frac{-r}{K}y + r \\ &= r\left(\frac{-y}{K} + 1\right) \\ &= r\left(1 - \frac{y}{K}\right). \end{aligned}$$

Our differential equation is then

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y.$$

This particular autonomous differential equation is an example of what's known as a ***logistic equation*** which is often used to model the growth in a population that can't grow indefinitely. The value K is sometimes called the ***carrying capacity*** and represents the largest possible population that can be sustained, while r is called the ***intrinsic growth rate*** and represents the fastest possible growth rate.

In our yeast example, let's again suppose the intrinsic growth rate is $\ln(3)$, corresponding to the yeast tripling every day under ideal conditions, and the carrying capacity is two million yeast cells. Starting off with 100,000 yeast cells initially we would have the following initial value problem:

$$\frac{dy}{dt} = \ln(3) \cdot \left(1 - \frac{y}{2 \times 10^6}\right) y, \qquad y(0) = 1 \times 10^5.$$

This is a separable differential equation, and in principle is something we can solve. We'll see how to solve these logistic equations shortly, but for the purposes of our example let's just go ahead and mention that the solution will be

$$\begin{aligned} y &= \frac{100,000 \cdot 2,000,000}{100,000 + (2,000,000 - 100,000)e^{-\ln(3)t}} \\ &= \frac{2 \times 10^{11}}{1 \times 10^5 + 1.9 \times 10^6 \cdot 3^{-t}}. \end{aligned}$$

After 1.5 days (36 hours), the total number of yeast cells in our fermenting beer would then be about

$$y(1.5) \approx 4.295 \times 10^5$$

so a little less than half a million cells. (Notice this is fewer cells than the number we had in our earlier example since the growth rate of cells slows over time.)

Let's now make some general observations about solutions to our general logistic equation,

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right) y.$$

Notice that certain values of y will make $\frac{dy}{dt} = 0$. Just from our observations before, we see that $y = 0$ and $y = K$ will both make $\frac{dy}{dt} = 0$. The "intuitive" reason for this is that our population size can't grow if the population size is zero (e.g., if there are no yeast cells to begin with we don't magically suddenly have more yeast cells); and by construction we're setting things up so that the population does not grow if we reach our carrying capacity of K. Notice that these are exactly the roots of the polynomial on the right-hand side of the equation,

$$r\left(1 - \frac{y}{K}\right) y.$$

If it happened to be that our initial condition was $y(0) = 0$ or $y(0) = K$, then the solution to the initial value problem would be the constant function $y(t) = 0$ or $y(t) = K$ (depending on which initial condition we had), since the derivative is zero. Solutions such as this which are constant for all time are called ***equilibrium solutions*** of the differential equation.

Notice also that if $y > K$, then $\frac{dy}{dt}$ would be negative since $\frac{y}{K} > 1$. (At least, assuming r is positive which it should be for problems where we model population growth since r represents the intrinsic growth rate). This means that if we start with a population that's higher than the carrying capacity, our population will shrink since the environment can't support more individuals than the carrying capacity. (E.g., our yeast cells would die off if we put too many into our beer and they couldn't all find sugar to consume.) For any $y(0)$ between 0 and K, however, we will have $\frac{dy}{dt} > 0$ and so the population will continue to increase until it approaches the carrying capcity.

If we were to graph solutions $y = f(t)$ to our logistic equation for various initial values, what would we see? Put another way, what do the integral curves of the slope field associated to the differential equation look like? Plotting several different solutions for different initial values $y(0)$ would give us Figure 2.3.

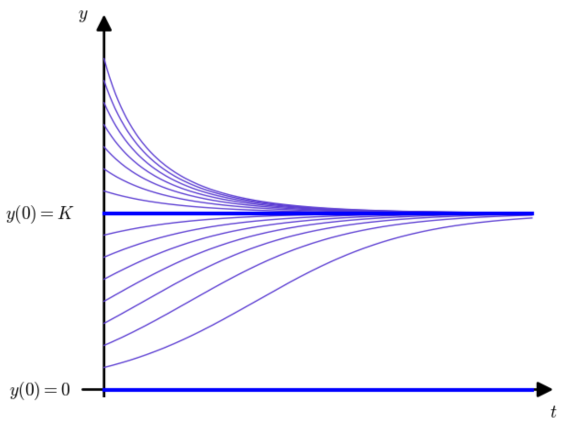


Figure 2.3: Solutions to the logistic equation $\frac{dy}{dt} = r(1 - {}^{y}/{}_{K})y$.

In Figure 2.3 we have plotted integral curves to solutions of the logistic equation for various choices of the initial condition $y(0)$. The solutions corresponding to initial conditions $y(0) = K$ and $y(0) = 0$ are plotted in thick blue; notice these solutions remain constant, since as we noted above for these initial conditions we would have $\frac{dy}{dt} = 0$. All of the other solutions, however, converge towards the solution $y = K$ and away from the solution $y = 0$. The intuitive reason for this is that for any other population size, the population either grows towards K (if $y(0) < K$), or shrinks towards K (if $y(0) > K$)

Solving the logistic equation

Now we turn our attention to seeing how to solve the logistic equation,

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y.$$

Notice this is a separable equation, which we may rewrite as

$$\frac{1}{\left(1 - \frac{y}{K}\right)y}\frac{dy}{dt} = r.$$

We then integrate both sides,

$$\int \frac{1}{\left(1 - \frac{y}{K}\right)y}\,dy = \int r\,dt.$$

The right hand side is of course very easy:

$$\int r\,dt = rt + C.$$

The left hand side, however, requires partial fractions:

$$\begin{aligned}\frac{1}{\left(1 - \frac{y}{K}\right)y} &= \frac{a}{1 - \frac{y}{K}} + \frac{b}{y} \\ &= \frac{ay + b\left(1 - \frac{y}{K}\right)}{\left(1 - \frac{y}{K}\right)y} \\ &= \frac{\left(a - \frac{b}{K}\right) + b}{\left(1 - \frac{y}{K}\right)y}.\end{aligned}$$

This leads us to the system of equations

$$\begin{aligned}a - \frac{b}{K} &= 0 \\ b &= 1.\end{aligned}$$

Of course since $b = 1$, we see $a = \frac{1}{K}$ and we can now compute the integral by rewriting it as

$$\int \frac{1}{\left(1 - \frac{y}{K}\right)y}\,dy = \int \frac{1/K}{1 - \frac{y}{K}}\,dy + \int \frac{1}{y}\,dy.$$

The second term is simply $\int \frac{1}{y}\,dy = \ln|y|$. For the first term, we perform the substitution

$$u = 1 - \frac{y}{K}, \qquad du = \frac{-1}{K}\,dy.$$

The integral then becomes $-\int \frac{1}{u}\,du = -\ln|u|$, and so

$$\int \frac{1/K}{1-\frac{y}{K}}\,dy = -\ln\left|1-\frac{y}{K}\right|.$$

Putting all of this together we have

$$-\ln\left|1-\frac{y}{K}\right| + \ln|y| = rt + C.$$

Taking advantage of properties of logarithms we may write this as

$$\ln\left|\frac{y}{1-\frac{y}{K}}\right| = rt + C.$$

Exponentiating each side yields

$$\left|\frac{y}{1-\frac{y}{K}}\right| = e^{rt+C} = Ce^{rt}.$$

We can now easily drop the absolute values to pick up a $\pm$ which is absorbed into the C,

$$\frac{y}{1-\frac{y}{K}} = Ce^{rt}.$$

Now we can solve for y without too much trouble:

$$\begin{aligned}
&\frac{y}{1-\frac{y}{K}} = Ce^{rt} \\
\implies &y = Ce^{rt}\left(1-\frac{y}{K}\right) = Ce^{rt} - \frac{Ce^{rt}}{K}y \\
\implies &y + \frac{Ce^{rt}}{K}y = Ce^{rt} \\
\implies &y\left(1+\frac{Ce^{rt}}{K}\right) = Ce^{rt} \\
\implies &y = \frac{Ce^{rt}}{1+\frac{Ce^{rt}}{K}} = \frac{KCe^{rt}}{K+Ce^{rt}} = \frac{CK}{Ke^{-rt}+C}
\end{aligned}$$

Now, given an initial population size $y(0) = y_0$, we still need to find the corresponding value of C, but of course this is just algebra:

$$\begin{aligned} y_0 &= \frac{CK}{K+C} \\ \implies CK &= y_0 K + y_0 C \\ \implies CK - y_0 C &= y_0 K \\ \implies C(K - y_0) &= y_0 K \\ \implies C &= \frac{y_0 K}{K - y_0}. \end{aligned}$$

Thus for an initial population size of y_0, the size of the population at time t will be

$$\begin{aligned} y &= \frac{CK}{Ke^{-rt} + C} \\ &= \frac{\left(\frac{y_0 K}{K} - y_0\right) \cdot K}{Ke^{-rt} + \frac{y_0 K}{K - y_0}} \\ &= \frac{\left(\frac{y_0 K}{K - y_0}\right) \cdot K}{\left(\frac{K(K-y_0)e^{-rt} + y_0 K}{K - y_0}\right)} \\ &= \frac{y_0 K}{(K - y_0)e^{-rt} + y_0}. \end{aligned}$$

Notice that, provided y_0 is not zero, the limit as t goes to infinity is

$$\lim_{t \to \infty} \frac{y_0 K}{(K - y_0)e^{-rt} + y_0} = K.$$

So, any solution, except the equilibrium solution $y = 0$, gets attracted to the solution $y = K$, as we had noticed in the picture of integral curves above. For this reason, $y = K$ is called the ***asymptotically stable solution***. Conversely, every solution exception $y = 0$ is repelled away from $y = 0$, and so we call $y = 0$ the ***unstable equilibrium solution***.

A very simple change to our differential equation can reverse the roles of stable and unstable equilibria. If we negate the right-hand side of our earlier logistic equation to obtain

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right) y$$

(we also changed the K to a T for reasons that will be explained in a moment) then our integral curves appear as in Figure 2.4.

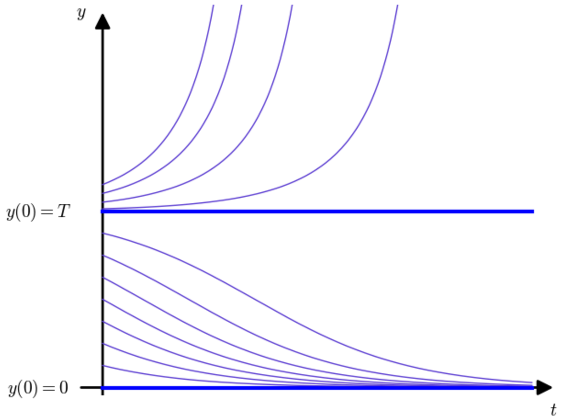


Figure 2.4: The modified logistic equation has has the roles of asymptotically stable and unstable equilibria reversed.

In terms of populations, in this system we have some ***threshhold level*** T below which no growth is possible and the population size shrinks to zero. Above the threshhold level, however, the population size grows exponentially.

The same sort of separation of varaiables and integration from before can be used to solve our modified logistic equation, and an initial population size of y_0 will yield the solution

$$y = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}.$$

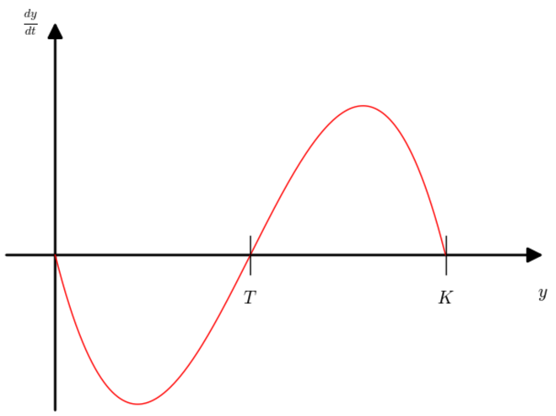
We see here that as t goes to infinity, the long term, asymptotic behavior again depends on y_0), and $y = T$ and $y = 0$ as the equilibria solutions.

Notice that, in fact, the population blows up yo infinity in a finite amount of time, since the denominator can become zero in our solution to the differential equation. If we wish to disallow this from happening, we can combine our original and modified logistic equations to obtain a model for population size where no growth occurs below the threshhold T (e.g., if there are too few organisms to reproduce quickly enough, the population size may shrink), but where we also have a maximum carrying capactiy of K:

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y$$

where we assume $r > 0$ and $0 < T < K$. Notice here there weill be three equilibrium solutions, $y = 0$, $y = T$, and $y = K$ (these three values will make $\frac{dy}{dt} = 0$).

Plotting $\frac{dy}{dt}$ we have the following:



Notice that our graph is below the horizontal axis, meaning $\frac{dy}{dt} < 0$, when y is between 0 and T. This means that our derivative will be negative, and so y is shrinking. If $T < y < K$, however, we have $\frac{dy}{dt}$ is positive and so the population size is growing. Notice that the population size will change and move away from T if it not equal to T, and so $y = T$ is an asymptotically unstable equilibrium. However, $y = 0$ and $y = K$ are both asymptotically stable equilibria, since the population size will converge to those values.

2.6 Exact equations

At this point we really only know how to solve two types of first-order ordinary differential equatinos: the linear and separable equations. We now begin extending our repertoire of "solvable" differential equations by considering a special family of differential equations which are neither linear nor separable. To do this, though, we'll need to know a little bit of multivariable calculus.

Some multivariable calculus

Recall that if $f(x, y)$ is a function of two variables, then the partial derivatives of f with respect to x, denoted $\frac{\partial f}{\partial x}$ or f_x, is obtained by treating y as a constant in our derivative rules. Similarly, the partial derivative of f with respect to y, denoted $\frac{\partial f}{\partial y}$ or f_y, is obtained by treating x as a constant and then applying the dervative rules.

For example, if

$$f(x, y) = x^2 y + \sqrt{y} - x^y,$$

then

$$\frac{\partial f}{\partial x} = 2xy - yx^{y-1}$$
$$\frac{\partial f}{\partial y} = x^2 + \frac{1}{2\sqrt{y}} - x^y \ln(x)$$

We can likewise reverse the process of partial differentiation by integrating, keeping the "other" variable (the one we're not integrating) constant. Recall that when we integrate a function of one variable we pick up a "$+C$" as a constant of integration. The reason for this "$+C$" is that the derivative of a constant is zero. When we perform partial differentiation on a function of two variables, any expression involving only the "other" variable (the one we are not differentiating with respect to) will go to zero. For example, when differentiating $f(x,y) = x^2y + \sqrt{y} - x^y$ with respect to x above, the $\sqrt{y}$ term had derivative zero since it depends only on y which we are treating as a constant. Thus when we integrate with respect to x, we should pick up an arbitrary function of y instead of a constant; and when we integrate with respect to y, we should pick up an arbitrary function of x.

For example, if we integrate $xy^2 + cos(xy)$ with respect to x we obtain

$$\int \left(xy^2 + \cos(xy)\right)\, dx = \frac{x^2y^2}{2} + \frac{\sin(xy)}{y} + g(y),$$

but integrating with respect to y yields

$$\int \left(xy^2 + \cos(xy)\right)\, dy = \frac{xy^3}{3} + \frac{\sin(xy)}{x} + h(x).$$

Notice that if we perform partial differentation of these expressions with respect to the variable we just integrated, we get back the expression we started with:

$$\frac{\partial}{\partial x}\left(\frac{x^2y^2}{2} + \frac{\sin(xy)}{y} + g(y)\right) = xy^2 + \cos(xy)$$
$$\frac{\partial}{\partial y}\left(\frac{xy^3}{3} + \frac{\sin(xy)}{x} + h(x)\right) = xy^2 + \cos(xy)$$

The multivariable chain rule

If x and y are functions of some other variable t, $x(t)$ and $y(t)$, then any function of x and y is also a function of t. That is, a function of two variables $f(x,y)$ becomes a function of just t, $f(x(t), y(t))$. Hence it makes sense to discuss the (ordinary) derivative of this function with respect to t, $\frac{d}{dt}f(x(t), y(t))$. The multivariable chain rule says that this derivative can be computed in terms of the partial derivatives of f with respect to x and y. In particular, we have

$$\frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

For example, if

$$f(x, y) = x^2y - x\sin(y)$$

and if

$$x(t) = t^3, \quad \text{and } y(t) = 2t,$$

then

$$\begin{aligned}\frac{d}{dt}f(x(t), y(t)) &= (2xy - \sin(y)) \cdot 3t^2 + \left(x^2 - x\cos(y)\right) \cdot 2 \\ &= \left(2t^3 \cdot 2t - \sin(2t)\right) \cdot 3t^2 + \left(t^6 - t^3\cos(2t)\right) \cdot 2.\end{aligned}$$

Remark.
If you've taken some linear algebra, you might be interested to learn that this formula for the multivariable chain rule is really matrix multiplication in disguise. In particular, a function $f : \mathbb{R}^2 \to \mathbb{R}$ has total derivative given by the 1×2 matrix,

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

whereas the function $x(t)$ and $y(t)$ define a function $g : \mathbb{R} \to \mathbb{R}^2$ by $g(t) = (x(t), y(t))$, and the total derivative is the 2×1 matrix,

$$\begin{pmatrix} \frac{dx}{dt} \\ \\ \frac{dy}{dt}. \end{pmatrix}$$

The general chain rule says that the derivative of the composition $f \circ g$, which is exactly our function $f(x(t), y(t))$, is obtained by multiplying these two derivatives together,

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{dx}{dt} \\ \\ \frac{dy}{dt}. \end{pmatrix} = \frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

Notice that if y is a function of x, then $f(x, y)$ is really a function just of x, $f(x, y(x))$. By the multivariable chain rule this becomes

$$\frac{d}{dx} f(x, y(x)) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}.$$

For example, if $f(x, y) = xy^2 + y - x$ and $y = \sin(x)$, then the above formula tells us

$$\begin{aligned} \frac{d}{dx} f(x, y(x)) &= y^2 - 1 + (2xy + 1)\cos(x) \\ &= \sin^2(x) - 1 + (2x\sin(x) + 1)\cos(x) \\ &= \sin^2(x) - 1 + 2x\sin(x)\cos(x) + \cos(x) \end{aligned}$$

Exact equations

With these facts from calculus in mind, let's consider the following differential equation:

$$x^2 y + y^2 + \left(\frac{x^3}{3} + 2xy \right) \frac{dy}{dx} = 0.$$

This equation is obviously not linear, and (perhaps less obviously) not separable. So, how can we solve this equation? The next step we'll perform is not immediately obvious, though we'll see where it comes from soon.

Let's notice that the function

$$\psi(x, y) = \frac{x^3 y}{3} + xy^2$$

has the property that

$$\begin{aligned} \psi_x(x, y) &= x^2 y + y^2 \\ \psi_y(x, y) &= \frac{x^3}{3} + 2xy. \end{aligned}$$

With this in mind, our differential equation may be written as

$$\psi_x + \psi_y \frac{dy}{dx} = 0.$$

Since our differential equation is first order and everything is continuous, it must have a solution. That is, y is some function of x. By the multivariable chain rule, this means our differential equation may be written as

$$\frac{d}{dx}\psi(x, y) = 0,$$

or more explicitly,

$$\frac{d}{dx}\left(\frac{x^3 y}{3} + xy^2\right) = 0.$$

Integrating both sides of this equation gives us

$$\frac{x^3 y}{3} + xy^2 = C$$

and this implicitly defines y as a function of x solving the differential equation.

What happened above was that we recognized that there was a function ψ with the special property that our differential equation could be written as

$$\psi_x + \psi_y \frac{dy}{dx} = 0.$$

The multivariable chain rule then said we could write this as

$$\frac{d}{dx}\psi = 0$$

and integrating both sides showed that the level curves

$$\psi(x, y) = C$$

solve the differential equation.

Of course in the example above the ψ appeared out of the blue. How can we go about finding a ψ in general, or determining if such a ψ even exists? The key to answering this is the following theorem from multivariable calculus.

Theorem 2.3 (Clairaut's theorem).
If $\psi(x, y)$ is a continuous function defined in a rectangle $(a, b) \times (c, d)$ (that is, in the set of (x, y) points satisfying $a < x < b$ and $c < y < d$), and if the second-order partial derivatives of ψ are defined and continuous in that rectangle, then we must have $\psi_{xy} = \psi_{yx}$.

As a consequence of Clairaut's theorem, if we have a differential equation of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

and we want to find a ψ such that $\psi_x = M$ and $\psi_y = N$ (if such ψ exists, then by the argument above we have that the level curves $\psi(x, y) = C$ give implicit solutions to our differential equation), then Clairaut's theorem tells us we must have

$$M_y = N_x$$

as this is equivalent to $\psi_{xy} = \psi_{yx}$.

As another example, consider the differential equation

$$3x^2 - 2xy + 2 + \left(6y^2 - x^2 + 3\right)\frac{dy}{dx} = 0.$$

Is it possible to find a function $\psi(x, y)$ such that

$$\begin{aligned} \psi_x &= 3x^2 - 2xy + 2 \\ \psi_y &= 6y^2 - x^2 + 3 \end{aligned}$$

If so, then the differential equation above could be rewritten as

$$\begin{aligned} \psi_x + \psi_y\frac{dy}{dx} &= 0 \\ \implies \frac{d}{dx}\psi(x, y) &= 0 \end{aligned}$$

Integrating both sides of the equation with respect to x tells us that the differential equation would have implicit solutions given by

$$\psi(x, y) = C.$$

But, the question still remains if there is in fact a ψ whose partial derivatives the required functions. Clairaut's theorem would tell us that if such a ψ existed then we'd need $\psi_{xy} = \psi_{yx}$, and in this example that means we would require

$$\frac{\partial}{\partial y}\left(3x^2 - 2xy + 2\right) = \frac{\partial}{\partial x}\left(6y^2 - x^2 + 3\right),$$

and it's easy to see that this equation holds since both partial derivatives will equal $2x$.

If it turned out these partial derivatives did not agree, then that would mean there is no ψ with the required properties! Since the partial derivatives did agree, there is some hope such a ψ could exist, and now we try to determine what that ψ might be.

Since that function ψ would have to satisfy $\psi_x = 3x^2 - 2xy + 2$, we can try to compute ψ by working backwards and integrating ψ_x with respect to x, which would give us

$$\begin{aligned}\psi(x,y) &= \int \psi_x \, dx \\ &= \int \left(3x^2 - 2xy + 2\right) \, dx \\ &= x^3 - x^2y + 2x + h(y).\end{aligned}$$

We know that ψ_y should equal $6y^2 - x^2 + 3$, and we can use this to try to compute what $h(y)$ must be. That is, if

$$\psi(x,y) = x^3 - x^2y + 2x + h(y)$$

then by differentiating this expression we can compute

$$\psi_y = -x^2 + h'(y),$$

but we ψ_y should be $6y^2 - x^2 + 3$. Thus we can equate these expressions and solve for $h'(y)$:

$$\begin{aligned}&- x^2 + h'(y) = 6y^2 - x^2 + 3 \\ \Longrightarrow\ & h'(y) = 6y^2 + 3.\end{aligned}$$

We need to compute $h(y)$, but we know that $h'(y) = 6y^2 + 3$, so we can now integrate to compute $h(y) = 2y^3 + 3y + C$. That is, we claim the function

$$\psi(x,y) = x^3 - x^2y + 2x + 2y^3 + 3y + C$$

has our desired properties, which we can easily double-check:

$$\psi_x(x,y) = \frac{\partial}{\partial x}\left(x^3 - x^2y + 2x + 2y^3 + 3y + C\right) = 3x^2 - 2xy + 2$$
$$\psi_y(x,y) = \frac{\partial}{\partial y}\left(x^3 - x^2y + 2x + 2y^3 + 3y\right) = -x^2 + 6y^2 + 3.$$

Putting all of this together, our original differential equation,

$$3x^2 - 2xy + 2 + \left(6y^2 - x^2 + 3\right)\frac{dy}{dx} = 0$$

may be written as

$$\psi_x(x,y) + \psi_y(x,y)\frac{dy}{dx} = 0$$

which we can further rewrite, by the multivariable chain rule, as

$$\frac{d}{dx}\psi(x,y) = 0.$$

(In fact, differentiating $\psi(x,y)$ – thinking of y as some unknown function of x, $y(x)$, so we're really differentiating $\psi(x,y(x))$ – gives us back the original differential equation.) Integrating both sides of the equation tells us the differential equation is solved implicitly by

$$\psi(x,y) = C.$$

Here let's notice we had a C that appeared earlier in our $h(y)$ when we integrated, and a C in this equation above as well. These are arbitrary C's, so we can combine them together as one C, and our differential equation is solved implicitly by

$$x^3 - x^2y + 2x + 2y^3 + 3y = C.$$

The process we've described through our examples above generalizes. In general, we say that a first-order differential equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is said to be ***exact*** if there exists a function ψ such that $\psi_x = M$ and $\psi_y = N$. Exact differential equations are solved implicitly by $\psi(x,y) = C$. Of course, the question that we should address now is how do we know if a given differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is exact or not – i.e., how do we know that there exists a ψ with $\psi_x = M$ and $\psi_y = N$. Of course, Clairaut's theorem provides one important piece of the puzzle, but is it enough? That is, is knowing $M_y = N_x$ enough to conclude the equation is exact? In general, no: there do exist functions M and N such that $M_y = N_x$, yet no ψ with $\psi_x = M$ and $\psi_y = N$ exist. However, the following theorem tells us that *if* our functions M and N are defined in a "nice enough" domain, such a ψ is guaranteed to exist.

Theorem 2.4.
Suppose that functions $M(x, y)$ and $N(x, y)$ are continuous and have continuous partial derivatives defined in a rectangle $(a, b) \times (c, d)$. If $M_y = N_x$, then the differential equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is exact. That is, there exists a function $\psi(x, y)$ with $\psi_x = M$ and $\psi_y = N$, so solutions to the differential equation are provided (implicitly) by $\psi(x, y) = C$.

Remark.
When we mention a rectangle $(a, b) \times (c, d)$ above, this includes the infinite rectangle, $(-\infty, -\infty) \times (\infty, \infty)$, which is the entire (x, y)-plane.

Functions satisfying the condition that their first partial derivatives for each variable exist and are continuous are often referred to as C^1 ***functions***.

Example 2.11.
Find the implicit solution to the following initial value problem:

$$2xy + y^2 - 2 + \left(x^2 + 2xy + 3\right)\frac{dy}{dx} = 0, \quad \text{and} \quad y(2) = 3.$$

Notice ouf M and N functions here are simply polynomials, and so they are defined and continuous everywhere, and their partial derivatives are defined and continuous everywhere as well (since these will be polynomials too).

We first check if the partial derivatives agree in the way they are required to for an equation to be exact:

$$\frac{\partial}{\partial y}\left(2xy + y^2 - 2\right) = 2x + 2y$$
$$\frac{\partial}{\partial x}\left(x^2 + 2xy + 3\right) = 2x + 2y$$

By Theorem 2.4 above, there must exist a ψ so that the level curves $\psi(x, y) = C$ represent solutions to our differential equation. To find such a ψ we integrate its partial derivatives, which by Theorem 2.4, we know correspond to terms of our differential equation.

$$\begin{aligned}\psi(x, y) &= \int \psi_x \, dx \\ &= \int \left(2xy + y^2 - 2\right) \, dx \\ &= x^2y + xy^2 - 2x + h(y).\end{aligned}$$

As we require $\psi_y(x, y) = x^2 + 2xy + 3$, we must have the following:

$$\begin{aligned}&\frac{\partial}{\partial y}\psi(x, y) = x^2 + 2xy + 3 \\ \implies &\frac{\partial}{\partial y}\left(x^2y + xy^2 - 2x + h(y)\right) = x^2 + 2xy + 3 \\ \implies &x^2 + 2xy + h'(y) = x^2 + 2xy + 3 \\ \implies &h'(y) = 3 \\ \implies &h(y) = 3y.\end{aligned}$$

Thus our $\psi(x, y)$ is

$$\psi(x, y) = x^2y + xy^2 - 2x + 3y.$$

Let's quickly double check that

$$x^2y + xy^2 - 2x + 3y = C$$

would give a solution to our differential equation by differentiating both sides of the equation with respect to x:

$$\begin{aligned} &x^2y + xy^2 - 2x + 3y = C \\ \implies &\frac{d}{dx}\left(x^2y + xy^2 - 2x + 3y\right) = \frac{d}{dx}C \\ \implies &2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} - 2 + 3\frac{dy}{dx} = 0 \\ \implies &2xy + y^2 - 2 + \left(x^2 + 2xy + 3\right)\frac{dy}{dx} = 0. \end{aligned}$$

Our differential equation is thus (implicitly) solved. To solve the initial value problem, we simply need to find the choice of C so that the point $(2, 3)$ is on the curve

$$x^2y + xy^2 - 2x + 3y = C.$$

This is a simple matter of plugging in $x = 2$, $y = 3$ and computing C:

$$\begin{aligned} &2^2 \cdot 3 + 2 \cdot 3^2 - 2 \cdot 2 + 3 \cdot 3 = C \\ \implies &C = 35. \end{aligned}$$

And so the implicit solution to our initial value problem is

$$x^2y + xy^2 - 2x + 3y = 35.$$

Sometimes we can take a non-exact equation and modify it to become an exact equation. For instance, consider the differential equation

$$(x + 2)\sin(y) + x\cos(y)\frac{dy}{dx} = 0.$$

Notice this equation *is not* exact:

$$\begin{aligned} \frac{\partial}{\partial y}(x + 2)\sin(y) &= (x + 2)\cos(y) \\ \frac{\partial}{\partial x}x\cos(y) &= \cos(y) \end{aligned}$$

Notice, however, if we were multiply through by the function xe^x, the differential equation becomes

$$xe^x(x + 2)\sin(y) + xe^x \cdot x\cos(y)\frac{dy}{dx} = 0,$$

or simply

$$\left(x^2e^x + 2xe^x\right)\sin(y) + x^2e^x\cos(y)\frac{dy}{dx} = 0,$$

and this equation *is* exact:

$$\begin{aligned}\frac{\partial}{\partial y}\left(x^2e^x + 2xe^x\right)\sin(y) &= \left(x^2e^x + 2xe^x\right)\cos(y)\\ \frac{\partial}{\partial x}x^2e^x\cos(y) &= \left(x^2e^x + 2xe^x\right)\cos(y)\end{aligned}$$

The choice to multiply by xe^x seems out of the blue right now, but we'll see where it came from soon. Let's notice, though, that since the partial derivatives and everything is continuous and defined everywhere, Theorem 2.4 tells us there must exist a function $\psi(x, y)$ whose partial derivatives are the terms in our modified differential equation,

$$\begin{aligned}\psi_x &= \left(x^2e^x + 2xe^x\right)\sin(y)\\ \psi_y &= x^2e^x\cos(y).\end{aligned}$$

To find such a ψ we can compute either $\int \psi_x\,dx$ or $\int \psi_y\,dy$. Here the integral with respect to y is slightly easier:

$$\begin{aligned}\psi(x, y) &= \int \psi_y\,dy\\ &= \int x^2e^x\cos(y)\,dy\\ &= x^2e^x\sin(y) + g(x).\end{aligned}$$

Now we need to determine the correct choice of $g(x)$ by noting

$$\begin{aligned}&\frac{\partial}{\partial x}\psi(x, y) = \left(x^2e^x + 2xe^x\right)\sin(y)\\ \implies &\frac{\partial}{\partial x}\left(x^2e^x\sin(y) + g(x)\right) = \left(x^2e^x + 2xe^x\right)\sin(y)\\ \implies &\left(x^2e^x + 2xe^x\right)\sin(y) + g'(x) = \left(x^2e^x + 2xe^x\right)\sin(y)\\ \implies &g'(x) = 0.\end{aligned}$$

Hence $g(x)$ is just a constant, and any constant will suffice, so we may as well take $g(x) = 0$.

So, the equation

$$x^2e^x\sin(y) = C$$

solves our modified differential equation,

$$\left(x^2e^x + 2xe^x\right)\sin(y) + x^2e^x\cos(y)\frac{dy}{dx} = 0,$$

but (perhaps surprisingly) it *also* solves our original differential equation because we can cancel out the xe^x. That is, if we take our implicit solution above and differentiate it, then we have

$$\begin{aligned}
&x^2e^x\sin(y) = C\\
\implies &\frac{d}{dx}x^2e^x\sin(y) = \frac{d}{dx}C\\
\implies &2xe^x\sin(y) + x^2e^x\sin(y) + x^2e^x\cos(y)\frac{dy}{dx} = 0\\
\implies &xe^x(2+x)\sin(y) + xe^x\cdot x\cos(y)\frac{dy}{dx} = 0
\end{aligned}$$

Notice, though, that since each term on the left-hand side has a factor of xe^x and the right-hand side is zero, by dividing by xe^x on both sides we arrive at our original differential equation,

$$(x+2)\sin(y) + x\cos(y)\frac{dy}{dx} = 0$$

and so $x^2e^x\sin(y) = C$ also solves the original equation!

What we did above was take a non-exact equation and turn into an exact equation by multiplying through by some function. Can we always do this? If not always, are there special instances when we can? How do we find the right function to multiply by?

In general, suppose we have a differential equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

which *is not* exact. We seek a function $\mu(x,y)$ such that if we multiply through bo μ to obtain

$$M(x,y)\mu(x,y) + N(x,y)\mu(x,y)\frac{dy}{dx} = 0,$$

then this new modified equation is exact. Such a function μ, if it exists, is called an ***integrating factor*** for our differential equation.

If an integrating factor μ exists, notice that means we must have

$$\frac{\partial}{\partial y}\mu(x,y)M(x,y) = \frac{\partial}{\partial x}\mu(x,y)N(x,y).$$

Applying the product rule this gives us

$$\begin{aligned} &\mu_y M + \mu M_y = \mu_x N + \mu N_x \\ \implies &\mu_y M + \mu M_y - \mu_x N - \mu N_x = 0 \\ \implies &\mu_y M - \mu_x N + (M_y - N_x)\mu = 0. \end{aligned}$$

So, μ must satisfy a certain (partial) differential equation. For our purposes right now this means we usually won't be able to solve this equation since this course is about ordinary differential equations and not partial differential equations. There is, however, one special case when we *can* find the μ.

Though in general the μ we need will be a function of both x and y, our example above shows that it may sometimes be a function just of x (or just of y). If that happens, our partial differential equation above

$$\mu_y M - \mu_x N + (M_y - N_x)\mu = 0$$

simplifies since μ_y will be zero. Writing $\mu_x = \frac{d\mu}{dx}$, we then have

$$\begin{aligned} &-\frac{d\mu}{dx}N + (M_y - N_x)\mu = 0 \\ \implies &\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu. \end{aligned}$$

Keeping in mind that we are assuming μ is a function *only* of x, its derivative $\frac{d\mu}{dx}$ must also be a function only of x, which means that the right-hand side of the equation is a function only of x, and so in particular the expression

$$\frac{M_y - N_x}{N}$$

must be a function only of x. If that's the case, then our differential equation involving μ above,

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

is both linear and separable, and so its something we can solve.

To summarize the above discussion, if $M + n\frac{dy}{dx} = 0$ is not exact but the expression $\frac{M_y - N_x}{N}$ depends only on x, then there exists an integrating factor μ such that $\mu M + \mu N \frac{dy}{dx} = 0$ is exact, and we can find μ by solving $\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$.

In our example from before,

$$(x+2)\sin(y) + x\cos(y)\frac{dy}{dx} = 0$$

notice we have

$$\begin{aligned}
&\frac{\frac{\partial}{\partial y}(x+2)\sin(y) - \frac{\partial}{\partial x}x\cos(y)}{x\cos(y)} \\
=&\frac{(x+2)\cos(y) - \cos(y)}{x}\cos(y) \\
=&\frac{x\cos(y) + 2\cos(y) - \cos(y)}{x\cos(y)} \\
=&\frac{(x+1)\cos(y)}{x\cos(y)} \\
=&\frac{x+1}{x}.
\end{aligned}$$

Thus the differential we need to solve to find μ is $\frac{d\mu}{dx} = \frac{x+1}{x}\mu$, but again, this is separable and so we can solve it:

$$\begin{aligned}
&\frac{d\mu}{dx} = \frac{x+1}{x}\mu \\
\implies &\frac{1}{\mu}\frac{d\mu}{dx} = \frac{x+1}{x} = 1 + \frac{1}{x} \\
\implies &\int \frac{1}{\mu}\frac{d\mu}{dx}\,dx = \int \left(1 + \frac{1}{x}\right)dx \\
\implies &\int \frac{1}{\mu}\,d\mu = \int \left(1 + \frac{1}{x}\right)dx \\
\implies &\ln|\mu| = x + \ln|x| + C \\
\implies &|\mu| = e^{x + \ln|x| + C} = e^x e^{\ln|x|} e^C \\
\implies &\mu = Cxe^x.
\end{aligned}$$

Since we just need *a* solution to $\frac{d\mu}{dx} = \frac{x+1}{x}$, we can take $C = 1$ and use $\mu = xe^x$. This, of course, is the function we multiplied by in our earlier example.

Example 2.12.
Solve the differential equation

$$3x^2y + 2xy + y^3 + \left(x^2 + y^2\right)\frac{dy}{dx} = 0.$$

Notice this is not a differential equation as

$$\frac{\partial}{\partial y}\left(3x^2y + 2xy + y^3\right) = 3x^2 + 2x + 3y^2$$
$$\frac{\partial}{\partial x}\left(x^2 + y^2\right) = 2x$$

We check to see if the corresponding $\frac{M_y - N_x}{N}$ expression depends only on x:

$$\frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = \frac{3x^2 + 3y^2}{x^2 + y^2} = 3.$$

Even though this depends on x in a trivial way, it still depends only on x (i.e., does not involve y), and so we should be able to find an integrating factor μ to make our differential equation exact. To do this we need to solve $\frac{d\mu}{dx} = 3\mu$, but this is easily seen to be solved by $\mu = e^{3x}$. Multiplying through by μ now gives us the differential equation

$$e^{3x}\left(3x^2y + 2xy + y^3\right) + e^{3x}\left(x^2 + y^2\right)\frac{dy}{dx} = 0.$$

We can easily check that this differential equation is in fact exact:

$$\frac{\partial}{\partial y}e^{3x}\left(3x^2y + 2xy + y^3\right) = e^{3x}\left(3x^2 + 2x + 3y^2\right)$$
$$\begin{aligned}\frac{\partial}{\partial x}e^{3x}\left(x^2 + y^2\right) &= 3e^{3x}\left(x^2 + y^2\right) + e^{3x} \cdot 2x \\ &= e^{3x}\left(3x^2 + 2x + 3y^2\right)\end{aligned}$$

We now find ψ by integrating either of its partial derivatives:

$$\begin{aligned}\psi &= \int \psi_y \, dy \\ &= \int e^{3x}\left(x^2+y^2\right)\, dy \\ &= \int \left(e^{3x}x^2 + e^{3x}y^2\right)\, dy \\ &= e^{3x}x^2y + \frac{e^{3x}y^3}{3} \\ &= \frac{e^{3x}}{3}\left(3x^2y+y^3\right).\end{aligned}$$

The claim now is that our differential equation is solved implicitly by

$$\frac{e^{3x}}{3}\left(3x^2y+y^3\right) = C$$

which we can double-check using implicit differentiation:

$$\begin{aligned}&\frac{e^{3x}}{3}\left(3x^2y+y^3\right) = C \\ \Longrightarrow\ &\frac{d}{dx}\frac{e^{3x}}{3}\left(3x^2y+y^3\right) = \frac{d}{dx}C \\ \Longrightarrow\ &e^{3x}\left(3x^2y+y^3\right) + \frac{e^{3x}}{3}\left(6xy+3x^2\frac{dy}{dx}+3y^2\frac{dy}{dx}\right) = 0 \\ \Longrightarrow\ &e^{3x}\left(3x^2y+y^3+2xy\right) + e^{3x}\left(x^2+y^2\right)\frac{dy}{dx} = 0 \\ \Longrightarrow\ &3x^2y+y^3+2xy+\left(x^2+y^2\right)\frac{dy}{dx} = 0.\end{aligned}$$

2.7 Practice problems

Problem 2.1. Solve each of the differential equations below.

(a) $\dfrac{dy}{dx} + x^2y = 0$

(b) $\dfrac{dy}{dx} - 2y = 4 - x$

Problem 2.2. Solve each of the initial value problems below. After computing the solution, use Sage to create a plot of the differential equation's vector field together with your solution.

(a) $\frac{dy}{dx} = 3y$, $y(0) = 2$

(b) $\frac{dy}{dx} = -y + 7$, $y(1) = -2$

(c) $\dfrac{dy}{dx} - y = 2xe^{2x}$, $y(0) = 1$

Problem 2.3. Suppose that $b(t)$ represents the number of cells of a certain strain of bacteria in an infected individual t days after an antiobiotic is administered changes according to the differential equation

$$\frac{db}{dt} = \frac{b - 900}{2}.$$

Supposing the initial number of bacteria is $0 < b_0 < 900$, at what time will there be no more bacteria in the individual?

Problem 2.4. Find the general solution to each of the differential equations below.

(a) $\frac{dy}{dx} = \frac{x^2}{y}$

(b) $\frac{dy}{dx} = \frac{3x^2-1}{3+2y}$

(c) $\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$

Problem 2.5. Find the explicit solutions to each initial value problem below.

(a) $\frac{dy}{dx} = 2xy$, $y(0) = 2e$

(b) $\frac{dy}{dx} - 2y + 1 = 0$, $y(1) = 1$

(c) $\frac{dy}{dx} = 6e^{2x-y}$, $y(0) = 0$

Problem 2.6. A tank contains 100 gallons of fresh water, and salt water with a concentration of 1/2 lb of salt per gallon is poured into the tank at a rate of two gallons per minute. Simultaneously, well mixed water is drained from the tank at a rate of two gallons per minute. What is the concentration of salt in the tank after 10 minutes?

Problem 2.7. Determine the largest interval where each initial value problem below is solved.

(a) $\dfrac{dy}{dx} + \dfrac{x}{\sqrt{4-x^2}}y = x^2$, $y(1) = 2$

(b) $\dfrac{dy}{dx} + \dfrac{x}{\sqrt{4-x^2}}y = \dfrac{1}{x^2}$, $y(1) = 2$

(c) $\dfrac{dy}{dx} + \dfrac{1}{\sqrt{1+x^2}}y = x^2$, $y(0) = -2$

Problem 2.8. For each of the autonomous differential equations below, create a graph with $\frac{dy}{dt}$ on the vertical axis and y on the horizontal axis. Determine the equilibrium solutions of the differential equation (if there are any), and classify each as either asymptotically stable or asymptotically unstable. Then use Sage to plot the slope field for the differential equation and verify that your determination of the stable/unstable equilibria is correct.

(a) $\dfrac{dy}{dt} = y^2 - 2y$

(b) $\dfrac{dy}{dt} = y^2 - 4$

(c) $\dfrac{dy}{dt} = 4 - y^2$

Problem 2.9. Compute the antiderivative of each of the functions below with respect to both x and y.

(a) $x^2y - y + 3x$

(b) $\frac{x}{y^2+1}$

Problem 2.10. Determine if there exists a function $\psi(x, y)$ such that ψ_x and ψ_y are the given functions. If such a ψ exists, determine it.

(a) $\psi_x = e^x \sin(y)$, $\psi_y = e^x \cos(y)$

(b) $\psi_x = x^2$, $\psi_y = y^2$

(c) $\psi_x = 3x^2y$, $\psi_y = 3xy^2$

Problem 2.11. Find implicit solutions to each of the initial value problems below.

(a) $(y+2x)+\left(x+3y^2\right)\dfrac{dy}{dx}=0$ where $y(2)=3$

(b) $\left(2+3y^2-x\sin(xy)\right)\dfrac{dy}{dx}=y\sin(xy)-\dfrac{2x}{1+x^2}$ where $y(0)=0$

3

Second order linear differential equations

Ours, according to Leibniz, is the best of all possible worlds, and the laws of nature can therefore be described in terms of extremal principles. Thus, arising from corresponding variational problems, the differential equations of mechanics have invariance properties relative to certain groups of coordinate transformations.

Carl Ludwig Siegel
Lectures on Celestial Mechanics

We now move on from first-order differential equations to second-order differential equations – i.e., we will begin to consider differential equations involving a second derivative. In particular, for the time being we will concentrate on second order linear differential equations. After giving some basic terminology, we'll then start to investigate solutions, working our way up from the simplest possible cases.

3.1 Homogeneous second order linear differential equations

A ***second-order*** differential equation is just a differential equation which may be written as

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right).$$

For example,

$$\frac{d^2y}{dx^2} = x^2y + \cos(x)\frac{dy}{dx}, \text{ and } \frac{d^2y}{dx^2} = y$$

are both second order ODE's. Such a second-order differential equation is called ***linear*** if the function $f(x, y, {}^{dy}/_{dx})$ mentioned above may be written as

$$g(x) + h(x)y + k(x)\frac{dy}{dx},$$

and the differential equation is called ***non-linear*** otherwise. So, a general second-order linear differential equation may be written as

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x).$$

If $r(x)$ is the zero function, then we say this differential equation is ***homogeneous***, and otherwise it is ***nonhomogeneous***.

Let's begin by considering a particular second-order linear differential equation,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0.$$

In this particular case we can turn the differential equation into a first-order equation by introducing a new variable. In particular, if we let $v = \frac{dy}{dx}$, then the above will become

$$\frac{dv}{x} + 2v = 0.$$

This is a first-order linear differential equation, so we can solve it with an integrating factor. In particular, multiplying through by

$$\mu(x) = e^{\int 2\,dx} = e^{2x}$$

we have

$$\begin{aligned} e^{2x}\frac{dv}{dx} + 2e^{2x}v &= 0 \\ \implies \frac{d}{dx}e^{2x}v &= 0 \\ \implies e^{2x}v &= C \\ \implies v &= Ce^{-2x}. \end{aligned}$$

Of course, $v = \frac{dy}{dx}$ and so we really have

$$\begin{aligned} \frac{dy}{dx} &= Ce^{-2x} \\ \implies y &= \int Ce^{-2x}\,dx = -\frac{C_1}{2}e^{-2x} + C_2. \end{aligned}$$

Notice that we wound up getting two constants here. Intuitively, both of the derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ will require an integration in solving the differential equation, and so each one gives us a constant.

Thus an initial value problem for a second-order differential equation requires *two* initial conditions. Usually these are given to us as either

$$y(x_0) = y_0 \text{ and } y'(x_0) = v_0, \text{ or}$$
$$y(x_0) = y_0 \text{ and } y(x_1) = y_1.$$

Plugging in both initial conditions, in whichever form they are given to us, gives a system of equations whose solution will tell us the constants.

Example 3.1.
Solve the initial value problem with differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

and initial conditions $y(0) = 1$ and $y(\ln(\sqrt{2})) = 2$.

We know that the general form of the solution is $y = \frac{-C_1}{2}e^{-2x} + C_2$. Our first initial condition, $y(0) = 1$, then yields the following:

$$1 = \frac{-C_1}{2} + C_2$$

the second initial condition, $y(\ln(\sqrt{2})) = 2$ gives us

$$2 = -C_1 + C_2.$$

We need to solve both of these equations simultaneously, and so we must solve the system of equations

$$\begin{aligned} \frac{-1}{2}C_1 + C_2 &= 1 \\ -C_1 + C_2 &= 2 \end{aligned}$$

We can solve this by first subtracting the second equation from the first, giving

$$\begin{aligned} &\frac{-1}{2}C_1 + C_2 - (-C_1 + C_2) = 1 - 2 \\ \implies &\frac{1}{2}C_1 = -1 \\ \implies &C_1 = -2 \end{aligned}$$

Now that we know $C_1 = -2$, we can easily plug into either of the equations above to determine C_2. Replacing C_1 by -2 in the second equation, $-C_1 + C_2 = 2$, we have $2 + C_2 = 2$ and so $C_2 = 0$. Thus our initial value problem is solved by

$$y = e^{-2x}.$$

Intuitively, the existence of these two constants that appear in the general solution to our second-order differential equation tells us that the space of all solutions to a given second-order ODE is two-dimensional. We can be a little bit more precise with the following theorem:

Theorem 3.1.
If $y_1(x)$ and $y_2(x)$ are two solutions to a second-order linear homogeneous differential equation, and if λ is any real number, then $y_1(x) + y_2(x)$ and $\lambda y_1(x)$ are also solutions.

Proof.
Consider the differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

By assumption we have

$$\frac{d^2y_1}{dx^2} + p(x)\frac{dy_1}{dx} + q(x)y_1 = 0$$
$$\frac{d^2y_2}{dx^2} + p(x)\frac{dy_2}{dx} + q(x)y_2 = 0$$

We simply note now that

$$\begin{aligned}
&\frac{d^2}{dx^2}(y_1+y_2)+p(x)\frac{d}{dx}(y_1+y_2)+q(x)(y_1+y_2)\\
=&\frac{d^2y_1}{dx^2}+p(x)\frac{dy_1}{dx}+q(x)y_1+\frac{d^2y_2}{dx^2}+p(x)\frac{dy_2}{dx}+q(x)y_2\\
=&0+0\\
=&0
\end{aligned}$$

Similarly we can compute

$$\begin{aligned}
&\frac{d^2}{dx^2}\lambda y_1+p(x)\frac{d}{dx}\lambda y_1+q(x)\lambda y_1\\
=&\lambda\left(\frac{d^2y_1}{dx^2}+p(x)\frac{dy_1}{dx}+q(x)y_1\right)\\
=&\lambda\cdot 0\\
=&0.
\end{aligned}$$

□

Remark.
Theorem 3.1 together with the fact that $y=0$ is obviously a solution to a second-order linear homogeneous differential equation shows that the set of solutions is a vector space.

An important corollary of the above theorem is the "principle of superposition:"

Corollary 3.2 (The principle of superposition)**.**
If $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ are all solutions to

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=0,$$

then for any collection of real numbers λ_1, λ_2, ..., λ_n, the linear combination

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x)$$

is also a solution.

Of course, the above theorems assume the existence of solutions to our second-order linear homogeneous differential equation, and we may want to know whether solutions are actually guaranteed to exist or not. Conveniently, the following theorem answers this for us.

Theorem 3.3.
Suppose $a < x_0 < b$ and $p(x)$, $q(x)$, $r(x)$ are all defined and continuous on (a, b). Then, for any y_0 and v_0, there exists a unique solution to the second-order linear differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

with initial conditions $y(x_0) = y_0$, $y'(x_0) = v_0$, defined on (a, b).

This theorem as stated doesn't tell us how to find a solution to the differential equation, it only promises us that a solution exists. So, our next goal is to actually find solutions to some of these second-order differential equations, now that we know (at least, if we're willing to take the above theorem on faith) the solutions exist.

To get started, we consider a special case where the functions $p(x)$ and $q(x)$ appearing in our homogeneous differential equation are just constants,

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

By Theorem 3.3, the solution to this differential equation exists for any choice of initial conditions $y(x_0) = y_0$, $y'(x_0) = v_0$, and the solution is unique and defined on the entire real line (since the constant functions $p(x) = b$ and $q(x) = c$ are defined and continuous on the entire real line). Furthermore, we have two degrees of freedom in choosing solutions because we have two parameters, x_0 and v_0, which describe the space of solutions. So, how do we go about finding these solutions?

Let's first notice that if there was a solution of the form $y = e^{rx}$ for some constant r, we would then have

$$\begin{aligned} &\frac{d^2}{dx^2}e^{rx} + b\frac{d}{dx}e^{rx} + ce^{rx} = 0 \\ \implies &r^2e^{rx} + bre^{rx} + ce^{rx} = 0 \\ \implies &\left(r^2 + br + c\right)e^{rx} = 0. \end{aligned}$$

But since e^{rx} is never equal to zero for any x, we must have that $r^2 + br + c = 0$, and so the r appearing in $y = e^{rx}$ must be a root of this quadratic polynomial. Furthermore, if r is a root of this polynomial, then $y = e^{rx}$ is a solution to this differential equation. Unwinding all of this essentially proves the following proposition.

Proposition 3.4.
The function $y = e^{rx}$ is a solution to

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

if and only if r is a root of the polynomial $ar^2 + br + c$.

As an example, consider

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$$

A solution of the form $y = e^{rx}$ is given only for r's that satisfy

$$r^2 - r - 2 = 0.$$

This quadratic factors as

$$(r+1)(r-2) = 0,$$

and hence $y = e^{-x}$ and $y = e^{2x}$ are solutions, which we can easily verify:

$$\begin{aligned} &\frac{d^2}{dx^2}e^{-x} - \frac{d}{dx}e^{-x} - 2e^{-x} \\ =&e^{-x} - \left(-e^{-x}\right) - 2e^{-x} \\ =&2e^{-x} - 2e^{-x} \\ =&0. \end{aligned}$$

$$\begin{aligned}
&\frac{d^2}{dx^2}e^{2x} - \frac{d}{dx}e^{2x} - 2e^{2x} \\
=&4e^{2x} - 2e^{2x} - 2e^{-x} \\
=&0.
\end{aligned}$$

Notice that not all of the solutions to the differential equation above have the form $y = e^{rx}$; by the principle of superposition, any linear combination of our two solutions above is also a solution. For example, $y = 3e^{-x} + 7e^{2x}$ is also a solution to

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

which is easy to check:

$$\begin{aligned}
&\frac{d^2}{dx^2}\left(3e^{-x} + 7e^{2x}\right) - \frac{d}{dx}\left(3e^{-x} + 7e^{2x}\right) - 2\left(3e^{-x} + 7e^{2x}\right) \\
=&3e^{-x} + 28e^{2x} - \left(-3e^{-x} + 14e^{2x}\right) - 6e^{-x} - 14e^{2x} \\
=&0.
\end{aligned}$$

Given a second-order homogeneous linear differential equation with constant coefficients,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

the quadratic equation

$$ar^2 + br + c = 0$$

is called the ***characteristic equation*** of the differential equation, and generalizing the discussion above, we see if r_1 and r_2 are two (real) roots of this quadratic, then every function of the form

$$y = \lambda_1 e^{r_1 x} + \lambda_2 e^{r_2 x}$$

for any choice of (real) λ_1, λ_2 is a solution to the differential equation.

Example 3.2.
Solve the initial value problem

$$\begin{gathered}
2\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 6y = 0 \\
y(0) = 2,\ y'(0) = -1
\end{gathered}$$

Our goal is to first find the values of r_1 and r_2 such that $y = e^{r_1 x}$

and $y = e^{r_2 x}$ solve the differential equation without regard to the initial conditions. If we can do that, then for every choice of constants c_1 and c_2 we will have that

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

also solves the differential equation, and we can try to see if there is a choice of c_1 and c_2 that will satisfy our initial conditions.

To find r_1 and r_2, we solve the characteristic equation which in this case is $2r^2 + 8r + 6 = 0$. We can divide through by 2 to obtain $r^2 + 4r + 3 = 0$, and now factor the left-hand side to obtain $(r+3)(r+1) = 0$, meaning $y = e^{-3x}$ and $y = e^{-x}$ are solutions to the differential equation, as are all choices of

$$y = c_1 e^{-3x} + c_2 e^{-x}.$$

Now we want to determine a c_1 and c_2 so that our initial conditions are satisfied. To do this, notice

$$\frac{dy}{dx} = \frac{d}{dx}\left(c_1 e^{-3x} + c_2 e^{-x}\right) = -3c_1 e^{-3x} - c_2 e^{-x}.$$

Our initial conditions then give us the equations

$$\begin{aligned} c_1 + c_2 &= 2 \\ -3c_1 - c_2 &= 1 \end{aligned}$$

Adding the equations together tells us $-2c_1 = 3$ and so $c_1 = - {-2}/{3}$. Plugging this into the second equation gives us $2 - c_2 = 1$, or $c_2 = 1$. Thus our initial value problem is solved by

$$y = \frac{-2}{3} e^{-3x} + e^{-x}.$$

Of course, there's are some obvious questions that comes to mind with the process described above: What if the characteristic only has one root (this happens, for example, with $r^2 - 2r + 1 = 0$), or if it only has complex roots (such as $r^2 + 4x + 13 = 0$)? We will address both of these issues soon, but first we will discuss some generalities about second-order linear differential equations which will be helpful.

First, let's introduce some notation. Given an interval I, let $C^2(I)$ be the set of all twice continuously differentiable functions defined on I. Sometimes we'll just write C^2 if the I is understood from context. Given any two functions $p, q \in C^2$, we can define a function $L : C^2 \to C^2$ as follows:

$$L(\varphi) = \varphi'' + p\varphi' + q\varphi.$$

Notice that for any two $\varphi, \psi \in C^2$ and any constant $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} L(\lambda\varphi) &= (\lambda\varphi)'' + p \cdot (\lambda\varphi)' + q \cdot (\lambda\varphi) \\ &= \lambda\varphi'' + \lambda p\varphi' + \lambda q\varphi \\ &= \lambda L(\varphi) \end{aligned}$$

$$\begin{aligned} L(\varphi + \psi) &= (\varphi + \psi)'' + p \cdot (\varphi + \psi)' + q(\varphi + \psi) \\ &= \varphi'' + p\varphi' + q\varphi + \psi'' + p\psi' + q\psi \\ &= L(\varphi) + L(\psi). \end{aligned}$$

This means that L is a linear transformation of C^2. Note too that $L(\varphi) = 0$ if and only if φ is a solution to the differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

That is, the set of solutions to the homogeneous second-order linear differential equation is exactly the kernel of L.

Now, we had seen previously that if y_1 and y_2 were two solutions to

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

then so are $c_1y_1 + c_2y_2$. Now consider the following question: suppose two solutions y_1 and y_2 to the differential equation are known. Can *every* solution be written as $c_1y_1 + c_2y_2$ for some choice of c_1 and c_2?

If this were true, then that would mean every initial condition

$$y(x_0) = y_0,\ y'(x_0) = v_0$$

could be satisfied by $c_1y_1 + c_2y_2$ for some choice of c_1 and c_2. Notice, though, that this gives us a system of linear equations,

$$\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) &= y_0 \\ c_2y_1'(x_0) + c_2y_2'(x_0) &= v_0 \end{aligned}$$

Or, written in terms of matrices,

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}.$$

For a given x_0, we want to solve this system *for all* choices of y_0 and v_0. The algebra for seeing exactly how to do this isn't too bad in terms of elementary row operations of matrices, though it looks ugly. Putting the augmented coefficient matrix of the above system,

$$\left(\begin{array}{cc|c} y_1(x_0) & y_2(x_0) & y_0 \\ y_1'(x_0) & y_2'(x_0) & v_0 \end{array}\right)$$

into row-reduced echelon form gives us

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{y_0 y_2'(x_0) - v_0 y_2(x_0)}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)} \\ 0 & 1 & \frac{y_1(x_0)v_0 - y_1'(x_0)y_0}{y_2'(x_0)y_1(x_0) - y_1'(x_0)y_2(x_0)} \end{array}\right)$$

This tells us the solution to

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= y_0 \\ c_2 y_1'(x_0) + c_2 y_2'(x_0) &= v_0 \end{aligned}$$

is given by

$$\begin{aligned} c_1 &= \frac{y_0 y_2'(x_0) - v_0 y_2(x_0)}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)} \\ c_2 &= \frac{y_1(x_0)v_0 - y_1'(x_0)y_0}{y_2'(x_0)y_1(x_0) - y_1'(x_0)y_2(x_0)} \end{aligned}$$

Of course, this only makes sense if the denominator in the fractions above is non-zero:

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0.$$

I.e., if the determinant of the coefficient matrix,

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix}$$

is non-zero. This quantity, the determinant above, is called the ***Wronskian*** of the solutions y_1 and y_2, and is often denoted by $W(y_1, y_2)(x_0)$, $W(x_0)$, or simply W:

$$W(x_0) = \det \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} = y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0).$$

Proposition 3.5.
Given two functions y_1, y_2, solving our linear second-order homogeneous differential equation above (i.e., $L(y_1) = L(y_2) = 0$), every solution to the differential equation may be written as

$$y = c_1 y_1(x) + c_2 y_2(x)$$

for some choice of c_1 and c_2 if and only if the Wronskian of y_1 and y_2 is non-zero for some choice of point x.

In particular, if x_0 is a point such that $W(x_0) \neq 0$, then any initial condition

$$y(x_0) = y_0,\ y'(x_0) = v_0$$

is solved by $y = c_1 y_1(x) + c_2 y_2(x)$ where

$$c_1 = \frac{y_0 y_2'(x_0) - v_0 y_2(x_0)}{y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0)}$$
$$c_2 = \frac{y_1(x_0) v_0 - y_1'(x_0) y_0}{y_2'(x_0) y_1(x_0) - y_1'(x_0) y_2(x_0)}$$

Remark.
The above proposition shows that the space of solutions to the differential equation $L(y) = 0$ is a two-dimensional real vector space with basis $\{y_1, y_2\}$, provided the corresponding Wronskian is not identically zero.

Example 3.3.
Consider the differential equation

$$x^2 \frac{d^2 y}{dx^2} - 2y = 0.$$

Show that every solution to this differential equation on $(0, \infty)$ may be written as

$$y = c_1 x^2 + c_2 \frac{1}{x},$$

and find the solution which satisfies

$$y(1) = 2 \text{ and } y'(1) = 3.$$

First we note that $y_1(x) = x^2$ and $y_2(x) = \frac{1}{x}$ are in fact solutions:

$$\begin{aligned} x^2 \frac{d^2 y_1}{dx^2} - 2y_1(x) &= x^2 \cdot 2 - 2 \cdot x^2 = 0 \\ x^2 \frac{d^2 y_2}{dx^2} - 2y_2(x) &= x^2 \cdot 2x^{-3} - 2x^{-1} = 0. \end{aligned}$$

To see that every solution may be written as a linear combinaton of our two solutions $y_1(x) = x^2$ and $y_2(x) = x^{-1}$, we need to verify that the Wronskian is non-zero:

$$\begin{aligned} W(x) &= \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \\ &= \det \begin{pmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{pmatrix} \\ &= x^2 \cdot (-x^{-2}) - x^{-1} \cdot 2x \\ &= -1 - 2 \\ &= -3 \end{aligned}$$

As the Wronskian is non-zero at every $x_0 \neq 0$, we may solve any initial value problem involving our given differential equation using the two solutions above. In particular, for the initial conditions given in the example we have

$$\begin{aligned} c_1 y_1(1) + c_2 y_2(1) &= 2 \\ c_1 y_1'(1) + c_2 y_2'(1) &= 3 \end{aligned}$$

That is,

$$\begin{aligned} c_1 + c_2 &= 2 \\ 2c_1 - c_2 &= 3 \end{aligned}$$

We can solve this system by adding the two equations together to obtain $3c_1 = 5$, so $c_1 = 5/3$. Plugging this back into the first equation tells us $5/3 + c_2 = 2$, and so $c_2 = 2 - 5/3 = 1/3$. Thus the solution to our initial value problem is

$$y = \frac{5}{3}x^2 + \frac{1}{3x}.$$

If all solutions to our differential equation $L(y) = 0$ may be written as linear combinations of $y_1(x)$ and $y_2(x)$ (which, again, occurs if and only if the Wronskian is not identically zero), we say that y_1 and y_2 form a ***fundamental set of solutions*** of the differential equation. This justifies our earlier claim that if r_1 and r_2 are distinct real roots of the characteristic equation $ar^2 + br + c = 0$, then every solution to

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

may be written as $y = c_1e^{r_1x} + c_2e^{r_2x}$ for some cohice of c_1 and c_2; we just need to verify that the Wronskian of e^{r_1x} and e^{r_2x} is non-zero, but this is easy to check:

$$\begin{aligned} \det \begin{pmatrix} e^{r_1x} & e^{r_2x} \\ r_1e^{r_1x} & r_2e^{r_2x} \end{pmatrix} &= r_2e^{r_1x}e^{r_2x} - r_1e^{r_1x}e^{r_2x} \\ &= (r_2 - r_1)e^{(r_1+r_2)x} \end{aligned}$$

And so the Wronskian is non-zero provided r_1 and r_2 are distinct.

The obvious question to consider now is: Does every second-order homogeneous linear differential equation actually have a fundamental set of solutions? This is easy to answer with the theory we have developed.

Proposition 3.6.
Suppose $p(x)$ and $q(x)$ are defined and continuous in an interval I. Then the second-order homogeneous linear differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

has a fundamental set of solutions.

Proof.
Pick any point x_0 in I. By the theorem on the existence and uniqueness of second-order linear differential equations, there must exist a solution of the differential equation satisfying $y_1(x_0) = 1$ and $y_1'(x_0) = 0$, and a second solution which satisfies $y_2(x_0) = 0$ and $y_2'(x_0) = 1$. We claim these form a fundamental set of solutions, and this is easy to verify by computing the Wronskian of these two solutions at x_0:

$$\begin{aligned} W(x_0) &= \det \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \\ &= y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \\ &= 1 \cdot 1 - 0 \cdot 1 \\ &= 1. \end{aligned}$$

□

Previously we said the values c_1 and c_2 for finding a solution to our second-order linear homogeneous differential equation can be found given the initial conditions

$$y(x_0) = y_0 \text{ and } y'(x_0) = v_0$$

provided the Wronskian of two fundamental solutions was non-zero at x_0. Could it happen that the Wronskian of these solutions, though not zero everywhere, is zero for some choices of x_0? That is, are there "bad" choices of x_0 where we can't fin d the c_1 and c_2? We can easily answer this question by appealing to the following theorem, named after Norwegian mathematician Niels Henrik Abel (the same mathematician whose name is associated with the abelian groups of abstract algebra).

Theorem 3.7 (Abel's theorem)**.**
If $y_1(x)$ and $y_2(x)$ are any two solutions to

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x) = 0$$

where $p(x)$ and $q(x)$ are continuous in an interval I, then their Wronskian *is equal to*

$$W(x) = ce^{-\int p(x)\,dx}$$

for some constant c.

Proof.
Consider the system of equations

$$\begin{aligned} y_1''(x) + p(x)y_1'(x) + q(x)y_1(x) = 0 \\ y_2''(x) + p(x)y_2'(x) + q(x)y_2(x) = 0 \end{aligned}$$

We obtain a new equation by consider $-y_2$ times the first equation plus y_1 times the second equation. This then gives us

$$-y_2(y_1''(x)+p(x)y_1'(x)+q(x)y_1(x))+y_1(x)(y_2''(x)p(x)y_2'(x)+q(x)y_2(x)) = 0.$$

After expanding and simplifying this becomes

$$y_1(x)y_2''(x) - y_1''(x)y_2(x) + (y_1(x)y_2'(x) - y_1'(x)y_2(x))\,p(x) = 0.$$

Notice that the second term contains the Wronskian, and so we may write this equation as

$$y_1(x)y_2''(x) - y_1''(x)y_2(x) + W(x)p(x) = 0.$$

Furthermore, as $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$, we have

$$\begin{aligned} \frac{dW}{dx} &= y_1(x)y_2''(x) + y_1'(x)y_2'(x) - (y_1'(x)y_2'(x) + y_1''(x)y_2(x)) \\ &= y_1(x)y_2''(x) + y_1''(x)y_2(x). \end{aligned}$$

Hence our equation above can be written as

$$\frac{dW}{dx} + W(x)p(x) = 0.$$

This is a first-order linear differential equation which can be solved by an integrating factor, and the result is $W(x) = Ce^{-\int p(x)\,dx}$. □

Let's notice that this means $W(x)$ is either zero everywhere (if $C = 0$), or it's zero nowhere (since e^x is never zero). It also means for any two different fundamental sets of solutions, their Wronskians differ only by a constant multiple. We can also take advantage of this alternate version of the Wronskian to solve certain differential equations.

Example 3.4.
Find the solution to

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4 = 0$$

satisfying $y(0) = 3$ and $y'(0) = 1$.

First we look for solutions of the form e^{rx} by considering roots of the characteristic equation,

$$r^2 - 4r + r = 0.$$

Notice this factors as $(r-2)^2 = 0$, and so there is only one root, $r = 2$. Thus $y_1 = e^{2x}$ solves the differential equation, but *not* the initial conditions since $y_1(0) = 1$ and $y_1'(0) = 2$. So, we need to determine a second solution to our differential equation. If we had a y_2 such that $y_1 = e^{2x}$ and y_1 formed a fundamental set of solutions, we could then do the algebra to find c_1 and c_2 so that $c_1y_1 + c_2y_2$ solved our initial value problem.

To find the second solution, we take advantage of Abel's theorem which tells us the Wronskian of any fundamental solution is

$$W(x) = Ce^{-\int(-4)\,dx} = Ce^{4x}.$$

That is, our y_2 must satisfy

$$\det\begin{pmatrix} e^{2x} & y_2 \\ 2e^{2x} & y_2' \end{pmatrix} = Ce^{4x}.$$

Computing the determinant, this means

$$y_2'e^{2x} - 2y_2e^{2x} = Ce^{4x}.$$

Dividng both sides of the equation by e^{2x} we then have

$$y_2' - 2y_2 = Ce^{2x}.$$

This is a first-order linear differential equation which we can easily solve. We first multiply through by the integrating factor of $\mu = e^{\int -2\,dx} = e^{-2x}$ to obtain the following:

$$\begin{aligned} e^{-2x}y_2' - 2e^{-2x}y_2 &= C \\ \implies \frac{d}{dx}e^{-2x}y_2 &= C \\ \implies e^{-2x}y_2 &= Cx \\ \implies y_2 &= Cxe^{2x}. \end{aligned}$$

Hence for any non-zero number C, $y_2 = Cxe^{2x}$ will provide our other fundamental solution. It's easy to double-check that $y_2 = Cxe^{2x}$ is in fact a solution to the differential equation:

$$\begin{aligned} &\frac{d^2y_2}{dx^2} - 4\frac{dy_2}{dx} + 4y_2 \\ =&2Ce^{2x} + 2C\left(e^{2x} + 2xe^{2x}\right) - 4C\left(e^{2x} + 2xe^{2x}\right) + 4Cxe^{2x} \\ =&2Ce^{2x} + 2Ce^{2x} + 4Cxe^{2x} - 4Ce^{2x} - 8Cxe^{2x} + 4Cxe^{2x} \\ =&0 \end{aligned}$$

So, the general solution to our differential equation is given by

$$y = c_1e^{2x} + c_2xe^{2x}.$$

The first derivative is then

$$y'(x) = 2c_1e^{2x} + c_2\left(e^{2x} + 2xe^{2x}\right).$$

Thus, if $y(0) = 3$ and $y'(0) = 1$, we have the following system of equations:

$$\begin{aligned} c_1 &= 3 \\ 2c_1 + c_2 &= 1 \end{aligned}$$

Of course, the first equation instantly tells us $c_1 = 3$, and the second equation then becomes $6 + c_2 = 1$, so $c_2 = -5$ and our initial value problem is solved by

$$y = 3e^{2x} - 5xe^{2x}.$$

3.2 Complex numbers and Taylor series

Before going any further in our discussion of differential equations, we should recall a few facts about complex numbers and Taylor series which will be necessary for what's to come.

Recall that a complex number is an expression which may be written as $a + ib$ where a and b are real numbers and $i^2 = -1$. We may denote a complex number by $z = a + ib$ and refer to a as the ***real part*** of z, written $\mathrm{Re}(z)$, and b is the ***imaginary part*** of z, written as $\mathrm{Im}(z)$.

To every complex number $z = a+ib$ there is another complex number obtained by changing the sign of the imaginary part, called the ***complex conjugate*** of z and written $\overline{z}$,

$$\overline{z} = a - ib.$$

We can add complex numbers by adding their real and imaginary parts separately. That is, if z_1 and z_2 are two complex numbers with $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = a_1 + a_2 + i(b_1 + b_2).$$

We multiply complex numbers by distributing the real and imaginary parts, and then simplifying with $i^2 = -1$:

$$\begin{aligned} z_1 z_2 &= (a_1 + ib_1)(a_2 + ib_2) \\ &= a_1 a_2 + i a_1 b_2 + i a_2 b_1 + i^2 b_1 b_2 \\ &= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1). \end{aligned}$$

Notice that complex numbers can be the roots of polynomials with real coefficients. For example, $2 \pm 3i$ are the roots of $x^2 - 4x + 13$. This is easy to check plugging each number, $2 + 3i$ and $2 - 3i$ into the polynomial. Plugging in $2 + 3i$, for example, gives us

$$\begin{aligned} &(2 + 3i)^2 - 4(2 + 3i) + 13 \\ =&4 + 12i + 9i^2 - 8 - 12i + 13 \\ =&4 + 12i - 9 - 8 - 12i + 13 \\ =&0. \end{aligned}$$

Notice too that these two roots are complex conjugates of one another. This is not a fluke: in general every polynomial has complex roots (which occassionally are real – i.e., have zero imaginary parts).

Theorem 3.8 (Fundamental theorem of algebra).
Every polynomial of degree n with real or complex coefficients has n roots counted by multiplicity. Furthermore, if the polynomial has only real coefficients, then the complex roots come in complex conjugate pairs.

Let's also recall that for each point a in the domain of an infinitely differentiable function $f(x)$, there exists an interval around a where $f(x)$ equals its Taylor series centered at a[1],:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The Taylor series for e^x, $\sin(x)$, and $\cos(x)$, centered at $a = 0$, for example are the following:

$$\begin{aligned} e^x &= \sum_{k=0}^{\infty} \frac{1}{k!} x^k \\ \sin(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ \cos(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \end{aligned}$$

Notice that these expressions for e^x, $\sin(x)$, and $\cos(x)$ make sense even

[1]There is a technical point here: the width of the interval may be zero! This is generally not the situation we care about. Functions which can be represented as a convergent Taylor series with positive radius of convergence around each point are called ***analytic***. All of the functions we typically care about are analytic, but non-analytic functions do exist!

when x is a complex number! For example, we can compute e^i by

$$\begin{aligned}
e^i &= \sum_{k=0}^{\infty} \frac{1}{k!} i^k \\
&= 1 + i - \frac{1}{2} - \frac{i}{6} + \frac{1}{24} + \frac{i}{120} - \frac{1}{720} + \cdots \\
&= \left(1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \cdots\right) + i\left(1 - \frac{1}{6} + \frac{1}{120} + \cdots\right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \\
&= \cos(1) + i\sin(1).
\end{aligned}$$

In general, writing down the Taylor series for $e^{i\theta}$, where θ is any real number, and separating the odd-order and even-order terms shows

$$\begin{aligned}
e^{i\theta} &= \sum_{k=0}^{\infty} \frac{1}{k!} (i\theta)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} i^k \theta^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \theta^k + i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \theta^{2k+1} \\
&= \cos(\theta) + i\sin(\theta).
\end{aligned}$$

We can use this to easily exponentiate any complex number $z = a + ib$:

$$\begin{aligned}
e^z &= e^{a+ib} \\
&= e^a e^{ib} \\
&= e^a (\cos(b) + i\sin(b))
\end{aligned}$$

Note too that

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta).$$

From these expressions we can write

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

(This, by the way, gives one easy way or proving certain trig identities like the double-angle identities: convert to exponentials and then use simple algebraic properties.)

3.3 Characteristic polynomials with complex roots

What does all of this have to do with solving differential equations? Suppose we had a second-order homogeneous linear differential equation whose characteristic polynomial only had complex conjugate roots. For example,

$$\frac{d^2y}{dx^2} + y = 0.$$

The characteristic equation is then

$$r^1 + 1 = 0$$

which has roots $r = \pm i$. Do $y = e^{\pm ix}$ solve the differential equation? Let's check in the case of $y = e^{ix}$

$$\begin{aligned}
&\frac{d^2}{dx^2}e^{ix} + e^{ix} \\
=&\frac{d}{dx}ie^{ix} + e^{ix} \\
=&i^2e^{ix} + e^{ix} \\
=&-e^{ix} + e^{ix} \\
=&0
\end{aligned}$$

That $y = e^{-ix}$ case is very similar.

Since these solve two complex-valued functions solve our differential equation, so do any linear combination of these functions, such as

$$\frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix} = \cos(x), \text{ and}$$
$$\frac{1}{2}e^{ix} - \frac{1}{2}e^{-ix} = \sin(x),$$

which is easy to directly verify:

$$\begin{aligned}
\frac{d^2}{dx^2}\cos(x) + \cos(x) &= \frac{d}{dx}(-\sin(x)) + \cos(x) \\
&= -\cos(x) + \cos(x) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\frac{d^2}{dx^2}\sin(x) + \sin(x) &= \frac{d}{dx}\cos(x) + \sin(x) \\
&= -\sin(x) + \sin(x) \\
&= 0
\end{aligned}$$

More generally, each complex-valued solution of a second-order homogeneous linear differential equation gives us two real valued solutions: its real and imaginary parts.

Theorem 3.9.
Suppose $u(x)$ and $v(x)$ are real-valued functions such that the complex-valued function $u(x) + iv(x)$ solves the differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

where $p(x)$ and $q(x)$ are continuous. Then $u(x)$ and $v(x)$ also solve the differential equation.

Proof.
By assumption we have

$$\frac{d^2}{dx^2}\left(u(x) + iv(x)\right) + p(x)\frac{d}{dx}\left(u(x) + iv(x)\right) + q(x)\left(u(x) + iv(x)\right) = 0$$

But writing out the left-hand side and combining the real and imaginary parts this becomes

$$u''(x) + p(x)u'(x) + q(x)u(x) + i\left(v''(x) + p(x)v'(x) + q(x)v(x)\right) = 0.$$

This is a complex number that equals zero, so its real and imaginary parts must be zero which means

$$u''(x) + p(x)u'(x) + q(x)u(x) = 0, \text{ and}$$
$$v''(x) + p(x)v'(x) + q(x)v(x) = 0.$$

□

So, to summarize, if we have a differential equation of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where $ar^2 + br + c$ has only complex conjugate roots, say $r = u \pm iv$, then we see that $e^{(u+iv)x}$ and $e^{(u-iv)x}$ are complex-valued solutions of the differential equation:

$$\begin{aligned}
&a\frac{d^2}{dx^2}e^{(u\pm iv)x} + b\frac{d}{dx}e^{(u\pm iv)x} + ce^{(u\pm iv)x} \\
=&a(u \pm iv)^2 e^{(u\pm iv)x} + b(u \pm iv)e^{(u\pm iv)x} + ce^{(u\pm iv)x} \\
=&\left(a(u \pm iv)^2 + b(u \pm iv) + c\right) e^{(u\pm iv)x} \\
=&0
\end{aligned}$$

But then so are the real and imaginary parts of these solutions, since we can write the real and imaginary parts as linear combinations of our initial complex-valued solutions. Notice that

$$e^{(u\pm iv)x} = e^{ux}e^{\pm ivx} = e^{ux}\left(\cos(vx) \pm i\sin(vx)\right)$$

and so we have the following two real-valued solutions to the differential equation,

$$\begin{aligned}
y_1 &= e^{ux}\cos(vx), \text{ and} \\
y_2 &= e^{ux}\sin(vx).
\end{aligned}$$

Notice the Wronskian of these solutions is non-zero:

$$\begin{aligned}
W(x) &= \det\begin{pmatrix} e^{ux}\cos(vx) & e^{ux}\sin(vx) \\ ue^{ux}\cos(vx0 - ve^{ux}\sin(vx) & ue^{ux}\sin(vx) + ve^{ux}\cos(vx) \end{pmatrix} \\
&= e^{ux}\left(ue^{ux}\sin(vx) + ve^{ux}\cos(vx)\right) - e^{ux}\sin(vx)\left(ue^{ux}\cos(vx) - ve^{ux}\sin(vx)\right) \\
&= ve^{2ux}.
\end{aligned}$$

We have thus proven the following proposition:

Proposition 3.10.
If the roots of $ar^2 + br + c$ are complex conjugates $u \pm iv$, then the general real-valued solution to

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

is given by

$$y = c_1e^{ux}\cos(vx) + c_2e^{ux}\sin(vx).$$

Example 3.5.
Find the solution to the initial value problem

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

where $y(0) = -2$ and $y'(0) = 5$.

The characteristic equation is $r^2 - 4r + 13 = 0$, and this does not factor in any obvious way, so we compute the roots with the quadratic formula:

$$\begin{aligned} r &= \frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot 13}}{2 \cdot 1} \\ &= \frac{4 \pm \sqrt{16 - 52}}{2} \\ &= \frac{4 \pm \sqrt{-36}}{2} \\ &= \frac{4 \pm 6i}{2} \\ &= 2 \pm 3i \end{aligned}$$

Thus a complex-valued solution to the differential equation would be

$$e^{(2+3i)x} = e^{2x+3ix} = e^{2x}e^{3ix} = e^{2x}\left(\cos(3x) + i\sin(3x)\right).$$

Taking the real and imaginary parts of this, we see that

$$y_1 = e^{2x}\cos(3x) \text{ and } y_2 = e^{2x}\sin(3x)$$

for a fundamental set of real solutions, so every real-valued solution may be written as

$$y = c_1e^{2x}\cos(3x) + c_2e^{2x}\sin(3x)$$

for some choice of c_1 and c_2. We need to find these choices so our solution satisfies the initial conditions, $y(0) = -2$ and $y'(0) = 5$. This will revolve around solving a system of linear equations, but first we go ahead and compute the derivative of our putative solution:

$$y' = 2c_1e^{2x}\cos(3x) - 3c_1e^{2x}\sin(3x) + 2c_2e^{2x}\sin(3x) + 3c_2e^{2x}\cos(3x).$$

Now our initial conditions give us the following system of linear equations:

$$\begin{aligned} c_1 &= -2 \\ 2c_1 + 3c_2 &= 5 \end{aligned}$$

Of course, we instantly have $c_1 = -2$ and can then easily compute, using the second equation, that $c_2 = 3$.

The solution to our initial value problem is thus

$$y = -2e^{2x}\cos(3x) + 3e^{2x}\sin(3x)$$

3.4 The method of undetermined coefficients

Though we have developed some general theory at this point, the only second order differential equations we have seen how to solve are the homogeneous equations with constant coefficients. We now begin to develop the methods to solve general non-homogeneous linear second order differential equations. I.e., general differential equations of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

where p, q, and r are continuous functions – *not* necessarily constants.

We begin by relating the solutions to non-homogeneous equations to the corresponding homogeneous equation.

Lemma 3.11.
If $y_1(x)$ and $y_2(x)$ are two solutions to the non-homogeneous equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x),$$

then their difference, $y_1(x) - y_2(x)$, is a solution to the homogeneous equa-

tion

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

Proof.
(Easy) Exercise. □

Recall that we let $L : C^2(I) \to C^2(I)$ represent the differential operator

$$L(\varphi) = \varphi''(x) + p(x)\varphi'(x) + q(x)\varphi(x)$$

to simplify our notation. That is, the differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x),$$

can be written more simply as $L(y) = r(x)$ using this operator.

Lemma 3.12.
If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions to the homogeneous equation $L(y) = 0$, and if $\varphi(x)$ is some function satisfying $L(\varphi) = r(x)$, then every solution so $L(y) = r(x)$ may be written as

$$y = c_1y_1(x) + c_2y_2(x) + \varphi(x).$$

Proof.
Simply observer that $L(y - \varphi)$ equals zero:

$$\begin{aligned} L(y-\varphi) &= L(c_1y_1 + c_2y_2 + \varphi - \varphi) \\ &= L(c_1y_1 + c_2y_2) + L(\varphi) - L(\varphi) \\ &= 0 + r(x) - r(x) \\ &= 0 \end{aligned}$$

But, by linearity, $L(y - \varphi) = L(y) - L(\varphi)$. Thus $L(y) = L(\varphi) = r(x)$.

□

This lemma tells us that if we know how to solve the corresponding homogeneous equation and can find *just one* solution to the non-homogeneous equation, then we can combine these to get all of the solutions. That is, to solve

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x),$$

the very first we must do is find a pair of fundamental solutions to the corresponding homogeneous equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

For example, to solve

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2x,$$

we first solve

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0,$$

which is easily done by considering the characteristic equation,

$$r^2 + 2r + 1 = 0$$

which factors as $(r+1)^2 = 0$, and so the homogeneous equation has the general solution

$$c_1e^{-x} + c_2xe^{-x}.$$

Now we have to find *some* solution to the non-homogeneous equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2x.$$

Noticing that the right-hand side is a polynomial, it seems reasonable to expect that there should be a polynomial solution. So, let's suppose y was some polynomial, let's say of degree three just for the sake of being concrete. Then y would have the form

$$y = \alpha_3x^3 + \alpha_2x^2 + \alpha_1x + \alpha_0$$

and we can easily compute its derivatives,

$$y' = 3\alpha_3 x^2 + 2\alpha_2 x + \alpha_1, \text{ and}$$
$$y'' = 6\alpha_3 x + 2\alpha_2.$$

We can plug this into our differential equation to obtain

$$\begin{aligned} & y'' + 2y' + y = 2x \\ \implies & (6\alpha_3 x + 2\alpha_2) + 2\left(3\alpha_3 x^2 + 2\alpha_2 x + \alpha_1\right) + \left(\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0\right) = 2x \\ \implies & \alpha_3 x^3 + (6\alpha_3 + \alpha_2)\, x^2 + (6\alpha_3 + 4\alpha_2 + \alpha_1)\, x + (2\alpha_2 + 2\alpha_1 + \alpha_0) = 2x. \end{aligned}$$

From this we obtain a system of linear equations,

$$\begin{aligned} 2\alpha_2 + 2\alpha_1 + \alpha_0 &= 0 \\ 6\alpha_3 + 4\alpha_2 + \alpha_1 &= 2 \\ 6\alpha_3 + \alpha_2 &= 0 \\ \alpha_3 &= 0 \end{aligned}$$

This system can be solved easily with back substitution to determine

$$\begin{aligned} \alpha_0 &= -4 \\ \alpha_1 &= 2 \\ \alpha_2 &= 0 \\ \alpha_3 &= 0 \end{aligned}$$

Hence we claim

$$y = 2x - 4$$

is one particular solution to our non-homogeneous equation, which is easy to check:

$$\begin{aligned} & \frac{d^2}{dx^2}(2x-4) + 2\frac{d}{dx}(2x-4) + 2x - 4 \\ & = 0 + 4 + 2x - 4 \\ & = 2x. \end{aligned}$$

By our lemma above, this means that *every* solution to

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2x$$

can be written as

$$y = c_1 e^{-x} + c_2 x e^{-x} + 2x - 4.$$

To find the c_1 and c_2 for a given solution we of course need some initial conditions. So suppose we wanted the solution with $y(0) = 3$ and $y'(0) = 5$. To get our system of equations to find c_1 and c_2 we will need to compute the derivative

$$y' = -c_1e^{-x} + c_2e^{-x} - c_2xe^{-x} + 2$$

The initial conditions then give us

$$\begin{aligned} c_1 - 4 &= 3 \\ -c_1 + c_2 + 2 &= 5 \end{aligned}$$

This is easily solved by $c_1 = 7$ and $c_2 = 10$, and so our initial value problem is solved by

$$y = 7e^{-x} + 10xe^{-x} + 2x - 4$$

In the previous example we assumed our solution to the non-homogeneous equation had the form

$$\alpha_3x^3 + \alpha_2x^2 + \alpha_1x + \alpha_0$$

and found $\alpha_3 = \alpha_2 = 0$; notice the polynomial solution had the same degree as the right-hand of the differential equation, which was $2x$. Was this a fluke, or does this always happen? Before answering this in general, let's consider another example, say

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2x.$$

Could there be a degree one solution to this differential equation? I.e., something of the form $y = \alpha_1x + \alpha_0$? If there was such a solution, then we would have

$$0 - \alpha_1 = 2x.$$

This has no solution! So, can we find a degree two solution? If $y = \alpha_2x^2 + \alpha_1x + \alpha_0$, then $y' = 2\alpha_2x + \alpha$ and $y'' = 2\alpha_2$, thus we have

$$\begin{aligned} 2\alpha_2 - (2\alpha_2x + \alpha_1) &= 2x \\ \implies -2\alpha_2x + (2\alpha_2 - \alpha_1) &= 2x \end{aligned}$$

This gives us the system of equations

$$\begin{aligned} -2\alpha_2 &= 2 \\ 2\alpha_2 - \alpha_1 &= 0 \end{aligned}$$

which is solved by $\alpha_1 = -2$, $\alpha_2 = -1$. Thus we claim that $-x^2 - 2x$ is a solution to our differential equation, which is easy to verify.

to understand what's going on, suppose we're trying to solve

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = r(x)$$

where r is a polynomial of degree n. Notice that if y is a polynomial of degree m, then $\frac{dy}{dx}$ is a polynomial of degree $m-1$, and $\frac{d^2y}{dx^2}$ has degree $m-2$. If $c = 0$ above, then

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx}$$

has degree $m-1$, and so we would require $m = n+1$ so that the degrees of the left-hand and right-hand sides (which equals $r(x)$ in the differential equation above) are equal. If both b and c were equal to zero, however, then $a\frac{d^2y}{dx^2}$ would have degree $m-2$, and we would need $m = n+2$. This tells us what the degree or our solution to the differential equation should be.

Example 3.6.
Solve the initial value problem

$$2\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = x^3 - 6x + 4$$

where $y(0) = 2$ and $y'(0) = 3$.

We first solve the complementary homogeneous equation,

$$2\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 0.$$

The characteristic polynomial is $2r^2 - 64$ which has roots $r_1 = 0$ and $r_2 = 3$ and so the homogeneous equation is solved by

$$y = c_1e^{0x} + c_2e^{3x} = c_1 + c_2e^{3x}.$$

Now we need to also find some solution to the non-homogeneous equation. As the right-hand side of the equation is degree three, but there is a missing "cy" term on the left-hand side, we expect that there is a polynomial of degree 4 solving the differential equation.

Let's suppose

$$y = \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0$$

solves our non-homogeneous equation,

$$2\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = x^3 - 6x + 4.$$

Plugging in the first and second derivatives of our polynomial above would give us

$$2(12\alpha_4 x^2 + 6\alpha_3 x + 2\alpha_2) - 6(4\alpha_4 x^3 + 3\alpha_3 x^2 + 2\alpha_2 x + \alpha_1) = x^3 - 6x + 4.$$

Combining like-terms on the left-hand side would give us

$$-24\alpha_4 x^3 + (24\alpha_4 - 18\alpha_3)x^2 + (12\alpha_3 - 12\alpha_2)x + 4\alpha_2 - 6\alpha_1 = x^3 - 6x + 4.$$

We are thus lead to the following system of linear equations,

$$\begin{aligned} -24\alpha_4 &= 1 \\ 24\alpha_4 - 18\alpha_3 &= 0 \\ 12\alpha_3 - 12\alpha_2 &= 6 \\ 4\alpha_2 - 6\alpha_1 &= 4 \end{aligned}$$

(Notice there are no restrictions on α_0 here since the α_0 will disappear when differentiating. We can thus take α_0 to be any constant, and for simplicity we will take it to be zero.) The arithmetic for solving this system is a bit tedious, but the solution is

$$\begin{aligned} \alpha_4 &= -{}^1/_{24} \\ \alpha_3 &= -{}^1/_{18} \\ \alpha_2 &= -{}^5/_9 \\ \alpha_1 &= -{}^{28}/_{27} \end{aligned}$$

and so one particular solution to our non-homogeneous equation is

$$y = \frac{-1}{24}x^4 - \frac{1}{18}x^3 - \frac{5}{9}x^2 - \frac{28}{27}x.$$

The general solution to our non-homogeneous differential equation is then

$$y = c_1 + c_2 e^{3x} - \frac{1}{24}x^4 - \frac{1}{18}x^3 - \frac{5}{9}x^2 - \frac{28}{27}x.$$

To satisfy our initial condition $y(0) = 2$ and $y'(0) = 3$, we need to find c_1 and c_2 so that

$$\begin{aligned} c_1 + c_2 &= 2 \\ 3c_2 - \frac{28}{27} &= 3. \end{aligned}$$

Solving this tells us $c_2 = 1 + \frac{28}{81}$ and $c_2 = 1 - \frac{28}{81}$, and so our initial value problem is solved by

$$y = 1 - \frac{28}{81} + \left(1 + \frac{28}{81}\right) e^{3x} - \frac{1}{24}x^4 - \frac{1}{18}x^3 - \frac{5}{9}x^2 - \frac{28}{27}x.$$

The key observation we made to solve the non-homogeneous differential equations above was that the derivative of a polynomial is a polynomial, and so if the right-hand side of our differential equation is a polynomial, it's reasonable to expect there to be a polynomial solution. Writing out a polynomial of the "correct" degree with unknown coefficients and plugging it into the differential equation gave us a system of linear equations whose solution gave the coefficients of our polynomial. This method is called ***the method of undetermined coefficients***, and it applies to more than just polynomials.

In order to use the method of undetermined coefficients, what we need is a family of functions whose derivatives belong to the same family. There are two more "obvious" families with these properties: exponentials and trig functions.

By "exponentials" we mean functions of the form ae^{bx}. What's nice about these functions is that each time you differentiate you simply pick up a factor of b. That is, the first derivative is abe^{bx}, the second derivative is ab^2e^{bx}, the third derivative is ab^3e^{bx}, and so on.

Example 3.7.

Find a soluton to the equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 6e^{7x}.$$

Suppose that a solution y had the form $y = \alpha e^{7x}$, so $y' = 7\alpha e^{7x}$ and $y'' = 49\alpha e^{7x}$. Hence if y solves our differential equation above, we must have

$$\begin{aligned} &49\alpha e^{7x} - 21\alpha e^{7x} + 2\alpha e^{7x} = 6e^{7x} \\ \implies &30\alpha e^{7x} = 6e^{7x} \\ \implies &\alpha = {}^{6}\!/_{30} = {}^{1}\!/_{5}. \end{aligned}$$

Trig functions also have the property that their derivatives are again trig functions. E.g., the derivative of $\sin(x)$ is $\cos(x)$ and the derivative of $\cos(x)$ is $-\sin(x)$. So, if we had a non-homogeneous equation where the right-hand side of the equation was a linear combination of sines and cosines, we should expect the solution is also a linear combination of sines and cosines.

For example, suppose we wanted to find a solution to

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 6\cos(5x).$$

then we might suppose the solution had the form

$$y = \alpha\sin(5x) + \beta\cos(5x).$$

Notice that we want both a sine and a cosine term. If we only had one or the other, then the $\frac{dy}{dx}$ term on our left-hand side would contain the other function, whereas the y and $\frac{d^2y}{dx^2}$ terms would contain the initial function. Thus we'll need to consider that y contains both types of terms to avoid resulting in a system of equations with no solutions.

Let's notice that if y is the linear combination of $\sin(5x)$ and $\cos(5x)$ above, then

$$\begin{aligned} \frac{dy}{dx} &= 5\alpha\cos(5x) - 5\beta\sin(5x) \\ \frac{d^2y}{dx^2} &= -25\alpha\sin(5x) - 25\beta\cos(5x). \end{aligned}$$

Plugging these into the left-hand side of the differential equation would give us

$$\begin{aligned}&\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y \\ =& \left(-25\alpha\sin(5x) - 25\beta\cos(5x)\right) - 7\left(5\alpha\cos(5x) - 5\beta\sin(5x)\right) + 12\left(\alpha\sin(5x) - \beta\cos(5x)\right) \\ =&(-13\alpha + 35\beta)\sin(5x) + (-13\beta - 35\alpha)\cos(5x).\end{aligned}$$

Keeping in mind this is supposed to equal $6\cos(5x)$, we have

$$(-13\alpha + 35\beta)\sin(5x) + (-13\beta - 35\alpha)\cos(5x) = 6\cos(5x),$$

and so we are lead to the following system of linear equations by equating the sine terms and cosine terms of the left- and right-hand sides of the equation above:

$$\begin{aligned}-13\alpha + 35\beta &= 0 \\ -13\beta - 35\alpha &= 6\end{aligned}$$

Solving this system is not particularly hard, but the numbers are ugly:

$$\alpha = \frac{-105}{697} \quad \text{and} \quad \beta = \frac{-39}{697}$$

and so a particular solution to our non-homogeneous equation above is

$$y = \frac{-105}{697}\sin(5x) - \frac{39}{697}\cos(5x).$$

Example 3.8.
Find a solution to the non-homogeneous equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 3\sin(2x).$$

Suppose there is a solution of the form

$$y = \alpha\sin(2x) + \beta\cos(2x)$$

The derivatives are then

$$\begin{aligned}\frac{dy}{dx} &= 2\alpha\cos(2x) - 2\beta\sin(2x) \\ \frac{d^2y}{dx^2} &= -4\alpha\sin(2x) - 4\beta\cos(2x)\end{aligned}$$

Plugging these into the left-hand side of our differential equation above we would have

$$\begin{aligned}
&\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y\\
=&-4\alpha\sin(2x)-4\beta\cos(2x)+2\left(2\alpha\cos(2x)-2\beta\sin(2x)\right)+5\left(\alpha\sin(2x)+\beta\cos(2x)\right)\\
=&(\alpha-4\beta)\sin(2x)+(\beta+4\alpha)\cos(2x)
\end{aligned}$$

Equating this with $3\sin(2x)$ gives us the system of equations

$$\begin{aligned}
\alpha-4\beta&=3\\
4\alpha+\beta&=0
\end{aligned}$$

Solving this system gives us $\alpha = 3/17$ and $\beta = -12/17$, and so a particular solution to our non-homogeneous differential equation above is

$$y=\frac{3}{17}\sin(2x)-\frac{12}{17}\cos(2x).$$

Let's now notice that if we multiply a polynomial and an exponential, their derivative is also a product of a polynomial and an exponential. Similarly, the product of a polynomial and a combination of sines and cosines has a derivative which is a product of polynomials and sines and cosines. We can use this to get yet another "family" of functions whose derivative is again a function in that family.

For example, suppose we wanted to find a particular solution to

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=(x^2+x)e^{3x}.$$

Since the right-hand side is a product of a polynomial and an exponential, we may suppose there is a solution of the form

$$y=(\alpha_2x^2+\alpha_1x+\alpha_0)\beta e^{3x}.$$

Computing the derivatives of this and putting them together according to the left-hand side of the differential equation would give us the following:

$$\left[16\alpha_2x^2+(12\alpha_2+13\alpha_1)\,x+2\alpha_2+8\alpha_1+13\alpha_0\right]\beta e^{3x}.$$

Equating this with $(x^2 + x)e^{3x}$ then gives us a system of equations,

$$\begin{aligned} 16\alpha_2 &= 1 \\ 12\alpha_2 + 13\alpha_1 &= 1 \\ 2\alpha_2 + 8\alpha_1 + 13\alpha_0 &= 0 \\ \beta &= 1 \end{aligned}$$

which is solved by

$$\begin{aligned} \alpha_2 &= \frac{1}{16} \\ \alpha_1 &= \frac{1}{52} \\ \alpha_0 &= \frac{-29}{1352} \\ \beta &= 1 \end{aligned}$$

and so a solution to our non-homogeneous equation is

$$y = \left(\frac{1}{16}x^2 + \frac{1}{52}x - \frac{29}{1352}\right)e^{3x}$$

More generally, if we have a linear non-homogeneous differential equation where the right-hand side is a sum where each term belongs to a family of functions we have described (exponentials, polynomials, sines/cosines, products of exponentials and polynomials, etc.), then the the solutions to that differential equation are sums of solutions to the same differential equation where each right-hand side is one of the terms of the right-hand side of the original equation.

That is, if our differential equation is of the form

$$L(y) = r_1 + r_2 + \cdots + r_n$$

and if $L(y_i) = r_i$ for each i, then the differential equation is solved by

$$y = y_1 + y_2 + \cdots + r_n$$

because L is linear:

$$\begin{aligned} L(y) &= L(y_1 + y_2 + \cdots + y_n) \\ &= L(y_1) + L(y_2) + \cdots + L(y_n) \\ &= r_1 + r_2 + \cdots + r_n. \end{aligned}$$

For example, consider the differential equation

$$\frac{d^2y}{dx^2} + y = x^2e^x + e^{2x}\cos(x) + x^2 + x - 2.$$

To find a particular solution to this equation, we solve each of the following equations:

$$\begin{aligned}
\frac{d^2y_1}{dx^2} + y_1 &= x^2e^x \\
\frac{d^2y_2}{dx^2} + y_2 &= e^{2x}\cos(x) \\
\frac{d^2y_3}{dx^2} + y_3 &= x^2 + x - 2
\end{aligned}$$

Exercise 3.1.
Solve each of the three differential equations above.

After solving these differential equations we find

$$\begin{aligned}
y_1 &= \left(\frac{1}{2}x^2 - x + \frac{1}{2}\right)e^x \\
y_2 &= \frac{1}{8}e^{2x}\cos(x) + \frac{1}{8}e^{2x}\sin(x) \\
y_3 &= \frac{1}{2}x^2 + x - 3
\end{aligned}$$

Putting these together, our differential equation above,

$$\frac{d^2y}{dx^2} + y = x^2e^x + e^{2x}\cos(x) + x^2 + x - 2.$$

is solved by

$$\begin{aligned}
y &= y_1 + y_2 + y_3 \\
&= \left(\frac{1}{2}x^2 - x + \frac{1}{2}\right)e^x + \frac{1}{8}e^{2x}\cos(x) + \frac{1}{8}e^{2x}\sin(x) + \frac{1}{2}x^2 + x - 3
\end{aligned}$$

3.5 Variation of parameters

In using the method of undetermined coefficients, we made an assumption about the form of our solution. We would like to now describe a method which does not require us to restrict ourselves to polynomials, exponents, or trig functions.

The idea is that if the homogeneous equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

has general solution $c_1y_1(x) + c_2y_2(x)$, then the non-homogeneous differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

has a solution of the form

$$y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

and we can try to solve for u_1 and u_2.

For example, suppose we wished to solve

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2+1}.$$

The corresponding homogeneous differential equation is solved by $y_1 = e^x$ and $y_2 = xe^x$. So we suppose there is a solution to the non-homogeneous differential equation of the form

$$y = u_1(x)e^x + u_2(x)xe^x$$

and we want to find u_1 and u_2. Notice the derivative of y is

$$y' = u_1'(x)e^x + u_1(x)e^x + u_2'(x)xe^x + u_2(x)(e^x + xe^x).$$

We can compute y'' and then plug y, y', and y'' into our equation above. Notice, though, that this will give us one equation in two variables which may have infinitely-many solutions. To winnow down to one solution, we will impose another condition that will make y'' easier to compute. Suppose that we assume

$$u_1'(x)e^x + u_2'(x)xe^x = 0.$$

Under this assumption, y' becomes

$$y' = u_1(x)e^x + u_2(x)(e^x + xe^x) = e^x(u_1(x) + u_2(x)(x+1)).$$

Then y'' can be written as

$$\begin{aligned} y'' &= e^x(u_1(x) + u_2(x)(x+1)) + e^x(u_1'(x) + u_2'(x)(x+1) + u_2(x)) \\ &= e^x(u_1(x) + u_2(x)(x+1)) + u_1'(x)e^x + u_2'(x)xe^x + u_2'(x)e^x + e^x u_2(x) \\ &= e^x(u_1(x) + u_2(x)(x+1)) + u_2'(x)e^x + u_2(x)e^x \\ &= e^x(u_1(x) + xu_2(x) + 2u_2(x) + u_2'(x)). \end{aligned}$$

Our differential equation then becomes

$$\begin{aligned} &e^x(u_1(x) + (x+2)u_2(x) + u_2'(x)) - 2e^x(u_1(x) + u_2(x)(x+1)) + u_1e^x + u_2xe^x \\ =&\frac{e^x}{x^2+1}. \end{aligned}$$

We may write this as

$$\begin{aligned} &u_1(x) + (x+2)u_2(x) + u_2'(x) - 2u_1(x) - 2xu_2(x)x + -2u_2(x) + u_1 + u_2x \\ =&\frac{1}{x^2+1}. \end{aligned}$$

This further simplifies to

$$u_2'(x) = \frac{1}{x^2+1}$$

which we can directly solve:

$$u_2(x) = \int \frac{dx}{x^2+1} = \tan^{-1}(x) + C_2.$$

Now we can go back and plug this into the equation that we introduced,

$$u_1'(x)e^x + u_2'(x)xe^x = 0$$

which we may rewrite as

$$u_1'(x) + xu_2'(x) = 0$$

and plug in $u_2'(x)$ to write

$$\begin{aligned} u_1'(x) + \frac{x}{x^2+1} &= 0 \\ \implies u_1(x) &= -\int \frac{x}{x^2+1}\,dx. \end{aligned}$$

Performing the substitution $u = x^2 + 1$, $du = 2xdx$, this becomes

$$\frac{-1}{2}\int \frac{du}{u} = \frac{-1}{2}\ln|u| + C_1$$

and so

$$u_1(x) = \frac{-1}{2}\ln(x^2+1) + C_1$$

and so we claim that the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2+1}$$

is solved by

$$y = \left(\frac{-1}{2}\ln(x^2+1) + C_1\right)e^x + \left(\tan^{-1}(x) + C_2\right)xe^x.$$

Exercise 3.2.
Verify that

$$y = \left(\frac{-1}{2}\ln(x^2+1) + C_1\right)e^x + \left(\tan^{-1}(x) + C_2\right)xe^x.$$

does indeed solve

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2+1}.$$

Generalizing this procedure gives us Lagrange's ***variation of parameters***. The general statement of which is the following:

Theorem 3.13 (Variation of parameters)**.**
If $p(x)$, $q(x)$, and $r(x)$ are continuous functions defined on an interval I and if $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions to the homogeneous differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

(note this means the Wronskian of y_1 and y_2 is non-zero), then every solution to the non-homogeneous equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

may be written as

$$y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where

$$u_1(x) = C_1 - \int \frac{y_2(x)r(x)}{W(x)}\,dx$$
$$u_2(x) = C_2 + \int \frac{y_1(x)r(x)}{W(x)}\,dx$$

Example 3.9.
Find the general solution to

$$\frac{d^2y}{dx^2} + y = \tan(x)$$

on the interval $I = (0, {}^{\pi}/_{2})$.

First we need to find a pair of fundamental solutions to

$$\frac{d^2y}{dx^2} + y = 0$$

which has characteristic equation $r^2 + 1 = 0$, which is solved by $r = \pm i$, and so $e^{\pm ix}$ is a complex-valued solution to the homogeneous differential equation, and the real and imaginary parts,

$$y_1(x) = \cos(x) \text{ and } y_2(x) = \sin(x)$$

form a fundamental set of real-valued solutions.

By Theorem 3.13, the general solution to the differential equation is then given by

$$y = \left(c_1 - \int \frac{\sin(x)\tan(x)}{W(x)}\,dx\right)\cos(x) + \left(c_2 + \int \frac{\cos(x)\tan(x)}{W(x)}\,dx\right)\sin(x).$$

Notice the Wronskian here is particularly nice:

$$\begin{aligned} W(x) &= \det\begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix} \\ &= \cos^2(x) + \sin^2(x) \\ &= 1 \end{aligned}$$

Now we simply need to compute the two integrals above:

$$\begin{aligned} \int \sin(x)\tan(x)\,dx &= \int \frac{\sin^2(x)}{\cos(x)}\,dx \\ &= \int \sec(x)\sin^2(x)\,dx \\ &= \int \sec(x)\cdot(1-\cos^2(x))\,dx \\ &= \int (\sec(x) - \cos(x))\,\,dx \\ &= \int \sec(x)\,dx - \int \cos(x)\,dx \end{aligned}$$

Of course, $\int \cos(x)\,dx = \sin(x) + C$. Integrating $\sec(x)$ requires one little trick:

$$\begin{aligned} \int \sec(x)\,dx &= \int \sec(x)\cdot\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx \\ &= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx. \end{aligned}$$

Now we let $u = \sec(x)+\tan(x)$ and $du = (\sec(x)\tan(x) + \sec^2(x))\,\,dx$, and so the integral becomes simply

$$\int \frac{1}{u}\,du = \ln|u| + C$$

which, after replacing u, gives us

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C.$$

On $(0, \pi/2)$ the function $\sec(x) + \tan(x)$ is positive, and so on this interval we have

$$\int \sin(x)\tan(x)\,dx = \ln(\sec(x) + \tan(x)) - \sin(x) + C.$$

The other integral we have to compute is even easier:

$$\int \cos(x)\tan(x)\,dx = \int \sin(x)\,dx = -\cos(x) + C$$

and so the general solution to

$$\frac{d^2y}{dx^2} + y = \tan(x)$$

on the interval $(0, \pi/2)$ is

$$\begin{aligned} y &= (c_1 - \ln(\sec(x) + \tan(x)) + \sin(x))\cos(x) + (c_2 - \cos(x))\sin(x) \\ &= (c_1 - \ln(\sec(x) + \tan(x)))\cos(x) + c_2\sin(x) \end{aligned}$$

4 Higher order differential equations

Do not worry about your difficulties in mathematics. I can assure you mine are still greater.

ALBERT EINSTEIN

In the last chapter we saw how to solve some second order differential equations, particularly in the case where all of the coefficients on the derivatives in the were constants. In this chapter we extend those ideas to higher order differential equations. This mostly follows the same format as the second order case, but there are some difficulties that are introduced. In particular, the characteristic polynomial can be more difficult to factor, and there are more possibilities for the roots. For instance, we could have a polynomial which has repeated complex conjugate roots. In this chapter we discuss the various possibilities that can arise in solving these higher order differential equations.

4.1 General remarks about linear homogeneous equations

Given a linear homogeneous differential equation of order n,

$$p_n(x)\frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_2(x)\frac{d^2 y}{dx^2} + p_1(x)\frac{dy}{dx} + p_0(x)y = 0,$$

there are functions $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ such that all solutions to the differential equation will have the form

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x).$$

In terms of linear algebra, this means that the set of all solutions forms an n-dimensional real vector space with basis $y_1(x)$, $y_2(x)$, ..., $y_n(x)$.

Given a collection of n functions $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ solving the equation, we want to know if these functions will form a fundamental set of solutions. *This is not a guarantee!* For example, if one of the functions turns out to be a sum of multiples of the others, we won't have a fundamental set of solutions. To determine if a given collection of n functions forms a fundamental set of solutions, we can determine if the Wronskian of those solutions is zero or not. The Wronskian here is the determinant of an $n \times n$ matrix whose rows are given by the original functions and the first $n-1$ derivatives:

$$W(x) = \det \begin{pmatrix} y_1(x) & y_2(x) & y_3(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & y_3'(x) & \cdots & y_n'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) & \cdots & y_n''(x) \\ & & & \vdots & \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & y_3^{(n-2)}(x) & \cdots & y_n^{(n-2)}(x) \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & y_3^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}$$

While there is a general formula for the determinant of a $n \times n$ matrix, in this course we will not ever need to consider matrices larger than 3×3, and here the formula isn't too terrible to remember: Given a 3×3 matrix A,

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

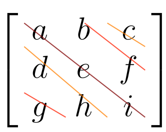
the determinant $\det(A)$ is computed by

$$\det(A) = aei + bfg + cdh - ceg - bdi - afh.$$

This looks like a complicated formula, but luckily there's a nice way to remember it. First, let's rearrange our formula just a tiny little bit.

$$\begin{aligned} &aei - afh - bei + bfg + ceh - ceg \\ =&aei + bfg + cdh - ceg - bdi - afh \end{aligned}$$

Now the way we remember this formula is that we look at lines through the matrix which go down and to the right (wrapping around if you hit the "edge" of the matrix):



We multiply the entries on each line, and then add them all up:

$$aei + bfg + bdh$$

Now to get the other entries, we draw lines through the matrix which go down and left:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Now we multiply the entries on each line, make them negative, and add them all up:

$$-ceg - bdi - afh$$

Now adding up these two quantities (the positives and the negatives), we have our formula for the determinant.

$$\det(M) = aei + bfg + bdh - ceg - bdi - afh$$

Supposing we have found a fundamental set of solutions to an n-th order differential equation, we may wish to find the values of c_1, ..., c_n which solve a given initial value problem. Ultimately this will come down to solving a system of equations, and since there are n unknowns, we should expect this sytem to have n equations. That is, just as an IVP for a first-order differential equation needed one initial condition, and we need two initial conditions to solve a second-order IVP, we will need three conditions for a third-order IVP, four conditions for a fourth-order IVP, and so on. Typically, but not always, these will be given by specifying a point which must be on the graph of the solution, as well as the first derivative, second derivative, third derivative, and so on, of the function at that point. That is, our initial conditions will often (but not always) be given as

$$y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2, ..., y^{(n-1)}(x_0) = y_{n-1}.$$

4.2 Solving homogeneous equations with constant coefficients

Suppose we have a differential equation of the form

$$c_n \frac{d^n y}{dx^n} + c_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + c_2 \frac{d^2 y}{dx^2} + c_1 \frac{dy}{dx} + c_0 y = 0.$$

As in the case of two variables, we can consider the characteristic polynomial,

$$c_n r^n + c_{n-1} r^{n-1} + \cdots + c_2 r^2 + c_1 r + c_0,$$

and the roots of this polynomial will tell us the solutions of the original differential equation. For example, the differential equation

$$\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 26\frac{dy}{dx} - 24y = 0$$

has characteristic polynomial

$$r^3 - 9r^2 + 26r - 24.$$

To determine the roots of this polynomial we would like to factor it. Here factoring can be more involved than with quadratic polynomials. In the case of a quadratic, we can always resort to using the quadratic formula, but there are not equivalent formulas for all higher-order polynomials. (There are actually formulas for polynomials of degree three and four, but they are considerably more complicated than the quadratic formula. There is not a general formula that works for all polynomials of degree five and higher.[1])

In order to help us factor the polynomial above, it is helpful to recall the following basic fact about polynomials.

Proposition 4.1.
If a is a root of a polynomial

$$c_n r^n + c_{n-1} r^{n-1} + \cdots + c_2 r^2 + c_1 r + c_0,$$

then $r - a$ is a factor of the polynomial.

This proposition can give us a tool for factoring the polynomial since if we find one root, we know one factor, and can use polynomial long

[1] The existence of formulas for degree five and higher was an open question in mathematics for a long time, but was eventually settled by Evariste Galois during the 19th century. The story goes that Galois challenged another man to a duel and the evening before the duel wrote down all of his thoughts about why such a formula could not exist, and mailed it to one of his friends. Although Galois was killed in the duel, at the age of 20, his friend forwarded his letter to leading mathematicians of the day who eventually deciphered his work. Galois' work is now often credited with beginning the study of abstract algebra (more specifically, group theory and Galois theory).

division to determine the factor. How can we go about finding a root, though? Here another proposition is helpful:

Proposition 4.2.
Given a monic polynomial

$$r^n + c_{n-1}r^{n-1} + \cdots + c_2r^2 + c_1r + c_0$$

(*i.e., a polynomial whose leading coefficient is* 1) *whose coefficients are all integers, any integer roots must divide the constant* c_0.

So, in the case of our polynomial above

$$r^3 - 9r^2 + 26r - 24,$$

a starting point is to find the integers that divide 24. Let's notice that $24 = 2 \cdot 12$, so 2 and 12 are both divisors of 24 (among others). It's easy to check if each of these makes the polynomial zero or not. Plugging 12 in gives

$$12^3 - 9 \cdot 12^2 + 26 \cdot 12 - 24 = 720 \neq 0$$

which tells us 12 is not a root of the polynomial, so $r - 12$ is not a factor. Plugging 2 in gives

$$2^3 - 9 \cdot 2^2 + 26 \cdot 2 - 24 = 0$$

Thus $r = 2$ is a root to the polynomial, and so $r - 2$ is a factor. That is, there is some polynomial $g(r)$ so that

$$r^3 - 9r^2 + 26r - 24 = g(r) \cdot (r - 2).$$

To determine $g(r)$, let's divide $r - 2$ over to obtain

$$g(r) = \frac{r^3 - 9r^2 + 26r - 24}{r - 2}.$$

We can compute this quotient by using polynomial long divison:

$$
\begin{array}{r}
r^2 \quad - 7r + 12 \\
r - 2 \overline{\smash{)}\, r^3 - 9r^2 + 26r - 24} \\
\underline{- r^3 + 2r^2 \qquad\qquad} \\
- 7r^2 + 26r \qquad \\
\underline{7r^2 - 14r \qquad} \\
12r - 24 \\
\underline{- 12r + 24} \\
0
\end{array}
$$

This tells us

$$r^3 - 9r^2 + 26r - 24 = (r - 2)(r^2 - 7r + 12).$$

Our goal now becomes factoring $r^2 - 7r + 12$, but this is much easier. Either by inspection or using the quadratic formula we can easily determine $r^2 - 7r + 12 = (r - 3)(r - 4)$. Thus our original polynomial factors as

$$r^3 - 9r^2 + 26r - 24 = (r - 2)(r - 3)(r - 4).$$

So, the characteristic polynomial of our differential equation above has roots $r = 2$, $r = 3$, and $r = 4$. Of course, these roots will tell us the solutions to our differential equation, and there are a few different cases to consider:

1. Distinct real roots
2. Repeated real roots
3. Complex conjugate roots
4. Repeated complex conjugate roots

Unlike the second order situation, with higher order equations we could have characteristic polynomials where all four cases occur simultaneously! The reason for this is the following important theorem:

Theorem 4.3 (The fundamental theorem of algebra).
Every polynomial of degree n with real (or complex) coefficients factors completely as a product of n linear factors, some of which may be repeated. If the polynomial has only real coefficients, any complex roots must occur

as complex conjugate pairs.

Grouping together identical factors, we can thus write every polynomial with real coefficients as

$$(r-r_1)^{m_1}(r-r_2)^{m_2}(r-r_3)^{m_3}\cdot\ldots\cdot(r-r_n)^{m_n}$$

and we call m_j the ***multiplicity*** of the j-th root.

Distinct real roots

If the characteristic polynomial for our differential equation has a real root a with multiplicity 1 (i.e., the root is not repeated), then just as in the second-order situation the function $y = e^{ax}$ will be a solution to the differential equation. For example, in our differential equation described earlier

$$\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 26\frac{dy}{dx} - 24y = 0,$$

we had characteristic polynomial

$$r^3 - 9r^2 + 26r - 24 = (r-2)(r-3)(r-4)$$

which has roots 2, 3, and 4. Each of these roots gives us a solution to the differential equation,

$$y_1(x) = e^{2x}, \quad y_2(x) = e^{3x}, \quad y_3(x) = e^{4x}.$$

While tedious, it's easy to compute the Wronskian of these solutions to see that they form a fundamental set of solutions:

$$\begin{aligned} W(x) &= \det\begin{pmatrix} e^{2x} & e^{3x} & e^{4x} \\ 2e^{2x} & 3e^{3x} & 4e^{4x} \\ 4e^{2x} & 9e^{3x} & 16e^{4x} \end{pmatrix} \\ &= 48e^{9x} + 16e^{9x} + 18e^{9x} - 12e^{9x} - 32e^{9x} - 36e^{9x} \\ &= 2e^{4x} \neq 0. \end{aligned}$$

Thus every solution to the differential equation

$$\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 26\frac{dy}{dx} - 24y = 0,$$

has the form

$$y = c_1e^{2x} + c_2e^{3x} + c_3e^{4x}.$$

Exercise 4.1.
Verify that functions of the form

$$y = c_1 e^{2x} + c_2 e^{3x} + c_3 e^{4x}$$

solve the differential equation

$$\frac{d^3 y}{dx^3} - 9\frac{d^2 y}{dx^2} + 26\frac{dy}{dx} - 24y = 0.$$

An initial value problem for this equation would require three initial conditions. For example, suppose we wished to find the solution to the equation above which satisfied $y(0) = 2$, $y'(0) = 0$, and $y''(0) = 1$. We would need to compute the derivatives of our general solution above:

$$\begin{aligned} y &= c_1 e^{2x} + c_2 e^{3x} + c_3 e^{4x} \\ y' &= 2c_1 e^{2x} + 3c_2 e^{3x} + 4c_3 e^{4x} \\ y'' &= 4c_1 e^{2x} + 9c_2 e^{3x} + 16c_3 e^{4x}. \end{aligned}$$

Now if we evaluate these at zero we have

$$\begin{aligned} y(0) &= c_1 + c_2 + c_3 \\ y'(0) &= 2c_1 + 3c_2 + 4c_3 \\ y''(0) &= 4c_1 + 9c_2 + 16c_3 \end{aligned}$$

But as we are told $y(0) = 2$, $y'(0) = 0$, and $y''(0) = 1$, this means we have a system of equations

$$\begin{aligned} c_1 + c_2 + c_3 &= 2 \\ 2c_1 + 3c_2 + 4c_3 &= 0 \\ 4c_1 + 9c_2 + 16c_3 &= 1 \end{aligned}$$

Since the Wronskian of our fundamental set of solutions was non-zero, this system must have a unique solution. There are a few different ways to solve this system, but perhaps the most direct way would be to use Cramer's rule which is described in Appendix **??**.

Cramer's rule tells us the solution to this system of equations can be computed in terms of determinants of certain matrices. In particular, the ***coefficient matrix*** of the system above is the 3×3 matrix that is obtained by looking at the coefficients of the unknowns c_1, c_2, and c_3, in our system above. Let's refer to this matrix as A:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{pmatrix}.$$

Now we construct three new matrices, which we will refer to as A_1, A_2, and A_3, which are given by taking the first, second, and third columns, respectively, of the coefficient matrix A, and replacing them by the column that contains the three numbers on the right-hand sides of our equations, which are 2, 0, and 1. That is, our matrices are:

$$A_1 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 4 \\ 1 & 9 & 16 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 4 \\ 4 & 1 & 16 \end{pmatrix} \quad A_1 = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 4 & 9 & 1 \end{pmatrix}$$

Again, A_1 was obtained by replacing the first column of A with the values 2, 0, 1, and these numbers are being used because they appear as the values on the right-hand side of our system of equations.

Cramer's rule says that the solutions c_1, c_2, and c_3 are equal to the ratios of determinants of our matrices above. In particular,

$$c_1 = \frac{\det(A_1)}{\det(A)} \quad c_2 = \frac{\det(A_2)}{\det(A)} \quad c_3 = \frac{\det(A_3)}{\det(A)}$$

Computing these determinants we have

$$\begin{aligned} c_1 &= \frac{\det\begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 4 \\ 1 & 9 & 16 \end{pmatrix}}{\det\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{pmatrix}} \\ &= \frac{96 + 4 + 0 - 3 - 0 - 72}{48 + 16 + 18 - 12 - 32 - 36} \\ &= \frac{25}{2} \end{aligned}$$

$$\begin{aligned} c_2 &= \frac{\det \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 4 \\ 4 & 1 & 16 \end{pmatrix}}{\det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{pmatrix}} \\ &= \frac{0 + 32 + 2 - 0 - 64 - 4}{2} \\ &= -17 \end{aligned}$$

$$\begin{aligned} c_3 &= \frac{\det \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 4 & 9 & 1 \end{pmatrix}}{\det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{pmatrix}} \\ &= \frac{3 + 0 + 36 - 24 - 2 - 0}{2} \\ &= \frac{13}{2} \end{aligned}$$

It is easy to verify that $c_1 = {}^{25}\!/_2$, $c_2 = -17$, and $c_3 = {}^{13}\!/_2$ do in fact solve our system above, and so the solution to our initial value problem with differential equation

$$\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 26\frac{dy}{dx} - 24y = 0,$$

and initial conditions

$$y(0) = 2,\ y'(0) = 0,\ \text{and}\ \ y''(0) = 1$$

is

$$y = \frac{25}{2}e^{2x} - 17e^{3x} + \frac{13}{2}e^{4x}.$$

To summarize, if the characteristic polynomial of an n-th order homogeneous differential equation with constant coefficients has distinct real roots r_1, r_2, ..., r_n, then the general solution to the differential equation is

$$y = c_1e^{r_1x} + c_2e^{r_2x} + \cdots + c_ne^{r_nx}.$$

Repeated real roots

Let's now consider a case where the characteristic polynomial has a repeated root. Suppose we want to solve the following fourth-order equation:

$$\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0.$$

The characteristic polynomial is

$$r^4 + 4r^3 + 6r^2 + 4r + 1.$$

Let's notice that since this is a monic polynomial (the coefficient on the leading term is 1), any integer roots must divide the constant term, 1. The only integers that divide 1 are 1 and -1. We can easily plug each of these into the polynomial to see if they are 1 or not:

$$\begin{aligned} &1^4 + 4 \cdot 1^3 + 6 \cdot 1^2 + 4 \cdot 1 + 1 = 16 \\ &(-1)^4 + 4 \cdot (-1)^3 + 6 \cdot (-1)^2 + 4 \cdot (-1) + 1 = 0. \end{aligned}$$

Thus $r = -1$ is a root, so $r + 1$ is a factor of the polynomial. To determine another factor we can use polynomial long division:

$$\begin{array}{r} r^3 + 3r^2 + 3r + 1 \\ r+1 \overline{)\, r^4 + 4r^3 + 6r^2 + 4r + 1} \\ \underline{-r^4 - r^3} \qquad\qquad\qquad \\ 3r^3 + 6r^2 \qquad\quad \\ \underline{-3r^3 - 3r^2} \qquad\quad \\ 3r^2 + 4r \qquad \\ \underline{-3r^2 - 3r} \qquad \\ r + 1 \\ \underline{-r - 1} \\ 0 \end{array}$$

Now our goal is to factor $r^3 + 3r^2 + 3r + 1$. Here we again notice there is a constant term of 1 in the monic polynomial, so if this has any integer roots, they must be ± 1. Checking each one we will see that $r = -1$ is a root, but $r = 1$ is not and so $r + 1$ is also a root of this polynomial.

Applying polynomial long division again we have

$$\begin{array}{rl} & r^2 + 2r + 1 \\ \cline{2-2} r+1 \big) & r^3 + 3r^2 + 3r + 1 \\ & \underline{-r^3 - r^2} \\ & 2r^2 + 3r \\ & \underline{-2r^2 - 2r} \\ & r + 1 \\ & \underline{-r - 1} \\ & 0 \end{array}$$

We may at this point recognize that $r^2+2r+1 = (r+1)^2$. Combining this with other factors of $r+1$, we see that the polynomial factors as $(r+1)^4$. That is, $r = -1$ is a root with multiplicity four.

In the case of a second order equation, we saw that when we had a repeated root we added a factor of x to e^{rx}, obtaining xe^{rx}, as a second solution. Continuing this pattern, for a repeated root we continue to add progressively higher powers of x: if we had a root r of multiplicity 3, we would have solutions e^{rx}, xe^{rx}, and x^2e^{rx}; for a root r of multiplicity 4, we have solutions e^{rx}, xe^{rx}, x^2e^{rx}, and x^3e^{rx}; and so on.

In the case of our particular differential equation above, we have that the general solution will be

$$y = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x} + c_4x^3e^{-x}.$$

Of course, it could happen that our characteristic polynomial has several roots of different multiplicities. For example, consider

$$(r+1)^3(r-5)^2 = r^5 - 7r^4 - 2r^3 + 46r^2 + 65r + 25.$$

This is the characteristic polynomial of the differential equation

$$y^{(5)} - 7y^{(4)} - 2y^{(3)} + 46y^{(2)} + 65y^{(1)} + 25y = 0.$$

The root $r = -1$ has multiplicity 3 and so gives us the solutions

$$e^{-x},\ xe^{-x},\ x^2e^{-x},$$

while the root $r = 5$ has multiplicity 2 and so gives us the solutions

$$e^{5x},\ xe^{5x}.$$

Putting this together, the differential equation has general solution

$$y = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x} + c_4e^{5x} + c_5xe^{5x}.$$

Distinct complex conjugate roots

Recall that for any polynomial with real coefficients, if complex roots exist they must come in complex conjugate pairs, $a \pm ib$. As we saw with second order equations, if $a \pm ib$ were roots of the characteristic polynomial, then $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$ were two linearly independent solutions. Something similar happens for higher-order equations, except we may have multiple complex conjugate roots to the characteristic polynomial. Each distinct complex conjugate pair $a \pm ib$ again gives us linearly independent real-valued solutions $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$. For example, the differential equation

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

has characteristic polynomial

$$r^4 - 2r^3 + 3r^2 - 2r + 2.$$

Even though all the coefficients here are integers, it is easy to check there are no integer roots: the only divisors of the constant term 2 are ± 1 and ± 2, and plugging each of those four numbers into the polynomial will reveal none of these are roots. Factoring this polynomial is not at all obvious, but let's observe we could rewrite the polynomial as follows by writing $3r^2 = 2r^2 + r^2$:

$$r^4 - 2r^3 + 2r^2 + r^2 - 2r + 2.$$

Notice the first three terms look very similar to the second three terms, except they each have an extra factor of r^2. Factoring r^2 out of the first three terms would then give us

$$\begin{aligned} &r^2(r^2 - 2r + 2) + r^2 - 2r + 2 \\ =&r^2(r^2 - 2r + 2) + 1 \cdot (r^2 - 2r + 2) \end{aligned}$$

We can now factor by grouping to obtain

$$(r^2 + 1)(r^2 - 2r + 2).$$

Determining the roots now becomes a question of finding the roots of each of these quadratics. The first one, $r^2 + 1$, of course has roots $\pm i$ (this is easily seen by solving $r^2 + 1 = 0$ for r). For the second one we can apply the quadratic formula to obtain

$$\begin{aligned} r &= \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} \\ &= \frac{2 \pm \sqrt{-4}}{2} \\ &= \frac{2 \pm 2i}{2} \\ &= 1 \pm i \end{aligned}$$

Our complex conjugate roots are thus $\pm i$ and $1 \pm i$.

Each of these complex conjugate pairs gives us a pair of linearly independent real-valued solutions to the differential equation, just as in the second-order case. The roots $\pm i$ give us the functions $\sin(x)$ and $\cos(x)$; and the roots $1 \pm i$ give us the functions $e^x \cos(x)$ and $e^x \sin(x)$. Thus the general solution to our differential equation is

$$y = c_1 \sin(x) + c_2 \cos(x) + c_3 e^x \sin(x) + c_4 e^x \cos(x).$$

Repeated complex conjugate roots

With second order equations it was impossible to have repeated complex roots, as the fundamental theorem of calculus tells us a quadratic equation can have only two roots. For higher order equations, however, complex roots can repeat. Just as in the case of repeated real roots, however, we simply attach factors of powers of x, as determined by the multiplicity of the root.

As an example, consider the differential equation

$$y^{(6)} - 12y^{(5)} + 87y^{(4)} - 376y^{(3)} + 1131y^{(2)} - 2028y' + 2197y = 0$$

with characteristic polynomial

$$r^6 - 12r^5 + 87r^4 - 376r^3 + 1131r^2 - 2028r + 2197.$$

Factoring this polynomial is not obvious, but it does factor

$$(r^2 - 4r + 13)^3$$

Using the quadratic formula, we see the roots of $r^2 - 4r + 13$ are $2 \pm 3i$. Because $r^2 - 4r + 13$ is cubed, however, each of these roots has multiplicity three. That is, factoring down to a product of linear terms (which the fundamental theorem of algebra says we can always do if we allow complex roots), the polynomial becomes

$$(r-(2+3i))^3(r-(2-3i))^3.$$

This simply means our fundamental set of solutions to the differential equation is given by

$$\begin{aligned} y_1 &= e^{2x}\cos(3x) & y_2 &= e^{2x}\sin(3x) \\ y_3 &= xe^{2x}\cos(3x) & y_4 &= xe^{2x}\sin(3x) \\ y_5 &= x^2e^{2x}\cos(3x) & y_6 &= x^2e^{2x}\sin(3x) \end{aligned}$$

and the general solution to the differential equation is

$$y = c_1e^{2x}\cos(3x)+c_2e^{2x}\sin(3x)+c_3xe^{2x}\cos(3x)+c_4xe^{2x}\sin(3x)+c_5x^2e^{2x}\sin(3x)+c_6x^2e^{2x}\cos(3x).$$

Combining the previous cases

Of course, all of the various possibilities described above can occur in a given differential equation. As an extreme example, consider the differential equation

$$y^{(9)}-y^{(8)}+4y^{(7)}-12y^{(6)}-21y^{(5)}-51y^{(4)}-104y^{(3)}-88y^{(2)}-80y'-48y=0.$$

The characteristic polynomial,

$$r^9-r^8+4r^7-12r^6-21r^5-51r^4-104r^3-88r^2-80r-48$$

factors (although it's not at all obvious at first glance) as

$$(r-3)(r+1)^2(r-i)(r+i)(r-2i)^2(r+2i)^2$$

and so we have a real root of 3 with multiplicity 1, a real root of -1 with multiplicity 2, a pair of complex conjugate roots $\pm i$ with multiplicity 1, and another pair of complex conjugate roots $\pm 2i$ with multiplicity 2. Combining the solutions of the differential equation for these various roots, as described in the sections above, the general solution to the differential equation is

$$y = c_1e^{3x}+c_2e^{-x}+c_3xe^{-x}+c_4\cos(x)+c_5\sin(x)+c_6\cos(2x)+c_7\cos(2x)+c_8x\cos(2x)+c_9x\sin(2x).$$

5 The Laplace Transform

Nature laughs at the difficulties of integration.

PIERRE-SIMON LAPLACE

5.1 What is the Laplace transform?

Recall that when differentiating certain complicated functions such as

$$f(x) = \frac{x^{\cos(x)}(x^3 + 3x - 1)}{\sqrt{x^4 + 1}}$$

it can be advantageous to first take the natural logarithm of the function before differentiating,

$$\begin{aligned}\ln(f(x)) &= \ln\left(\frac{x^{\cos(x)}(x^3 + 3x - 1)}{\sqrt{x^4 + 1}}\right) \\ &= \cos(x)\ln(x) + \ln(x^3 + 3x - 1) - \frac{1}{2}\ln(x^4 + 1)\end{aligned}$$

The reason for this is that certain difficult to differentiate functions (exponentiation of functions, products, and quotients) are transformed into functions which are much simpler to differentiate (products, sums, and differences). Thus instead of differentiating the original function directly, we "transform" the function to a new function which we can differentiate,

$$\frac{1}{f(x)}f'(x) = -\sin(x)\ln(x) + \frac{\cos(x)}{x} + \frac{3x^2 + 3}{x^3 + 3x - 1} - \frac{4x^3}{2(x^4 + 1)}.$$

After doing this we must "invert" our transformation by solving for what we originally cared about:

$$\begin{aligned}f'(x) &= f(x)\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x} + \frac{3x^2 + 3}{x^3 + 3x - 1} - \frac{4x^3}{2(x^4 + 1)}\right) \\ &= \frac{x^{\cos(x)}(x^3 + 3x - 1)}{\sqrt{x^4 + 1}}\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x} + \frac{3x^2 + 3}{x^3 + 3x - 1} - \frac{4x^3}{2(x^4 + 1)}\right)\end{aligned}$$

That is, this process of transforming our problem, solving the transformed problem, and then undoing the transformation to obtain the solution to the original problem makes the problem much easier. (Without taking logarithms and directly differentiating the function above we have a complicated mess of product rules and quotients rules.)

In this chapter we will discuss an operation which is similar in spirit called *the Laplace transform*. As we will see, the Laplace transform will convert functions into other functions in such a way that certain differential equations become algebraic equations which are hopfully simpler to solve. Once we solve the algebraic equation, we will then "undo" the transformation process and have a solution to our initial differential equation.

Given a continuous function $f(t)$ defined on $[0, \infty)$, we define its Laplace transform as the function

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Notice that the domains of these functions are different: the original function f has as its domain $[0, \infty)$ and we referred to the inputs to this function as t, whereas F has some other domain (to be discussed momentarily), and we refer to its input as s.

So, for example, $F(2)$ would be

$$\int_0^\infty e^{-2t} f(t)\, dt$$

whatever that quantity happens to be, and $F(-7)$ would be

$$\int_0^\infty e^{7t} f(t)\, dt$$

whatever that integral works out to be. Let's notice, though, that since these are improper integrals, they may not converge for all values of s. That is, our integral above is really a limit,

$$\int_0^\infty e^{-st} f(t)\, dt = \lim_{b \to \infty} \int_0^b e^{-st} f(t)\, dt$$

and this limit may fail to exist. However, supposing the limit exists for some value s_0, it is an easy calculus exercise (using the comparison test) to verify that the limit will exist for all $s > s_0$.

We will often refer to the function $F(s)$ above as $\mathscr{L}\{f(t)\}$ to help distinguish it from the antiderivative of f.

5.2 Examples of computing Laplace transforms

Let's go ahead and compute a few examples of Laplace transforms.

Example 5.1.
Compute the Laplace transform of a constant function $f(t) = k$.

$$\begin{aligned}
\mathscr{L}\{k\} &= \int_0^\infty e^{-st} k \, dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st} k \, dt \\
&= \lim_{b\to\infty} k \int_0^b e^{-st} \, dt \\
&= \lim_{b\to\infty} k \frac{e^{-st}}{-s}\Big|_0^b \\
&= \frac{-k}{s} \lim_{b\to\infty} \left(e^{-sb} - e^0\right)
\end{aligned}$$

Now let's notice that if $s > 0$, then $-s < 0$ and $\lim_{b\to\infty} e^{-sb} = 0$. If, however, $s < 0$, then $-s > 0$ and $\lim_{b\to\infty} e^{-sb}$ diverges to infinity. If $s = 0$, then our integral $\int_0^\infty e^{-st} k \, dt$ clearly diverges. That is, the Laplace transform $\mathscr{L}\{k\}$ will only be defined if $s > 0$. Continuing with the computation though, we may simplify this as follows, supposing $s > 0$:

$$\begin{aligned}
\mathscr{L}\{k\} &= \frac{-k}{s} \lim_{b\to\infty} \left(e^{-sb} - e^0\right) & &= \frac{-k}{s}(0 - 1) \\
&= \frac{k}{s}
\end{aligned}$$

Example 5.2.
Compute the Laplace transform of $f(t) = e^{at}$.

$$\begin{aligned}\mathscr{L}\{e^{at}\} &= \int_0^\infty e^{-st}e^{at}\,dt \\ &= \lim_{b\to\infty}\int_0^b e^{(a-s)t}\,dt \\ &= \lim_{b\to\infty}\left.\frac{e^{(a-s)t}}{a-s}\right|_0^b \\ &= \lim_{b\to\infty}\left(\frac{e^{(a-s)b}}{a-s}-\frac{1}{a-s}\right)\end{aligned}$$

The first term of this limit will converge to zero if $a-s<0$, meaning $s>a$, and in this case we have

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \text{ provided } s>a$$

The idea of a Laplace transform is not limited to simply real-valued functions, nor is the function that results from taking the Laplace transform limited to taking only real values.

For example, if k is a complex constant and s is a complex value, we may still compute

$$\begin{aligned}\mathscr{L}\{k\} &= \int_0^\infty e^{-st}k\,dt \\ &= \lim_{b\to\infty}\int_0^b e^{-st}k\,dt \\ &= \lim_{b\to\infty} k\int_0^b e^{-st}\,dt \\ &= \lim_{b\to\infty} k\left.\frac{e^{-st}}{-s}\right|_0^b \\ &= \frac{-k}{s}\lim_{b\to\infty}\left(e^{-sb}-e^0\right)\end{aligned}$$

Now to deal with the complex-valued s we use Euler's formula to write

$$e^{-sb} = e^{(\operatorname{Re}(s)+i\operatorname{Im}(s))(-b)} = e^{-b\operatorname{Re}(s)}e^{-ib\operatorname{Im}(s)} = e^{-b\operatorname{Re}(s)}(cos(b\operatorname{Im}(s))-i\sin(b\operatorname{Im}(s)))$$

and the last line of the derivation above becomes

$$\frac{-k}{s}\lim_{b\to\infty}\left(e^{-sb}-e^{0}\right)=\frac{-k}{s}\left(\lim_{b\to\infty}e^{-b\,\mathrm{Re}(s)}(\cos(b\,\mathrm{Im}(s)-i\sin(b\,\mathrm{Im}(s))))-1\right)$$

The cosine and sine oscillate between 1 and -1, so this expression converges only when $\mathrm{Re}(s)>0$ so that $e^{-b\,\mathrm{Re}(s)}$ goes to zero. Thus, in this more general setting of a complex-valued Laplace transform where s is allowed to be a complex number we have

$$\mathscr{L}\{k\}=\frac{k}{s}\quad\text{provided }\mathrm{Re}(s)>0.$$

The same kind of modification to our computation of $\mathscr{L}\{e^{at}\}$ will show that if we consider the complex Laplace transform and allow for complex valued a, then

$$\mathscr{L}\{e^{at}\}=\frac{1}{s-a}\quad\text{provided}\quad\mathrm{Re}(s)>\mathrm{Re}(a)$$

The last few examples have been relatively straight-forward computations, but as the next example shows, computing Laplace transforms can be involved.

Example 5.3.
Compute the Laplace transform of $f(t)=\sin(\omega t)$.

$$\begin{aligned}\mathscr{L}\{\sin(\omega t)\}&=\int_0^\infty e^{-st}\sin(\omega t)\,dt\\&=\lim_{b\to\infty}\int_0^b e^{-st}\sin(\omega t)\,dt.\end{aligned}$$

We will first compute the indefinite integral of $e^{-st}\sin(\omega t)$, then evaluate at b and 0, and finally take the limit as b goes to infinity.

To get started, we use integration by parts with

$$\begin{aligned}u&=\sin(\omega t) & dv&=e^{-st}\,dt\\du&=\omega\cos(\omega t) & v&=\frac{-e^{-st}}{s}\end{aligned}$$

We then have

$$\begin{aligned}\int e^{-st}\sin(\omega t)\,dt &= \frac{-e^{-st}\sin(\omega t)}{s} - \int \frac{-\omega e^{-st}\cos(\omega)}{s}\,dt\\ &= \frac{-e^{-st}\sin(\omega t)}{s} + \frac{\omega}{s}\int e^{-st}\cos(\omega t)\,dt\end{aligned}$$

We now compute $\int e^{-st}\cos(\omega t)$ by applying integration by parts again,

$$\int e^{-st}\cos(\omega t)\,dt = \frac{-e^{-st}\cos(\omega t)}{s} - \frac{\omega}{s}\int e^{-st}\sin(\omega t)\,dt$$

Now plugging this back into our earlier expression for $\int e^{-st}\sin(\omega t)\,dt$ we have

$$\begin{aligned}\int e^{-st}\sin(\omega t)\,dt &= \frac{-e^{-st}\sin(\omega t)}{s} + \frac{\omega}{s}\left(\frac{-e^{-st}\cos(\omega t)}{s} - \frac{\omega}{s}\int e^{-st}\sin(\omega t)\,dt\right)\\ &= \frac{-e^{-st}(s\sin(\omega t)+\omega\cos(\omega t))}{s^2} - \frac{\omega^2}{s^2}\int e^{-st}\sin(\omega t)\,dt\end{aligned}$$

Now we can solve for $\int e^{-st}\sin(\omega t)\,dt$:

$$\begin{aligned}&\int e^{-st}\sin(\omega t)\,dt = \frac{-e^{-st}(s\sin(\omega t)+\omega\cos(\omega t))}{s^2} - \frac{\omega^2}{s^2}\int e^{-st}\sin(\omega t)\,dt\\ \implies &\int e^{-st}\sin(\omega t)\,dt + \frac{\omega^2}{s^2}\int e^{-st}\sin(\omega t)\,dt = \frac{-e^{-st}(s\sin(\omega t)+\omega\cos(\omega t))}{s^2}\\ \implies &\left(1+\frac{\omega^2}{s^2}\right)\int e^{-st}\sin(\omega t)\,dt = \frac{-e^{-st}(s\sin(\omega t)+\omega\cos(\omega t))}{s^2}\\ \implies &\left(\frac{s^2+\omega^2}{s^2}\right)\int e^{-st}\sin(\omega t)\,dt = \frac{-e^{-st}(s\sin(\omega t)+\omega\cos(\omega t))}{s^2}\\ \implies &\int e^{-st}\sin(\omega t)\,dt = \frac{-e^{-st}(s\sin(\omega t)+\omega\cos(\omega t))}{\omega^2+s^2}\end{aligned}$$

Now we can evaluate the definite integral which gives us

$$\int_0^b e^{-st}\sin(\omega t)\,dt = \frac{-e^{-sb}(s\sin(\omega b)+\omega\cos(\omega b))}{\omega^2+s^2} + \frac{\omega}{\omega^2+s^2}$$

Finally, as b goes to infinity, the e^{-sb} factor that appears in the numerator of the first term will go to zero if $s > 0$. Thus, after all of that calculation we determine that the Laplace transform of $\sin(\omega t)$ is

$$\mathscr{L}\{\sin(\omega t)\} = \frac{\omega}{\omega^2 + s^2} \text{ provided } s > 0.$$

5.3 Properties of the Laplace transform

As the previous example likely should convince you, computing the Laplace transform of a function can be a rather difficult and lengthy process from the definition as an integral. Conveniently, however, the Laplace transform satisfies many nice properties which can be used to greatly simplify our computations.

One of the main properties of the Laplace transform is that it is linear, which is very easy to check.

Proposition 5.1.
Given two integrable functions f and g defined on $(0, \infty)$ and any constant λ (including complex constants), we have

$$\mathscr{L}\{f(t) + g(t)\} = \mathscr{L}\{f(t)\} + \mathscr{L}\{g(t)\}$$
$$\mathscr{L}\{\lambda f(t)\} = \lambda \mathscr{L}\{f(t)\}.$$

Proof.
Both of these properties easily follow from the fact that integration

is linear:

$$\begin{aligned}\mathscr{L}\{f(t)+g(t)\} &= \int_0^\infty e^{-st}(f(t)+g(t))\,dt\\ &= \int_0^\infty \left(e^{-st}f(t)+e^{-st}g(t)\right)\,dt\\ &= \int_0^\infty e^{-st}f(t)\,dt + \int_0^\infty e^{-st}g(t)\,dt\\ &= \mathscr{L}\{f(t)\}+\mathscr{L}\{g(t)\}\end{aligned}$$

$$\begin{aligned}\mathscr{L}\{\lambda f(t)\} &= \int_0^\infty e^{-st}\lambda f(t)\,dt\\ &= \lambda\int_0^\infty e^{-st}f(t)\,dt\\ &= \lambda\mathscr{L}\{f(t)\}\end{aligned}$$

□

To see how these simple properties can greatly simplify our computations, let's re-compute the Laplace transform of $\sin(\omega t)$ by recalling that for any complex number $z = x + iy$ we have

$$\text{Im}(z) = \frac{z-\overline{z}}{2i}$$

since

$$\frac{z-\overline{z}}{2i} = \frac{x+iy-(x-iy)}{2i} = \frac{x+iy-x-iy}{2i} = \frac{2iy}{2i} = y.$$

Now notice that, by Euler's formula, $\sin(\omega t) = \text{Im}(e^{i\omega t})$. That is, we may write

$$\sin(\omega t) = \frac{e^{i\omega t}-e^{-i\omega t}}{2i}$$

By linearity of the Laplace transform, we may then compute

$$\begin{aligned}\mathscr{L}\{\sin(\omega t)\} &= \mathscr{L}\left\{\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right\}\\ &= \frac{1}{2i}\left(\mathscr{L}\{e^{i\omega t}\}-\mathscr{L}\{e^{-i\omega t}\}\right).\end{aligned}$$

We had already computed the Laplace transform of e^{at} was $\frac{1}{s-a}$ when $\mathrm{Re}(s) > \mathrm{Re}(a)$, and so taking a to be $i\omega$, we have

$$\mathscr{L}\{e^{i\omega t}\} = \frac{1}{s - i\omega}$$

provided $\mathrm{Re}(s) > 0$ (as the real part of $i\omega$ is zero). Similarly, $\mathscr{L}\{e^{-i\omega t}\} = \frac{1}{s+i\omega}$ provided $\mathrm{Re}(s) > 0$. Continuing with our computation of $\mathscr{L}\{\sin(\omega t)\}$ we then have

$$\begin{aligned}
\mathscr{L}\{\sin(\omega t)\} &= \frac{1}{2i}\left(\mathscr{L}\{e^{i\omega t}\} - \mathscr{L}\{e^{-i\omega t}\}\right) \\
&= \frac{1}{2i}\left(\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right) \\
&= \frac{1}{2i}\left(\frac{1}{s - i\omega} \cdot \frac{s + i\omega}{s + i\omega} - \frac{1}{s + i\omega} \cdot \frac{s - i\omega}{s - i\omega}\right) \\
&= \frac{1}{2i} \cdot \frac{s + i\omega - s + i\omega}{s^2 + \omega^2} \\
&= \frac{\omega}{s^2 + \omega^2}
\end{aligned}$$

provided $\mathrm{Re}(s) > 0$ (since this was necessary in the Laplace transform of our exponential functions).

Exercise 5.1.
Combine linearity of the Laplace transform together with Euler's formula to compute

$$\mathscr{L}\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}$$

provided $\mathrm{Re}(s) > 0$.

Another useful property of the Laplace transform is the following: the derivative of the Laplace transform of $f(t)$ is the negative of the Laplace transform of $tf(t)$. Writing $F(s)$ for $\mathscr{L}\{f(t)\}$, this means

$$\mathscr{L}\{tf(t)\} = -F'(s).$$

This follows simply from differentiating $F(s)$ with respect to s:

$$\begin{aligned}\frac{d}{ds}F(s) &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt \\ &= \int_0^\infty \frac{d}{ds}e^{-st}f(t)\,dt \\ &= \int_0^\infty -te^{-st}f(t)\,dt \\ &= -\int_0^\infty e^{-st}tf(t)\,dt \\ &= -\mathscr{L}\{tf(t)\}.\end{aligned}$$

We can sometimes take advantage of this to help us compute Laplace transforms. For example, if we wanted to compute the Laplace transform of t, it is possible to use the integral definition, but it's rather tedious. If we instead think of t as the product $t \cdot 1$ use the fact $\mathscr{L}\{1\} = 1/s$, we then have that the Laplace transform of t must satisfy

$$\begin{aligned}\mathscr{L}\{t\} &= \mathscr{L}\{t \cdot 1\} \\ &= -\frac{d}{ds}\mathscr{L}\{1\} \\ &= -\frac{d}{ds}\frac{1}{s} \\ &= \frac{1}{s^2}\end{aligned}$$

Similarly, we can interpret t^2 as $t \cdot t$ and apply the same trick to compute

$$\begin{aligned}\mathscr{L}\{t^2\} &= \mathscr{L}\{t \cdot t\} \\ &= -\frac{d}{ds}\mathscr{L}\{t\} \\ &= -\frac{d}{ds}\frac{1}{s^2} \\ &= \frac{2}{s^3}\end{aligned}$$

We can continue in this way and compute the Laplace transform of t^3 as

$$\begin{aligned}\mathscr{L}\{t^3\} &= \mathscr{L}\{t \cdot t^2\} \\ &= -\frac{d}{ds}\mathscr{L}\{t^2\} \\ &= -\frac{d}{ds}\frac{2}{s^3} \\ &= \frac{6}{s^4}\end{aligned}$$

Notice that we have a 6 in the numerator of $\mathscr{L}\{t^3\}$ since we brought down a 3 and multiplied by the 2 that was already when computing the derivative. We continue to multiply by larger and larger numbers as we take Laplace transforms of t raised to higher and higher powers, so perhaps we should be surprised by the following:

Proposition 5.2.
For any non-negative integer n, the Laplace transform of t^n is

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Notice this, together with linearity, allows us to compute the Laplace transform of any polynomial:

Example 5.4.
Compute the Laplace transform of

$$5t^3 - 7t^2 + 2t + 3$$

$$\begin{aligned}\mathscr{L}\{5t^3 - 7t^2 + 2t + 3\} &= \mathscr{L}\left\{5t^3\right\} - \mathscr{L}\left\{7t^2\right\} + \mathscr{L}\left\{2t\right\} + \mathscr{L}\left\{3\right\} \\ &= 5\mathscr{L}\left\{t^3\right\} - 7\mathscr{L}\left\{t^2\right\} + 2\mathscr{L}\left\{t\right\} + 3\mathscr{L}\left\{1\right\} \\ &= 5 \cdot \frac{6}{s^4} - 7 \cdot \frac{2}{s^3} + 2 \cdot \frac{1}{s^2} + 3 \cdot \frac{1}{s} \\ &= \frac{30}{s^4} - \frac{14}{s^3} + \frac{2}{s^2} + \frac{3}{s}\end{aligned}$$

Translating Laplace transforms

If $F(s)$ is the Laplace transform of $f(t)$, then the translated function $F(s-a)$ is the Laplace transform of $e^{at}f(t)$. This follows easily from the integral definition of the Laplace transform. Notice first that, by the integral definition of the Laplace transform,

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt,$$

if we replace the argument s by $s-a$, we obtain the integral

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt.$$

We can easily check that this also equals the Laplace transform of $e^{at}f(t)$:

$$\begin{aligned}\mathscr{L}\left\{e^{at}f(t)\right\} &= \int_0^\infty e^{-st}e^{at} f(t)\,dt \\ &= \int_0^\infty e^{-st+at} f(t)\,dt \\ &= \int_0^\infty e^{(-s+a)t} f(t)\,dt \\ &= \int_0^\infty e^{-(s-a)t} f(t)\,dt \\ &= F(s-a)\end{aligned}$$

For example, consider the Laplace transform of $e^{3t}t$. Since $\mathscr{L}\{t\} = \frac{1}{s^2}$, the above tells us the Laplace transform of $e^{3t}t$ should be $\frac{1}{(s-3)^2}$, which we can easily verify:

$$\begin{aligned}
\mathscr{L}\left\{e^{3t}t\right\} &= \mathscr{L}\left\{te^{3t}\right\} \\
&= -\frac{d}{ds}\mathscr{L}\left\{e^{3t}\right\} \\
&= -\frac{d}{ds}\frac{1}{s-3} \\
&= -\frac{d}{ds}(s-3)^{-1} \\
&= -(-1)(s-3)^{-2} \\
&= \frac{1}{(s-3)^2}
\end{aligned}$$

In terms of inverse Laplace transforms, this tells us that if $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}^{-1}\{F(s-a)\} = e^{at}f(t)$.

For example, suppose we need to compute the inverse Laplace of $8/(s+5)^3$. Since this is a translated version of the function $\frac{8}{s^3}$, the inverse Laplace should be e^{-5t} times the inverse Laplace of $\frac{8}{s^3}$:

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{8}{(s+5)^3}\right\} &= \mathscr{L}^{-1}\left\{\frac{4\cdot 2}{(s+5)^3}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{4\cdot 2!}{(s+5)^3}\right\} \\
&= 4\mathscr{L}^{-1}\left\{\frac{2!}{(s+5)^3}\right\} \\
&= 4e^{-5t}\cdot t^2.
\end{aligned}$$

where above we used the fact $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$ in the case $n = 2$.

5.4 Solving initial value problems and the inverse Laplace transform

We now turn our attention to using the Laplace transform to help us solve problems in differential equations. The key result we need is the following theorem which will tell us that how to the Laplace transform of the derivative $f'(t)$ is related to the Laplace transform of the original $f(t)$.

Theorem 5.3.
Suppose f is a differentiable function with (piecewise) continuous derivative defined on $[0, \infty)$ such that there exists constants K, a, and M so that $|f(t)| \leq Ke^{at}$ for all $t > M$. Then $\mathscr{L}\{f'(t)\}$ will exist for all $s > a$ and

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0).$$

Proof.
For simplifying the proof we will suppose $f'(t)$ is continuous everywhere.

Let's notice

$$\int_0^b e^{-st} f'(t)\, dt$$

can be determined using integration by parts. We take

$$u = e^{-st} \qquad dv = f'(t)\, dt$$
$$du = -se^{-st} \qquad v = f(t)\, dt$$

to write

$$\int_0^b e^{-st} f'(t)\, dt = e^{-st} f(t)\Big|_0^b + s \int_0^b e^{-st} f(t)\, dt$$
$$= e^{-sb} f(b) - f(0) + s \int_0^b e^{-st} f(t)\, dt.$$

Taking the limit as b goes to infinity will give the Laplace transform for $f'(t)$. Notice, though, that $e^{-sb} f(b)$ will go to zero if $s > a$, as we have assumed that $|f(t)| \leq Ke^{at}$ for $t > M$, and so

$$|e^{-sb} f(b)| = e^{-sb}|f(b)| \leq e^{-sb} Ke^{ab} = Ke^{(a-s)b}.$$

The last term above goes to zero if $a - s < 0$, so $s > a$. But as b goes to infinity, $\int_0^b e^{-st} f(t)\, dt$ converges to the Laplace transform of $f(t)$, and the result is shown. □

We can easily iterate the the use of the theorem above to obtain a formula for the Laplace transform of $f''(t)$, $f'''(t)$, and higher order derivatives. For example, since $f''(t)$ is the derivative of $f'(t)$, the above theorem tells us

$$\mathscr{L}\{f''(t)\} = s\mathscr{L}\{f'(t)\} - f'(0)$$

but if we plug in that $\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)$, this becomes

$$\begin{aligned}\mathscr{L}\{f''(t)\} &= s\mathscr{L}\{f'(t)\} - f'(0) \\ &= s\left(s\mathscr{L}\{f(t)\} - f(0)\right) - f'(0) \\ &= s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0).\end{aligned}$$

We can easily repeat the process to compute

$$\mathscr{L}\{f'''(t)\} = s^3\mathscr{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0)$$

and so on. This is the key to using the Laplace transform to solve differential equations problems: the Laplace transform replaces derivatives with Laplace transforms of the original function.

A consequence of the fact that $f(0)$, $f'(0)$, and so on appear in our formulas for the Laplace transform of a derivative is that when we use the Laplace transform to solve a differential equation, we will actually obtain a solution to an initial value problem.

As an example, let's consider a simple first-order differential equation:

$$\frac{dy}{dt} + y = t$$

together with the initial condition $y(0) = 1$.

We will take the Laplace transform of both sides of the equation. Since we are doing the same thing to both sides of an equation, we still have an equation:

$$\mathscr{L}\left\{\frac{dy}{dt} + t\right\} = \mathscr{L}\{t\}.$$

On the left-hand side we can apply linearity to obtain

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} + \mathscr{L}\left\{t\right\} = \mathscr{L}\{t\}.$$

Now we simply fill in what $\mathscr{L}\{{dy}/{dt}\}$ and $\mathscr{L}\{t\}$. We had seen above that $\mathscr{L}\{{dy}/{dt}\}$ is $s\mathscr{L}\{y\} - y(0)$ and $\mathscr{L}\{t\} = {1}/{s^2}$, and so this becomes

$$s\mathscr{L}\{y\} - y(0) + \mathscr{L}\{y\} = \frac{1}{s^2}.$$

Notice that this is really just an algebra problem now where $\mathscr{L}\{y\}$ is the unknown. So, we just try to solve for $\mathscr{L}\{y\}$ and this gives us

$$\begin{aligned} s\mathscr{L}\{y\} - y(0) + \mathscr{L}\{y\} &= \frac{1}{s^2} \\ \implies s\mathscr{L}\{y\} + \mathscr{L}\{y\} &= \frac{1}{s^2} + y(0) \\ \implies (s+1)\mathscr{L}\{y\} &= \frac{1}{s^2} + 1 \\ \implies \mathscr{L}\{y\} &= \frac{\frac{1}{s^2}+1}{s+1} = \frac{1+s^2}{s^2(s+1)} \\ \implies \mathscr{L}\{y\} &= \frac{1+s^2}{s^3+s^2)} \end{aligned}$$

So, now we know what the Laplace transform of y needs to be. Can we somehow "work backwards" to determine what the original y was? That is, can we somehow "undo" what the Laplace transform did to y?

It will turn out the answer to this question is yes, there is a way for us to "invert" the Laplace transform. In order to understand how to do this, we need to know two things:

1. Laplace transforms are unique in the sense that if $\mathscr{L}\{f(t)\} = \mathscr{L}\{g(t)\}$, then $f(t) = g(t)$. (At least, this is true if we make some technical assumptions about the functions, which we will do implicitly.) This means the Laplace transform is a one-to-one function and so is invertible.

2. Inverses of linear transformations are also linear. That is, the inverse Laplace transform, $\mathscr{L}^{-1}$, also has the property that we can split up sums and factor out scalars: $\mathscr{L}^{-1}\{f(t)+g(t)\} = \mathscr{L}^{-1}\{f(t)\} + \mathscr{L}^{-1}\{g(t)\}$ and $\mathscr{L}^{-1}\{\lambda f(t)\} = \lambda\mathscr{L}^{-1}\{f(t)\}$.

How can we use these properties to "undo" the Laplace transform above,

$$\mathscr{L}\{y\} = \frac{1+s^2}{s^3+s^2}$$

to recover the original y? First we need to express the right-hand side, $\frac{1+s^2}{s^3+s^2}$, as a sum of simpler terms, and we can do this by computing its partial fraction decomposition. Since the denominator factors as $s^2(s+1)$, we need to find values of A, B, and C so that

$$\frac{1+s^2}{s^3+s^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

Adding these fractions on the right together gives us

$$\frac{(A+C)s^2 + (A+B)s + B}{s^3 + s^2}$$

Equating this with $\frac{1+s^2}{s^3+s^2}$ gives us a system of equations,

$$\begin{aligned} A + C &= 1 \\ A + B &= 0 \\ B &= 1 \end{aligned}$$

The second equation tells us B and A are negatives of one another, and hence $A = -1$. Now we can easily plug into the first equation to determine $C = 2$ and so we have

$$\frac{1+s^2}{s^3+s^2} = \frac{-1}{s} + \frac{1}{s^2} + \frac{2}{s+1}.$$

Thus we can write

$$\mathscr{L}\{y\} = \frac{-1}{s} + \frac{1}{s^2} + \frac{2}{s+1}.$$

To recover y we compute the Laplace inverse, $\mathscr{L}^{-1}$, of both sides of the equation:

$$\mathscr{L}^{-1}\{\mathscr{L}\{y\}\} = \mathscr{L}^{-1}\left\{\frac{-1}{s} + \frac{1}{s^2} + \frac{2}{s+1}\right\}.$$

On the left-hand side we have $\mathscr{L}^{-1}\{\mathscr{L}\{y\}\} = y$ since we are composing a transformation with its inverse. On the right-hand side we take advantage of linearity to write

$$\mathscr{L}^{-1}\left\{\frac{-1}{s} + \frac{1}{s^2} + \frac{2}{s+1}\right\} = -\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} + 2\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}.$$

We have already seen functions whose Laplace transform are each of these functions, however:

$$\begin{aligned} \mathscr{L}\{1\} = \frac{1}{s} &\implies \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \\ \mathscr{L}\{t\} = \frac{1}{s^2} &\implies \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \\ \mathscr{L}\{e^{-t}\} = \frac{1}{s+1} &\implies \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \end{aligned}$$

Thus we have computed

$$y = -1 + t + 2e^{-t}$$

is the solution to our initial value problem, which is easy to verify:

$$y(0) = -1 + 0 + 2e^0 = -1 + 2 = 1$$

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}\left(-1 + t + 2e^{-t}\right) = 1 - 2e^{-t} \\ \Longrightarrow \frac{dy}{dt} + y &= \left(1 - 2e^{-t}\right) + \left(-1 + t + 2e^{-t}\right) = t \end{aligned}$$

If we want to obtain the general solution to a differential equation instead of the solution to a particular initial value problem, we can leave the value of $y(0)$ (and any derivatives $y'(0)$, $y''(0)$, etc.) as parameters. For example, if we want to find the general solution to

$$\frac{dy}{dt} + y = t,$$

we repeat the same calculations above but leave $y(0) = C$ for some unspecified C. Taking the Laplace transform of each side of the equation then gives us

$$s\mathscr{L}\{y\} - C + \mathscr{L}\{y\} = \frac{1}{s^2}$$

which we can solve for $\mathscr{L}\{y\}$ to obtain

$$\mathscr{L}\{y\} = \frac{1 + Cs^2}{s^2(s+1)}.$$

Taking the partial fraction decomposition of the right-hand side, we could rewrite this as

$$\mathscr{L}\{y\} = \frac{-1}{s} + \frac{1}{s^2} + \frac{1+C}{s+1}$$

We then take the inverse Laplace transform of each side to obtain

$$y = -\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} + C\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

(The last two terms follow from writing $\frac{1+C}{s+1}$ as $\frac{1}{s+1} + \frac{C}{s+1}$.) Conveniently we know the functions which give us these Laplace transforms, so we have

$$y = -1 + t + e^{-t} + Ce^{-t} = -1 + t + (1 + C)e^{-t}.$$

Then we want to solve a particular initial value problem, we just choose what $y(0) = C$ is. For instance, if $y(0) = 1$ then $C = 1$, and we recover our earlier solution to that initial value problem.

As another example, consider the following second-order initial value problem:

$$\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} - 6y = 2 \quad \text{where } y(0) = 0, y'(0) = 0.$$

Notice that when we take the Laplace transform of each side of the equation we will need to to compute $\mathscr{L}\left\{t\frac{dy}{dt}\right\}$, but we can do this as follows:

$$\begin{aligned}\mathscr{L}\left\{t\frac{dy}{dt}\right\} &= -\frac{d}{ds}\mathscr{L}\left\{\frac{dy}{dt}\right\}\\ &= -\frac{d}{ds}\left(s\mathscr{L}\left\{y\right\} - y(0)\right)\\ &= -s\frac{d}{ds}\mathscr{L}\left\{y\right\} - \mathscr{L}\left\{y\right\}\end{aligned}$$

Notice that this gives us a derivative of the Laplace transform. This means our resulting equation will in fact be a differential equation involving the Laplace transform.

When we take the Laplace transform of our original second order equation above we have

$$s^2\mathscr{L}\left\{y\right\} - sy(0) - y'(0) + 3\left(-s\frac{d}{ds}\mathscr{L}\left\{y\right\} - \mathscr{L}\left\{y\right\}\right) - 6\mathscr{L}\left\{y\right\} = \frac{2}{s}$$

which we can simplify down to

$$\frac{d}{ds}\mathscr{L}\left\{y\right\} + \left(\frac{3}{s} - \frac{s}{3}\right)\mathscr{L}\left\{y\right\} = \frac{-2}{3s^2}.$$

Notice that to solve for $\mathscr{L}\left\{y\right\}$ here, we actually have to solve a differential equation! Conveniently this is a linear first order equation and so it's something we can solve with an integrating factor:

$$\mu = e^{\int\left(\frac{3}{s} - \frac{s}{3}\right)ds} = s^3e^{-s^2/6}$$

Multiplying our first-order equation above by this integrating factor gives us

$$\begin{aligned}
&s^3e^{-s^2/6}\frac{d}{ds}\mathscr{L}\{y\} + s^3e^{-s^2/6}\left(\frac{3}{s}-\frac{s}{3}\right)\mathscr{L}\{y\} = s^3e^{-s^2/6}\frac{-2}{3s^2} \\
\implies &\frac{d}{ds}s^3e^{-s^2/6}\mathscr{L}\{y\} = \frac{-2s}{3}e^{-s^2/6} \\
\implies &s^3e^{-s^2/6}\mathscr{L}\{y\} = 2e^{-s^2/6} \\
\implies &\mathscr{L}\{y\} = \frac{2}{s^3}
\end{aligned}$$

Thus $y = t^2$ solves our initial value problem as is easily verified.

As one final example, let's try to solve another second-order differential equation. We will try to solve the initial value problem

$$t\frac{d^2y}{dt^2} - (t+1)\frac{dy}{dt} + y = 0.$$

where $y(0) = 0$ and $y'(0) = 0$.

Taking the Laplace transform of each side of the equation gives us

$$\mathscr{L}\left\{t\frac{d^2y}{dt^2}\right\} - \mathscr{L}\left\{(t+1)\frac{dy}{dt}\right\} + \mathscr{L}\{y\} = \mathscr{L}\{0\}.$$

It's easy to see that $\mathscr{L}\{0\} = 0$, but we need to write the other terms as expressions involving $\mathscr{L}\{y\}$. Let's first consider

$$\mathscr{L}\left\{(t+1)\frac{dy}{dt}\right\}.$$

We can distribute and then apply linearity to write

$$\mathscr{L}\left\{(t+1)\frac{dy}{dt}\right\} = \mathscr{L}\left\{t\frac{dy}{dt}\right\} + \mathscr{L}\left\{\frac{dy}{dt}\right\}.$$

We know that $\mathscr{L}\left\{\frac{dy}{dt}\right\}$ is given by

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = s\mathscr{L}\{y\} - y(0) = s\mathscr{L}\{y\} - y(0).$$

Using our property that $\mathscr{L}\{tf(t)\} = -F'(s)$, where $\mathscr{L}\{f(t)\} = F(s)$, we can write

$$\mathscr{L}\left\{t\frac{dy}{dt}\right\} = -\frac{d}{ds}\mathscr{L}\left\{\frac{dy}{dt}\right\},$$

but as noted above, we have $\mathscr{L}\left\{\frac{dy}{dt}\right\} = s\mathscr{L}\{y\} - y(0)$, and so

$$\frac{d}{ds}\mathscr{L}\left\{\frac{dy}{dt}\right\} = \frac{d}{ds}(s\mathscr{L}\{y\} - y(0)) = \mathscr{L}\{y\} + s \cdot \frac{d}{ds}\mathscr{L}\{y\}.$$

(Keep in mind that $\mathscr{L}\{y\}$ is a function of s, so we needed to use the product rule to compute the derivative of $s\mathscr{L}\{y\}$.) Putting this together,

$$\begin{aligned}\mathscr{L}\left\{(t+1)\frac{dy}{dt}\right\} &= -\left(\mathscr{L}\{y\} + s\frac{d}{ds}\mathscr{L}\{y\}\right) + s\mathscr{L}\{y\} - y(0)\\ &= (s-1)\mathscr{L}\{y\} - s\frac{d}{ds}\mathscr{L}\{y\} - y(0)\end{aligned}$$

We still need to compute $\mathscr{L}\left\{t\frac{d^2y}{dt^2}\right\}$:

$$\begin{aligned}\mathscr{L}\left\{t\frac{d^2y}{dt^2}\right\} &= -\frac{d}{ds}\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\}\\ &= -\frac{d}{ds}\left(s^2\mathscr{L}\{y\} - sy(0) - y'(0)\right)\\ &= -\left(2s\mathscr{L}\{y\} + s^2\frac{d}{ds}\mathscr{L}\{y\} - y(0)\right)\\ &= -2s\mathscr{L}\{y\} - s^2\frac{d}{ds}\mathscr{L}\{y\} + y(0)\end{aligned}$$

Thus Laplace transform of the right-hand side of our differential equation above is

$$\begin{aligned}&\mathscr{L}\left\{t\frac{d^2y}{dt^2}\right\} - \mathscr{L}\left\{(t+1)\frac{dy}{dt}\right\} + \mathscr{L}\{y\}\\ =& -2s\mathscr{L}\{y\} - s^2\frac{d}{ds}\mathscr{L}\{y\} + y(0) - \left((s-1)\mathscr{L}\{y\} - s\frac{d}{ds}\mathscr{L}\{y\} - y(0)\right) + \mathscr{L}\{y\}\\ =&(s-s^2)\frac{d}{ds}\mathscr{L}\{y\} + (2-3s)\mathscr{L}\{y\} + 2y(0)\end{aligned}$$

Keeping in mind the right-hand side of our earlier equation was simply zero, and our initial conditions tell us $y(0) = 0$, we now have the following:

$$(s-s^2)\frac{d}{ds}\mathscr{L}\{y\} + (2-3s)\mathscr{L}\{y\} = 0.$$

Notice this is a first-order equation. To simplify notation, let's simply write L for $\mathscr{L}\{y\}$ and C for $y(0)$ so our equation is

$$(s-s^2)L' + (2-3s)L = 0.$$

This is a linear first-order equation which we can further rewrite as

$$L' + \frac{2-3s}{s-s^2}L = 0.$$

The integrating factor for this equation is

$$\begin{aligned}\mu &= e^{\int \frac{2-3s}{s-s^2}\,ds} \\ &= e^{\ln(1-s)+2\ln(s)} \\ &= s^2(1-s)\end{aligned}$$

Multiplying through by the integrating factor gives

$$(s^2 - s^3)L' + (2s - 3s^2)L = 0$$

which can be written as

$$\frac{d}{ds}(s^2 - s^3)L = 0$$

which is easily solved by integrating both sides and solving for L:

$$\begin{aligned}&\frac{d}{ds}(s^2 - s^3)L = 0 \\ \Longrightarrow &\int \frac{d}{ds}(s^2 - s^3)L\,ds = \int 0\,ds \\ \Longrightarrow &(s^2 - s^3)L = C \\ \Longrightarrow &L = \frac{C}{s^2 - s^3}\end{aligned}$$

That is, the Laplace transform of our solution y to the differential equation satisfies

$$\mathscr{L}\left\{y\right\} = \frac{C}{s^2 - s^3}$$

We need to compute the inverse Laplace transform. Since the inverse is linear, though, we have

$$y = C\mathscr{L}^{-1}\left\{\frac{1}{s^2 - s^3}\right\}$$

To compute this Laplace inverse we need the partial fraction decomposition,

$$\frac{1}{s^2 - s^3} = \frac{1}{s^2(1-s)} = \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s-1}.$$

Conveniently we know each of these inverse Laplace transforms:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$
$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$
$$\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$$

and so we can compute

$$y = C(t + 1 - e^t).$$

Now let's notice that regardless of what C is we will have

$$y(0) = C(0 + 1 - e^0) = C(1 - 1) = 0.$$

The derivative will be

$$y' = C(1 - e^t)$$

and so $y'(0) = C(1 - e^0) = 0$.

6 Systems of Differential Equations

6.1 Introduction to systems of differential equations

In general, a ***system of equations*** is a collection of several equations and a ***solution*** to the system is a choice of unknowns (e.g., variables) which simultaneously solve all of the equations at once. While systems of equations come in several forms (e.g., linear systems in linear algebra, polynomial systems in algebraic geometry), in this class we are specifically interested in systems of differential equations. For example, we may wish to find a pair of functions $x(t)$ and $y(t)$ whose derivatives satisfy an equation such as

$$\begin{aligned} x'(t) &= 3x(t) - 2y(t) \\ y'(t) &= -x(t) + 2y(t) \end{aligned}$$

To solve this system we require functions $x(t)$ and $y(t)$ such that both sides of each equation above are equal at the same time. Notice that the derivative of one function may depend on another. As an example, you can verify that the functions

$$\begin{aligned} x(t) &= -2e^{4t} \\ y(t) &= e^{4t} \end{aligned}$$

form a solution to this system as

$$\begin{aligned} 3x(t) - 2y(t) &= 3(-2e^{4t}) - 2e^{4t} = -6e^{4t} - 2e^{4t} = -8e^{4t} = x'(t) \\ -x(t) + 2y(t) &= -(-2e^{4t}) + 2e^{4t} = 2e^{4t} + 2e^{4t} = 4e^{rt} = y'(t). \end{aligned}$$

Systems such as this naturally arise in many different contexts: in ecology they are used to model interactions between predators and prey; in electrical engineering they arise in the analysis of circuits, particularly circuits with several components in parallel; in Riemannian and Lorentzian geometry (the mathematical foundations for Einstein's general theory of relativity) they arise in the study of geodesics which curves that locally minimize distance.

As we begin to study systems of differential equations, we will see that certain tools from linear algebra are indispensible. In particular, we will need to understand how to multiply a matrix with a vector and how to compute the eigenvalues and eigenvectors of a matrix. Here we provide a very quick summary of these ideas which developed more thoroughly in the appendix.

Vectors and matrices

A ***vector*** for our purposes is simply a list of values, which we will usually write in a column. The number of values that appears is called the ***dimension*** of the vector. For example,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

is a three-dimensional vector, whereas

$$\begin{pmatrix} -\pi \\ e \\ \sqrt{2} \\ 0 \\ 42 \end{pmatrix}$$

is a five-dimensional vector. While vectors occur in many different contexts and have various interpretations (e.g., forces, velocities, and accelerations in physics are all vector-valued quantities), for us they are mostly a convenient device for recording information.

When we assign a variable a vector, we often write the vector either with an arrow over it, as $\vec{v}$, or in bold, $\mathbf{v}$, to indicate this quantity is a vector and not a single number. Both of these conventions are common, though in handwritten work the arrow is usually preferred since it is easier to write than making a letter bold.

Sometimes we will refer to an individual number as a ***scalar*** to distinguish it from a vector. We will often use the Greek letter lambda, λ, as a generic scalar and the letter $\mathbf{v}$ as a generic vector. When we need to refer to the specific values contained in a vector (sometimes called the vector's ***components***) were often denote them by subscripts. For example,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}.$$

When the vector only contains two or three components, however, we often just refer to them as x, y, and z.

One of the most basic operations we can perform with vectors is ***scalar multiplication*** where we multiply a vector by a scalar to obtain a new vector of the same dimension. This is accomplished simply by multiplying each component by the scalar:

$$\lambda \mathbf{v} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \\ \vdots \\ \lambda v_n \end{pmatrix}.$$

For example, if we multiply the three-dimensional vector with components 4, -3 and 7 by the scalar 2 we have

$$2 \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \\ 14 \end{pmatrix}.$$

Given two vectors $\mathbf{v}$ and $\mathbf{w}$ of the same dimension, we can define ***vector addition*** by constructing a new vector whose components are just the sums of components of $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

For example,

$$\begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix}.$$

We can also define a notion of multiplication between a matrix and a vector. Though the definition looks a bit involved in general, the idea is that we are simply combining the two operations (scalar multipication and vector addition) together. In particular, if A is an $m \times n$ matrix and $\mathbf{v}$ is an n-dimensional vector, we define an m-dimensional vector by performing scalar multiplication between the first component of $\mathbf{v}$ and the first column of A, then scalar multiplication between the second component of $\mathbf{v}$ and the second column of A, and so on, and adding up the results:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = v_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + v_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + v_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

For example,

$$\begin{pmatrix} 2 & 4 & 3 \\ 5 & 1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix} = -3 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 3 \begin{pmatrix} 4 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ -15 \end{pmatrix} + \begin{pmatrix} 12 \\ 3 \end{pmatrix} + \begin{pmatrix} 12 \\ 8 \end{pmatrix} = \begin{pmatrix} 18 \\ -4 \end{pmatrix}$$

Eigenvalues and eigenvectors

Given a square $n \times n$ matrix A, we sometimes want to find a scalar λ together with a vector $\mathbf{v}$ such that the following equation is satisfied:

$$A\mathbf{v} = \lambda \mathbf{v}.$$

When we find a scalar λ and vector $\mathbf{v}$ related to one another in this way, we say that $\mathbf{v}$ is an ***eigenvector*** of the matrix A associated with ***eigenvalue*** λ.

Remark.
We will see why knowledge of eigenvectors and eigenvalues is useful in the context of solving systems of differential equations soon. However, eigenvectors and eigenvalues appear in many other types of problems. For example, the PageRank algorithm which Google uses to determine which web pages are most relevant for a given search uses eigenvectors and eigenvalues in a fundamental way. Thinking of all the pages on the Internet that contain a particular phrase, such as *differential equations*, the question is which of these web pages is the most helpful. You can think of this collection of web pages as a large graph with one vertex per page and an edge between two vertices if one page links to another. We then think of each web page as getting one "vote" for the other pages that are important, with pages being allowed to split their vote between multiple other pages, and the "votes" coming from the other pages a given page links to. Determining the most relevant page then becomes a question of counting the votes, where we weigh votes by the importance

of the pages which are voting. Writing out the details of this turns the question into a question of computing the eigenvector of some particularly large matrix with an associated eigenvalue of 1. There are lots of details to be filled in, but that is the essence of how Google search works.

(Fun fact: the PageRank algorithm is intellectual property of Stanford University where Larry Page and Sergey Brin, the founders of Google, were graduate students at the time they developed the algorithm. Stanford licenses the algorithm to Google.)

Notice that the vector of all zeros, $\mathbf{0}$, necessarily solves $A\mathbf{v} = \lambda\mathbf{v}$ for any scalar λ and any square matrix A. We consider this a trivial solution and so don't think of $\mathbf{0}$ as being an eigenvector. Thus we want to find all of the non-zero vectors $\mathbf{v}$ that solve $A\mathbf{v} = \lambda\mathbf{v}$. Rewriting this equation slightly we have

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

where I is the $n \times n$ identity matrix. All this means is that we subtract λ from each diagonal entry of the matrix A. This equation will have non-zero solution precisely when the determinant of $A - \lambda I$ is zero. That is, an eigenvalue of A is a scalar λ so that $\det(A - \lambda I) = 0$. Writing all of this out in detail basically gives us an algebra problem. For example, if we want to find the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 5 & -2 \end{pmatrix}$$

we need to find the values of λ so that

$$A - \lambda I = \begin{pmatrix} 1 - \lambda & 2 \\ 5 & -2 - \lambda \end{pmatrix}$$

has zero determinant. I.e.,

$$\begin{aligned} &(1 - \lambda)(-2 - \lambda) - 10 = 0 \\ \implies &-2 - \lambda + 2\lambda + \lambda^2 - 10 = 0 \\ \implies &\lambda^2 + \lambda - 12 = 0 \end{aligned}$$

This gives us a quadratic equation which we can easily solve since it factors as $(\lambda + 4)(\lambda - 3)$, and so the eigenvalues are $\lambda = -4$ and $\lambda = 3$.

Remark.
Notice that if A is an $n \times n$ matrix with all real entries, the equation we must solve to determine the eigenvalues λ is a polynomial of degree n with real coefficients. This means the roots of the polynomial (the eigenvalues of the matrix) can be distinct real numbers, repeated real numbers, or can come in complex conjugate pairs (possibly repeated).

Once we have the eigenvalues, we can work to find the associated eigenvectors. In the case of eigenvalue $\lambda = 3$ for our matrix above, for example, we seek vectors $\mathbf{v}$ so that

$$A\mathbf{v} = 3\mathbf{v},$$

or

$$(A - 3I)\mathbf{v} = \mathbf{0}.$$

Writing out the details this gives us a system of equations

$$\begin{aligned} -2v_1 + 2v_2 &= 0 \\ 5v_1 - 5v_2 &= 0 \end{aligned}$$

Unlike other systems that we have seen before this kind of system will never have a single, unique solution. There will be infinitely-many eigenvectors associated with each eigenvalue. Our goal, then, should be to parametrize the space of all eigenvectors for a given eigenvalue, called the ***eigenspace***. We can do this by describing some of the components of the vector $\mathbf{v}$ in terms of the other components. In particular, the two equations above tell us that if $\mathbf{v}$ is an eigenvector associated with eigenvalue $\lambda = 3$ for our matrix above, we must have $v_1 = v_2$. Thus the eigenspace for this eigenvalue is

$$\left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid v_1 = v_2 \right\}$$

So, for example, the vectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 7 \end{pmatrix}, \text{ and } \begin{pmatrix} -\pi \\ -\pi \end{pmatrix}$$

are all eigenvectors. In some problems we will just want to find one particular eigenvector, and so we often make the simplest possible choice of our components satisfying the necessary conditions. In other problems we want to describe all possible eigenvectors and so seek the list of equations the components of the eigenvectors must satisfy. (In terms of linear algebra we want a basis of the eigenspace.)

Exercise 6.1.
Find the eigenspace associated to eigenvalue $\lambda = -4$ of the matrix from before,

$$A = \begin{pmatrix} 1 & 2 \\ 5 & -2 \end{pmatrix}$$

6.2 Linear first-order homogeneous systems

Suppose that we have two functions, $x(t)$ and $y(t)$, and we are told that the derivative of each function depends not only on the function itself, but also on the other function. For example, maybe $x'(t)$ equals $2x(t) + 3y(t)$, and simultaneously $y'(t) = 2x(t) + y(t)$. That is, we have a system of differential equations:

$$\begin{aligned} x'(t) &= 2x(t) + 3y(t) \\ y'(t) &= 2x(t) + y(t) \end{aligned}$$

Solving this system means finding functions $x(t)$ and $y(t)$ so that both of these differential equations are satisfied simultaneously.

When studying systems of equations such as this, it can be convenient to rewrite the equation in terms of matrices and vectors. Let's let $\mathbf{x}(t)$ denote the two-dimensional vector whose components are $x(t)$ and $y(t)$:

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

The derivative of such a vector-valued function is simply the vector whose components are the derivatives of the components of the initial function:

$$\mathbf{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

Our system above can thus be written as

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Or, letting A denote the 2×2 matrix above,

$$\mathbf{x}'(t) = A\mathbf{x}(t).$$

How should we go about solving this system? Let's take a cue from the case of a single equation, $x'(t) = Ax(t)$ where $x(t)$ is a "normal" scalar-valued function and A is a constant (a single number). In this case we would divide $x(t)$ over and write our equation as

$$\frac{1}{x}\frac{dx}{dt} = A,$$

integrating both sides of the equation would yield

$$\begin{aligned} \int \frac{1}{x}\frac{dx}{dt}\,dt &= \int A\,dt \\ \implies \int \frac{1}{x}dx &= \int A\,dt \\ \implies \ln|x| &= At + C \\ \implies x &= Ce^{At}. \end{aligned}$$

In some very special situations our system of two (or more) differential equations is reallly two equations of this form in disguise. Consider, for example,

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Writing this out we have

$$\begin{aligned} x'(t) &= 3x(t) \\ y'(t) &= y(t) \end{aligned}$$

Here we are in a very special case because we see that x and y are independent of one another, so we can treat each of the two equations above as a "normal" differential equation whose solution we can compute as

$$\begin{aligned} x(t) &= c_1e^{3t} \\ y(t) &= c_2e^{t} \end{aligned}$$

Notice that we used two different arbitrary constants, c_1 and c_2, above as there's no great reason the constant that is used for the x should equal the constant that's used for the y.

If we were to write our above two equations in terms of vectors, we could do some simple algebra to write the solution as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{t} \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} e^{3t} + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} e^{t}$$

Let's observe that each term by itself gives us a solution to our system:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} e^{3t} \quad \text{and} \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ c_2 \end{pmatrix} e^{t}$$

are both solutions to the system. (In each case we just took one of our constants to be zero to kill off the other variable. This seems silly right now, but we're building up to something interesting.) Let's look at each of these solutions separately.

For the first solution,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} e^{3t}$$

notice if we differentiate each side of the equation we have

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} 3e^{3t}$$

which we could write as

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = 3 \begin{pmatrix} c_1 \\ 0 \end{pmatrix} e^{3t}.$$

This quantity on the right-hand side is closely related to our original matrix from before. It may not be a completely obvious thing to notice at first, but we could write the right-hand side of the equation above as

$$3 \begin{pmatrix} c_1 \\ 0 \end{pmatrix} e^{3t} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ 0 \end{pmatrix} e^{3t}$$

Dividing out the e^{3t} (which is a scalar-valued function that never equals zero, so there's no concern about dividing it through) we have

$$3 \begin{pmatrix} c_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ 0 \end{pmatrix}$$

Similarly for our second solution,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ c_2 \end{pmatrix} e^t,$$

differentiating each side gives us

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ c_2 \end{pmatrix} e^t$$

and this right-hand side we could write as

$$\begin{pmatrix} 0 \\ c_2 \end{pmatrix} e^t = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ c_2 \end{pmatrix} e^t$$

and canceling the e^t's we have

$$\begin{pmatrix} 0 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ c_2 \end{pmatrix}.$$

In both cases, notice that we have a constant λ (which is $\lambda = 3$ for the first solution and $\lambda = 1$ for the second) together with a vector $\mathbf{v}$ (in the first equation it's $\mathbf{v} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, and in the second it's $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$) so that using our matrix $A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ the following equation is satisfied:

$$\lambda \mathbf{v} = A\mathbf{v}.$$

Equations of this form and the constants λ and vectors $\mathbf{v}$ that appear are very special: they are called the ***eigenvalues*** and ***eigenvectors*** of the matrix A. Eigenvectors and eigenvalues are topics that traditionally first encountered in a course in linear algebra, which some students may have had, but some students may not have had. For this reason a more thorough discussion of eigenvectors and eigenvalues appears in Appendix A.4.

Though the above situation was very special in that $x(t)$ and $y(t)$ were independent of each other, they showed us that the solution to our system of differential equations was closely related to the eigenvectors and eigenvalues of the coefficient matrix A that appeared in our system. With this in mind, let's consider the general setting.

Consider a system of n first-order linear equations with constant coefficients,

$$\begin{aligned}
x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \\
x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) \\
x_3'(t) &= a_{31}x_1(t) + a_{32}x_2(t) + \cdots + a_{3n}x_n(t) \\
&\vdots \\
x_n'(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t)
\end{aligned}$$

Written in terms of vectors and matrices we can condense this to

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

$\mathbf{x}'(t)$ is the vector of derivatives, and A is the coefficient matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Now let's suppose that $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue λ. Our claim is that

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t},$$

which is really short-hand for

$$\begin{aligned}
x_1(t) &= v_1 e^{\lambda t} \\
x_2(t) &= v_2 e^{\lambda t} \\
&\vdots \\
x_n(t) &= v_n e^{\lambda t}
\end{aligned}$$

is a solution to our system of equations. To verify this, let's differentiate both sides of the equation. Since $\mathbf{v}$ is a vector of constants, we have

$$\mathbf{x}'(t) = \mathbf{v}\lambda e^{\lambda t}$$

Since $\mathbf{v}$ is an eigenvector of A with associated eigenvalue λ, we know $A\mathbf{v} = \lambda\mathbf{v}$, and so we may rewrite the above as

$$\mathbf{x}'(t) = A\mathbf{v}e^{\lambda t}$$

but $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$ and so we have

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

and thus verify that we have a solution to our system of differential equations.

The argument above is the proof of the following proposition:

Proposition 6.1.
If $\mathbf{v}$ is an eigenvector associated with eigenvalue λ for a matrix A, then $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$ is a solution to the system of differential equations given by $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Example 6.1.
Determine a pair of solutions to the following system of differential equations:

$$\begin{aligned} x'(t) &= 2x(t) + 3y(t) \\ y'(t) &= 2x(t) + y(t). \end{aligned}$$

In terms of matrices and vectors we write this as $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $\mathbf{x}(t)$ is the vector

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

and A is the coefficient matrix

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}.$$

We now determine the eigenvectors and eigenvalues of this matrix. If $\mathbf{v}$ is an eigenvector with associated eigenvalue λ, then we have

$A\mathbf{v} = \lambda\mathbf{v}$ which we can rewrite as $(A - \lambda I)\mathbf{v} = \mathbf{0}$. If this equation has non-zero solutions $\mathbf{v}$, then $A - \lambda I$ is non-invertible and so has determinant zero, so we first need to find the values of λ so that $\det(A - \lambda I) = 0$. In our particular example this means

$$\det\begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda) - 3\cdot 2 \neq 0.$$

Expanding this and combining like-terms, we have a quadratic in λ:

$$\lambda^2 - 3\lambda - 4 = 0$$

This factors as $(\lambda - 4)(\lambda + 1)$, and so our eigenvalues are $\lambda = 4$ and $\lambda = -1$. For each of these eigenvalues we have to determine the corresponding eigenvectors.

For each eigenvalue we determine the corresponding eigenvectors. In the case of $\lambda = 4$, we need to find vectors

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

so that $A\mathbf{v} = 4\mathbf{v}$:

$$\begin{aligned} &A\mathbf{v} = 4\mathbf{v} \\ \implies &\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 4\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{aligned}$$

Writing this out as a system of equations we have

$$\begin{aligned} 2v_1 + 3v_2 &= 4v_1 \\ 2v_1 + v_2 &= 4v_2 \end{aligned}$$

We move all the variables to the left-hand side to write

$$\begin{aligned} -2v_1 + 3v_2 &= 0 \\ 2v_1 - 3v_2 &= 0 \end{aligned}$$

Notice we can solve either of these equations for one variable or the other to obtain $2v_1 = 3v_2$, or $v_2 = \frac{2}{3}v_1$. This means that for *any* choice of v_1, the v_2 vector is completely determined: it must be two-thirds of v_1. Choosing any non-zero v_1, say $v_1 = 1$, we have that our eigenvector is

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}.$$

This means one solution to our system of differential equations is

$$\begin{aligned} x(t) &= e^{4t} \\ y(t) &= \frac{2}{3}e^{4t}. \end{aligned}$$

It's easy to verify this is in fact a solution to our system. Computing the derivatives we have

$$\begin{aligned} x'(t) &= 4e^{4t} \\ y'(t) &= \frac{8}{3}e^{4t} \end{aligned}$$

If we multiply the vector $\mathbf{x}(t)$ by the coefficient matrix A from above, we compute

$$\begin{aligned} A\mathbf{x}(t) &= \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^{4t} \\ \frac{2}{3}e^{4t} \end{pmatrix} \\ &= \begin{pmatrix} 2e^{4t} + 3\,\frac{2}{3}e^{4t} \\ 2e^{4t} + \frac{2}{3}e^{4t} \end{pmatrix} \\ &= \begin{pmatrix} 4e^{4t} \\ \frac{8}{3}e^{4t} \end{pmatrix} \end{aligned}$$

and so we do in fact have a solution to our system of equations.

Above we computed an eigenvector associated to eigenvalue $\lambda = 4$, but we also need to compute an eigenvector for our other eigenvalue, $\lambda = -1$. This means we need to find vectors $\mathbf{v}$ so that $A\mathbf{v} = -\mathbf{v}$, which we can write as $(A + I)\mathbf{v} = \mathbf{0}$, so we have a system of equations

$$\begin{aligned} 3v_1 + 3v_2 &= 0 \\ 2v_1 + 2v_2 &= 0 \end{aligned}$$

Solving either equations we see that $v_2 = -v_1$. So, our system of differential equations is solved by

$$x(t) = e^{-t}$$
$$y(t) = -e^{-t}$$

We can verify this is a solution to our system by computing

$$\begin{aligned} A\mathbf{x}(t) &= \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} 2e^{-t} - 3e^{-t} \\ 2e^{-t} - e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix} \end{aligned}$$

and noticing the components of this vector are precisely the derivatives of our $x(t) = e^{-t}$ and $y(t) = -e^{-t}$ above.

So, one we know that the eigenvalues of our coefficient matrix and the corresponding eigenvectors, we know some solutions to our system of differential equations. The following *principle of superposition* says if we have multiple solutions (e.g., coming from different eigenvectors and/or eigenvalues) we can take linear combinations of them to obtain another solution.

Proposition 6.2 (The principle of superposition)**.**
if $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, *...*, $\mathbf{x}_m(t)$ *are all solutions to the system of linear differential equations* $\mathbf{x}'(t) = A\mathbf{x}(t)$, *then so is any linear combination*

$$\mu_1\mathbf{x}_1(t) + \mu_2\mathbf{x}_2(t) + \ldots + \mu_m\mathbf{x}_m(t).$$

Proof.
We simply verify the linear combination is a solution which essen-

tially just relies on linearity:

$$\begin{aligned}
&\frac{d}{dt}\left(\mu_1\mathbf{x}_1(t)+\mu_2\mathbf{x}_2(t)+\ldots+\mu_m\mathbf{x}_m(t)\right)\\
=&\mu_1\mathbf{x}_1'(t)+\mu_2\mathbf{x}_2'(t)+\ldots+\mu_m\mathbf{x}_m'(t)\\
=&\mu_1A\mathbf{x}_1(t)+\mu_2A\mathbf{x}_2(t)+\ldots+\mu_mA\mathbf{x}_m(t)\\
=&A\left(\mu_1\mathbf{x}_1(t)+\mu_2\mathbf{x}_2(t)+\ldots+\mu_m\mathbf{x}_m(t)\right).
\end{aligned}$$

□

As a consequence of the principle of superposition, if our coefficient matrix for a system of n homogeneous linear differential equations with constant coefficients has n linearly independent eigenvectors, the general solution to the system has the form

$$\mathbf{x}(t)=c_1\mathbf{v}^{(1)}e^{\lambda_1 t}+c_2\mathbf{v}^{(2)}e^{\lambda_2 t}+c_n\mathbf{v}^{(n)}e^{\lambda_n t}$$

where $\mathbf{v}^{(1)}$, ..., $\mathbf{v}^{(n)}$ are the eigenvectors and $\mathbf{v}^{(j)}$ is associated to eigenvalue λ_j (we allow that a given eigenvalue λ_j may appear multiple times in the expression above).

In the problems above we have been in the special case where we had distinct real eigenvalues, but in general eigenvalues may be repeated and may be complex.

If we have an $n \times n$ matrix with all real entries, then any associated complex eigenvalues must have corresponding complex eigenvectors. Consider, for example, the matrix

$$\begin{pmatrix}-1 & 2\\ -2 & -1\end{pmatrix}$$

To find the eigenvalues we must solve

$$\det\begin{pmatrix}-1-\lambda & 2\\ -2 & -1-\lambda\end{pmatrix}=0.$$

Writing out the determinant this becomes

$$(-1-\lambda)^2+4=0$$

or

$$\lambda^2+2\lambda+5=0$$

Using the quadratic formula we can solve this for λ to obtain

$$\begin{aligned}\lambda &= \frac{-2 \pm \sqrt{4-20}}{2} \\ &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= \frac{-2 \pm 4i}{2} \\ &= -1 \pm 2i\end{aligned}$$

To find the eigenvector associated with $-1+2i$, we need a vector so that

$$\begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (-1+2i) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Multiplying out each side this becomes

$$\begin{pmatrix} -v_1 + 2v_2 \\ -2v_1 - v_2 \end{pmatrix} = \begin{pmatrix} (-1+2i)v_1 \\ (-1+2i)v_2 \end{pmatrix}$$

Equating components gives us a system

$$\begin{aligned} -v_1 + 2v_2 &= (-1+2i)v_1 \\ -2v_1 - v_2 &= (-1+2i)v_2 \end{aligned}$$

Now suppose we solve the first equation for v_2:

$$\begin{aligned} & -v_1 + 2v_2 = (-1+2i)v_1 \\ \Longrightarrow\ & 2v_2 = -v_1 + 2iv_1 + v_1 \\ \Longrightarrow\ & 2v_2 = 2iv_1 \\ \Longrightarrow\ & v_2 = iv_1 \end{aligned}$$

This tells us that entries in the corresponding eigenspace are given by

$$\left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid v_2 = iv_1 \right\}.$$

Or, written another way, eigenvectors have the form

$$\begin{pmatrix} v \\ iv \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} v.$$

Exercise 6.2.
Verify that the eigenvectors corresponding to eigenvalue $-1-2i$ have the form

$$\begin{pmatrix} 1 \\ -i \end{pmatrix} v.$$

To relate this back to systems of differential equations, if we had the system

$$\begin{aligned} x'(t) &= -x(t) + 2y(t) \\ y'(t) &= -2y(t) - x(t) \end{aligned}$$

the eigenvectors and eigenvalues above would tell us that complex-valued solutions to our system of differential equation would have the form

$$c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1+2i)t} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1-2i)t}$$

or

$$\begin{aligned} x(t) &= c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t} \\ y(t) &= i c_1 e^{(-1+2i)t} - i c_2 e^{(-1-2i)t} \end{aligned}$$

While these are perfectly legitimate solutions, we may sometimes wish to have real-valued solutions to our system. We can obtain real-valued solutions to our system by taking the real and imaginary parts of these solutions. Of course, the key to doing this is to use Euler's formula. By Euler, we may write

$$\begin{aligned} e^{(-1+2i)t} &= e^{-t} e^{i2t} = e^{-t}\cos(2t) + i e^{-t}\sin(2t) \\ e^{(-1-2i)t} &= e^{-t} e^{-i2t} = e^{-t}\cos(-2t) + i e^{-t}\sin(-2t) = e^{-t}\cos(2t) - i e^{-t}\sin(-2t) \end{aligned}$$

Our complex-valued solution from above,

$$\begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1+2i)t}$$

can thus be written as

$$\begin{pmatrix} 1 \\ i \end{pmatrix} \left(e^{-t}\cos(2t) + i e^{-t}\sin(2t)\right)$$

Performing the scalar multiplication we may write this as

$$\begin{pmatrix} e^{-t}\cos(2t) + ie^{-t}\sin(2t) \\ -e^{-t}\sin(2t) + ie^{-t}\cos(2t) \end{pmatrix}$$

or

$$\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} e^{-t} + i \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} e^{-t}$$

The real and imaginary parts of this complex-valued solution,

$$\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} e^{-t} \text{ and } \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} e^{-t}$$

form a basis for the real-valued solutions. I.e., every real-valued solution to our system has the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} e^{-t}$$

So far the examples we have seen have had 2×2 coefficient matrices with either two distinct real eigenvalues, or a pair of complex conjugate eigenvalues. We now turn our attention to the third possibility which is to have a repeated eigenvalue. As an example of a system where this occurs, consider

$$\begin{aligned} x'(t) &= x(t) - y(t) \\ y'(t) &= x(t) + 3y(t) \end{aligned}$$

To solve this system we are lead to find the eigenvalues and eigenvectors of the coefficient matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

We thus need to find the values of λ that solve the following equation:

$$\begin{aligned} &\det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} = 0 \\ \Longrightarrow\ &(1-\lambda)(3-\lambda) + 1 = 0 \\ \Longrightarrow\ &3 - \lambda - 3\lambda + \lambda^2 + 1 = 0 \\ \Longrightarrow\ &\lambda^2 - 4\lambda + 4 = 0 \end{aligned}$$

Notice the polynomial factors as $(\lambda - 2)^2$, and so $\lambda = 2$ is our only eigenvalue. We now seek its eigenvectors:

$$\begin{aligned} A\mathbf{v} &= 2\mathbf{v} \\ \implies \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= 2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ \implies \begin{pmatrix} v_1 - v_2 \\ v_1 + 3v_2 \end{pmatrix} &= \begin{pmatrix} 2v_1 \\ 2v_2 \end{pmatrix}. \end{aligned}$$

Equating components gives us the system

$$\begin{aligned} v_1 - v_2 &= 2v_1 \\ v_1 + 3v_2 &= 2v_2 \end{aligned}$$

Solving the first equation for v_2 gives us $v_2 = -v_1$ (the second equation would also give us this relationship). Thus our only eigenvectors are multiples of the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. This tells us one possible family of solutions would be

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

Since we have a system of two equations and two unknowns, though, we expect the space of solutions to be two-dimensional, not one-dimensional. Thus there should be some "missing" solutions.

Motivated by our "trick" for repeated roots of the characteristic polynomial for a higher-order differential equation, we may be tempted to look for a solution of the form

$$\mathbf{x}(t) = te^{2t}\mathbf{v}$$

for some unknown vector $\mathbf{v}$. If we compute the derivative of this, however, we have to apply the product rule and compute

$$\mathbf{x}'(t) = e^{2t}\mathbf{v} + 2te^{2t}\mathbf{v}.$$

If this is to solve our equation $\mathbf{x}'(t) = A\mathbf{x}(t)$, then we would have

$$e^{2t}\mathbf{v} + 2te^{2t}\mathbf{v} = Ate^{2t}\mathbf{v}.$$

Writing out the details in terms of the components, this means

$$\begin{aligned} v_1e^{2t} + 2v_2te^{2t} &= v_1te^{2t} - v_2te^{2t} = (v_1 - v_2)te^{2t} \\ v_2e^{2t} + 2v_2te^{2t} &= v_1te^{2t} + 3v_2te^{2t} = (v_1 + 3v_2)te^{2t} \end{aligned}$$

Since e^{2t} is never zero, we could divide that out from the equations above to obtain

$$\begin{aligned} v_1 + 2v_2t &= (v_1 - v_2)t \\ v_2 + 2v_2t &= (v_1 + 3v_2)t \end{aligned}$$

Since there are no constant terms on the right-hand sides of these equations, this would mean $v_1 = v_2 = 0$. Thus there are no non-trivial solutions of our system of equations of the form $te^{2t}\mathbf{v}$.

The "trick" around this is to instead consider possible solutions of the form

$$\mathbf{x}(t) = (t\mathbf{u} + \mathbf{w})e^{2t} = te^{2t}\mathbf{u} + e^{2t}\mathbf{w}.$$

Supposing such a solution were to exist, when computing the derivative we would have

$$\mathbf{x}'(t) = e^{2t}\mathbf{u} + 2te^{2t}\mathbf{u} + 2e^{2t}\mathbf{w} = (u + 2w)e^{2t} + \mathbf{u}2te^{2t}.$$

Plugging this into our system we would have

$$(u + 2w)e^{2t} + \mathbf{u}2te^{2t} = A\left((t\mathbf{u} + \mathbf{w})e^{2t}\right)$$

Rewriting the right-hand side slightly, this becomes

$$(u + 2w)e^{2t} + \mathbf{u}2te^{2t} = A\mathbf{w}e^{2t} + A\mathbf{u}te^{2t}$$

Equating coefficients between the e^{2t} and te^{2t} terms on the left- and right-hand sides of the system gives us

$$u + 2w = A\mathbf{w} \text{ and } 2\mathbf{u} = A\mathbf{u}.$$

The second equation is equivalent to saying that $\mathbf{u}$ is an eigenvector of A associated with eigenvalue $\lambda = 2$. The first equation we can rewrite as

$$(A - 2I)\mathbf{w} = \mathbf{u}.$$

This gives us a system of equations very similar to the eigenvalue-eigenvector system, except the right-hand side is an eigenvector of the original matrix instead of the zero vector. The solutions w to such a system are sometimes called ***generalized eigenvectors***.

Above we had computed that the eigenvectors of our matrix associated with eigenvalue $\lambda = 2$ were multiples of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Plugging that into the right-hand side of the system above, we have

$$\begin{pmatrix} 1-2 & -1 \\ 1 & 3-2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

As a system of equations,

$$\begin{aligned} -w_1 - w_2 &= 1 \\ w_1 + w_2 &= -1 \end{aligned}$$

Of course, these two equations are multiples of one another, and they tell us that w_1 and w_2 are related by

$$w_2 = -1 - w_1$$

That is, the vector $\mathbf{w}$ has the form

$$\mathbf{w} = \begin{pmatrix} n \\ -1-n \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + n \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus our general solution to the system of differential equations above is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} \right)$$

Or, in terms of the individual components,

$$\begin{aligned} x(t) &= c_1 e^{2t} + c_2 te^{2t} \\ y(t) &= -c_1 e^{2t} - c_2 e^{2t} - c_2 te^{2t} \end{aligned}$$

Summary

To summarize, suppose we have a *homogeneous system of linear, first-order differential equations* of the form

$$\begin{aligned} x'(t) &= ax(t) + by(t) \\ y'(t) &= cx(t) + dy(t) \end{aligned}$$

where a, b, c, and d are real constants. We may write this in terms of matrices and vectors as

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \mathbf{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

To determine the solutions to the system we need to find the eigenvalues and eigenvectors of the matrix A. The eigenvalues are the values of λ such that $\det(A - \lambda I) = 0$. Once the eigenvalues are known we may compute the eigenvectors by solving, for each eigenvalue λ, the vector equation $A\mathbf{v} = \lambda\mathbf{v}$ (this is really a system of equations in two unknowns, the components of $\mathbf{v}$). The corresponding system of equations which tell us the eigenvectors will not have a unique solution, so we seek a relationship between the components. This will always be a linear equation in the components, and we may write the equation as $v_2 = kv_1$ where k is some constant.

The equation we solve to determine the eigenvalues is a quadratic polynomial in the variable λ, and so there are three possibilities:

1. There are two distinct real eigenvalues.

2. There are two complex conjugate eigenvalues.

3. There is a repeated real eigenvalue.

Each case gives us a different possible set of solutions to the system of differential equations, elaborated on below.

Distinct real eigenvalues

Suppose the matrix A has two distinct real eigenvalues, call them λ_1 and λ_2. Each eigenvalue has a corresponding one-dimensional eigenspace consisting of multiples of some given non-zero eigenvector. Write $\mathbf{v}^{(1)}$ for an eigenvector associated with eigenvalue λ_1, and $\mathbf{v}^{(2)}$ the eigenvector associated with λ_2. The general solution to our system of differential equations is given by

$$\mathbf{x}(t) = c_1\mathbf{v}^{(1)}e^{\lambda_1 t} + c_2\mathbf{v}^{(2)}e^{\lambda_2 t}.$$

Or, written out in terms of the components,

$$\begin{aligned} x(t) &= c_1 v_1^{(1)} e^{\lambda_1 t} + c_2 v_1^{(2)} e^{\lambda_2 t} \\ y(t) &= c_1 v_2^{(1)} e^{\lambda_1 t} + c_2 v_2^{(2)} e^{\lambda_2 t} \end{aligned}$$

As an example, consider the system

$$\begin{aligned} x'(t) &= 5x(t) - 4y(t) \\ y'(t) &= -x(t) + 5y(t) \end{aligned}$$

The coefficient matrix here is

$$A = \begin{pmatrix} 5 & -4 \\ -1 & 5 \end{pmatrix}.$$

We first find the eigenvalues:

$$\begin{aligned} \det \begin{pmatrix} 5-\lambda & -4 \\ -1 & 5-\lambda \end{pmatrix} &= 0 \\ \implies (5-\lambda)^2 - 4 &= 0 \\ \implies 25 - 10\lambda + \lambda^2 - 4 &= 0 \\ \implies \lambda^2 - 10\lambda + 21 &= 0 \\ \implies (\lambda - 7)(\lambda - 3) &= 0 \end{aligned}$$

And so we have two distinct eigenvalues, $\lambda_1 = 7$ and $\lambda_2 = 3$. We now seek the eigenvectors associated to each eigenvalue.

For the eigenvalue $\lambda = 7$, we need to find vectors $\mathbf{v}$ that solve $A\mathbf{v} = 7\mathbf{v}$. This gives us the vector equation,

$$\begin{pmatrix} 5 & -4 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 7 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Performing the matrix and scalar multiplications and equating components, this becomes

$$\begin{aligned} 5v_1 - 4v_2 &= 7v_1 \\ -v_1 + 5v_2 &= 7v_2 \end{aligned}$$

moving all the variables to the left-hand side of the equation this is

$$\begin{aligned} -2v_1 - 4v_2 &= 0 \\ -v_1 - 2v_2 &= 0 \end{aligned}$$

Solving either of these equations for v_2 will tell us $v_2 = \frac{-1}{2} v_1$. Thus all the eigenvectors associated to eigenvalue $\lambda = 7$ for this matrix are multiples of

$$\mathbf{v}^{(1)} = \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

For the eigenvalue $\lambda = 3$, we solve $A\mathbf{v} = 3\mathbf{v}$:

$$\begin{pmatrix} 5 & -4 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 3 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

As a system of equations this is

$$\begin{aligned} 5v_1 - 4v_2 &= 3v_1 \\ -v_1 + 5v_2 &= 3v_2 \end{aligned}$$

Solving either of these equations for v_2 will give us $v_2 = \frac{1}{2}v_1$, and so the eigenvectors as multiples of

$$\mathbf{v}^{(2)} = \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

Ths general solution to our system of differential equations is then written in vector form as

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} e^{3t}$$

or, in components,

$$\begin{aligned} x(t) &= c_1 e^{7t} + c_2 e^{3t} \\ y(t) &= \frac{-1}{2} c_1 e^{7t} + \frac{1}{2} c_2 e^{3t} \end{aligned}$$

Complex conjugate eigenvalues

Suppose the matrix A has two complex conjugate eigenvalues, which we will suppose are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. Each eigenvalue has an associated complex eigenspace. That is, the eigenvectors will have complex-valued components. These yield complex-valued solutions to our system of differential equations determined by the same formula as in the case of distinct real eigenvalues:

$$\mathbf{x}(t) = c_1 \mathbf{v}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{v}^{(2)} e^{\lambda_2 t}.$$

where $\mathbf{v}^{(j)}$ is the complex eigenvector associated with eigenvalue λ_j.

To obtain real-valued solutions we must consider the real and imaginary parts of complex-valued solutions. We may do this for either of our eigenvector/eigenvalue pairs (since the eigenvalues are complex conjugates of one another they are closely related and we will obtain the same set of solutions). Supposing the eigenvector $\mathbf{v}$ associated with eigenvalue $\alpha + i\beta$ is

$$\mathbf{v} = \begin{pmatrix} \xi_1 + i\eta_1 \\ \xi_2 + i\eta_2 \end{pmatrix}$$

then one of our complex-valued solutions is

$$\mathbf{v}e^{(a+ib)t} = \begin{pmatrix} \xi_1 + i\eta_1 \\ \xi_2 + i\eta_2 \end{pmatrix} e^{at} e^{ibt}$$

Using Euler's formula we may write this as

$$\begin{pmatrix} \xi_1 + i\eta_1 \\ \xi_2 + i\eta_2 \end{pmatrix} \left(e^{at}\cos(bt) + ie^{at}\sin(bt) \right)$$

Performing the scalar multiplication and writing the i^2's that appear as -1, this becomes

$$\begin{pmatrix} \xi_1 e^{at}\cos(bt) - \eta_1 e^{at}\sin(bt) + i\left(\xi_1 e^{at}\sin(bt) + \eta_1 e^{at}\cos(bt)\right) \\ \xi_2 e^{at}\cos(bt) - \eta_2 e^{at}\sin(bt) + i\left(\xi_2 e^{at}\sin(bt) + \eta_2 e^{at}\cos(bt)\right) \end{pmatrix}$$

which we may further rewrite as

$$\begin{pmatrix} \xi_1\cos(bt) - \eta_1\sin(bt) \\ \xi_2\cos(bt) - \eta_2\sin(bt) \end{pmatrix} e^{at} + i \begin{pmatrix} \xi_1\sin(bt) + \eta_1\cos(bt) \\ \xi_2\sin(bt) + \eta_2\cos(bt) \end{pmatrix} e^{at}$$

The real and imaginary parts separately form a basis for the space of real-valued solutions to our system, and so the general real-valued solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \xi_1\cos(bt) - \eta_1\sin(bt) \\ \xi_2\cos(bt) - \eta_2\sin(bt) \end{pmatrix} e^{at} + c_2 \begin{pmatrix} \xi_1\sin(bt) + \eta_1\cos(bt) \\ \xi_2\sin(bt) + \eta_2\cos(bt) \end{pmatrix} e^{at}$$

As an example, consider the system

$$\begin{aligned} x'(t) &= x(t) - y(t) \\ y'(t) &= x(t) + y(t) \end{aligned}$$

The coefficient matrix here is

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

We need to determine the eigenvalues:

$$\begin{aligned} &\det(A - \lambda I) = 0 \\ \implies &\det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = 0 \\ \implies &(1-\lambda)^2 + 1 = 0 \\ \implies &1 - 2\lambda + \lambda^2 + 1 = 0 \\ \implies &\lambda^2 - 2\lambda + 2 = 0 \end{aligned}$$

We now solve for λ using the quadratic formula:

$$\begin{aligned}\lambda &= \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2} \\ &= \frac{2 \pm \sqrt{4 - 8}}{2} \\ &= \frac{2 \pm \sqrt{-4}}{2} \\ &= \frac{2 \pm 2i}{2} \\ &= 1 \pm i\end{aligned}$$

If we are only interested in real-valued solutions, note that we only need to find the eigenvectors associated to one of the eigenvalues. For this example we will use $\lambda = 1 + i$. We seek vectors $\mathbf{v}$ that solve $A\mathbf{v} = (1 + i)\mathbf{v}$. This becomes

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (1 + i) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

This gives us the system of equations

$$\begin{aligned} v_1 - v_2 &= (1 + i)v_1 = v_1 + iv_1 \\ v_1 + v_2 &= (1 + i)v_2 = v_2 + iv_2 \end{aligned}$$

Putting the variables all on the left-hand sides of the equations above, this may be rewritten as

$$\begin{aligned} -iv_1 - v_2 &= 0 \\ v_1 - iv_2 &= 0 \end{aligned}$$

Solving the first equation for v_2 gives us $v_2 = -iv_1$.

Remark.
If we were to solve the second equation for v_2 we will get the same answer, but there's a small technical point in that we would need to divide by a complex number. In general this is done by multiplying and dividing by the conjugate of the denominator in our fraction. In the case of our example this we would have $iv_2 = v_1$ and dividing

the i over we have:

$$
\begin{aligned}
v_2 &= \frac{v_1}{i} \\
&= \frac{v_1}{i} \cdot \frac{-i}{-i} \\
&= \frac{-iv_1}{i \cdot (-i)} \\
&= \frac{-iv_1}{-i^2} \\
&= \frac{-iv_1}{-(-1)} \\
&= -iv_1
\end{aligned}
$$

Thus, taking v_1 to be 1, we see that eigenvectors of our matrix associated with eigenvalue $1+i$ are multiples of

$$\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

In terms of our notation above, we have

$$\begin{pmatrix} \xi_1 + i\eta_1 \\ \xi_2 + i\eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

and so $\xi_1 = 1$, $\eta_1 = 0$, $\xi_2 = 0$ and $\eta_2 = -1$. The eigenvalue was $1+i$, so $a = 1$ and $b = 1$ as well and using the formula above, the solutions to our system of differential equations are

$$
\begin{aligned}
x(t) &= c_1 \cos(t)e^t + c_2 \sin(t)e^t \\
y(t) &= c_1 \sin(t)e^t - c_2 \cos(t)e^t
\end{aligned}
$$

Repeated real eigenvalues

Suppose the matrix A has a repated real eigenvalue λ with eigenvector $\mathbf{v}$. We then have that one solution to the system is given by

$$\mathbf{x}(t) = c_1 \mathbf{v} e^{\lambda t}.$$

To find a second linearly independent solution we suppose there is a solution of the form

$$\mathbf{x}(t) = (t\mathbf{u} + \mathbf{w})e^{\lambda t}.$$

The derivative of this is then

$$\begin{aligned}\mathbf{x}'(t) &= (e^{\lambda t} + \lambda t e^{\lambda t})\mathbf{u} + \lambda e^{\lambda t}\mathbf{w} \\ &= (\mathbf{u} + \lambda \mathbf{w})e^{\lambda t} + \mathbf{u}\lambda t e^{\lambda t}\end{aligned}$$

This, however, should equal $A\mathbf{x}(t)$ which is $A\mathbf{u}te^{\lambda t} + A\mathbf{w}e^{\lambda t}$, so we have

$$(\mathbf{u} + \lambda \mathbf{w})e^{\lambda t} + \mathbf{u}\lambda t e^{\lambda t} = A\mathbf{u}te^{\lambda t} + A\mathbf{w}e^{\lambda t}$$

Equating the $e^{\lambda t}$ terms on each side, as well as the $te^{\lambda t}$ terms, this gives us a pair of vector equations,

$$\begin{aligned}A\mathbf{u} &= \lambda \mathbf{u} \\ A\mathbf{w} &= u + \lambda \mathbf{w}\end{aligned}$$

The first equation simply tells us the $\mathbf{u}$ is in fact our eigenvector from before, and the second equation tells us $\mathbf{w}$ is a "generalized eigenvector." After doing the algebra to determine $\mathbf{w}$ (and keeping in mind the above equations tell us $\mathbf{u}$ is the eigenvector $\mathbf{v}$), the general solution to the system of equations is given in vector form by

$$\mathbf{x}(t) = c_1 \mathbf{v} e^{\lambda t} + c_2(t\mathbf{v} + \mathbf{w})e^{\lambda t}$$

As an example, consider the system

$$\begin{aligned}x'(t) &= 6x(t) + 3y(t) \\ y'(t) &= -3x(t) + 12y(t)\end{aligned}$$

The coefficient matrix is

$$A = \begin{pmatrix} 6 & 3 \\ -3 & 12 \end{pmatrix}$$

We compute the eigenvalues as follows:

$$\begin{aligned}&\det(A - \lambda I) = 0 \\ \implies &\det\begin{pmatrix} 6-\lambda & 3 \\ -3 & 12-\lambda \end{pmatrix} = 0 \\ \implies &(6-\lambda)(12-\lambda) + 9 = 0 \\ \implies &72 - 18\lambda + \lambda^2 + 9 = 0 \\ \implies &\lambda^2 - 18\lambda + 81 = 0 \\ \implies &(\lambda - 9)^2 = 0\end{aligned}$$

Our only eigenvalue here is $\lambda = 9$. Now we seek its eigenvectors by solving $A\mathbf{v} = 9\mathbf{v}$. In terms of matrices and vectors this gives us

$$\begin{pmatrix} 6 & 3 \\ -3 & 12 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 9 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

After performing the multiplication this gives us the system of equations

$$\begin{aligned} 6v_1 + 3v_2 &= 9v_1 \\ -3v_1 + 12v_2 &= 9v_2 \end{aligned}$$

which we may rewrite as

$$\begin{aligned} -3v_1 + 3v_2 &= 0 \\ -3v_1 + 3v_2 &= 0 \end{aligned}$$

and these equations tell us that $v_1 = v_2$, so our eigenvectors are multiples of the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This gives us some of the possible solutions to our system, but to find the remaining solutions we need to determine a generalized eigenvector. This means we need to find a vector $\mathbf{w}$ such that $A\mathbf{w} = \mathbf{v} + 9\mathbf{w}$, where $\mathbf{v}$ is our eigenvector above. This gives us a system of equations we must solve. Writing the above out in terms of matrices and vectors, we have

$$\begin{aligned} \begin{pmatrix} 6 & 3 \\ -3 & 12 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 9 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ \implies \begin{pmatrix} 6w_1 + 3w_2 \\ -3w_1 + 12w_2 \end{pmatrix} &= \begin{pmatrix} 1 + 9w_1 \\ 1 + 9w_2 \end{pmatrix} \end{aligned}$$

This corresponds to the system of equations

$$\begin{aligned} 6w_1 + 3w_2 &= 1 + 9w_1 \\ -3w_1 + 12w_2 &= 1 + 9w_2 \end{aligned}$$

Moving the variables to the left-hand side (but keeping the constants 1 on each right-hand side) this becomes

$$\begin{aligned} -3w_1 + 3w_2 &= 1 \\ -3w_1 + 3w_2 &= 1 \end{aligned}$$

Solving this for w_2 we have

$$\begin{aligned} & -3w_1 + 3w_2 = 1 \\ \implies & 3w_2 = 1 + 3w_1 \\ \implies & w_2 = \frac{1}{3} + w_1 \end{aligned}$$

So our generalized eigenvector $\mathbf{w}$ has the form

$$\begin{pmatrix} w \\ w + \frac{1}{3} \end{pmatrix}$$

We may write this as

$$w\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$$

Notice that in our formula above,

$$\mathbf{x}(t) = c_1\mathbf{v}e^{\lambda t} + c_2(t\mathbf{v} + \mathbf{w})e^{\lambda t}$$

the factor of $t\mathbf{v} + \mathbf{w}$ in the second term is the only place $\mathbf{w}$ appears. What we've seen, though, is that our $\mathbf{w}$ is the sum of an eigenvector (a multiple of our $\mathbf{v}$) and the vector $\begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$. This second vector is the thing we really care about, so we will use it as the $\mathbf{w}$ above.

To recap, we have that

$$\lambda = 9, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$$

Plugging this into our formula above, the solution to our system (as a vector) is

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{9t} + c_2\left(t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}\right)e^{9t}$$

which in terms of the components equals

$$\begin{aligned} x(t) &= c_1e^{9t} + c_2te^{9t} \\ y(t) &= c_1e^{9t} + c_2(t + 1/3)e^{9t} \end{aligned}$$

6.3 Phase potraits

In this section we finish up our discussion of systems of differential equations by mentioning a way to qualitative describe the behavior of a solution to a system. We will specifically consider the case of two-dimensional systems for simplicity, but the ideas are similar in higher dimensions.

Notice that if $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a solution to a system $\mathbf{x}'(t) = A\mathbf{x}(t)$, then what we have is a curve in the plane parametrized by $\mathbf{x}(t)$: at each moment in time t we are told the x and y coordinates of a point, $(x(t), y(t))$, and as the time t changes, taking these points in tandem gives us a curve. The system of differential equations tells us what the tangent vectors to the curves should be at every point. That is, the tangent vector of the curve parametrized by $(x(t), y(t))$ is exactly the derivative

$$\mathbf{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

but our differential equation tells us what these should derivatives should be at each point. That is, the system of differential equations actually determines a ***vector field***, an association of a vector to each point in the plane.

In particular, the system

$$\begin{aligned} x'(t) &= ax(t) + by(t) \\ y'(t) &= cx(t) + dy(t) \end{aligned}$$

is associated to the vector field

$$\mathbf{F}(x, y) = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$

which means at a point (x, y) in the plane we associate the vector $\mathbf{F}(x, y)$ described above.

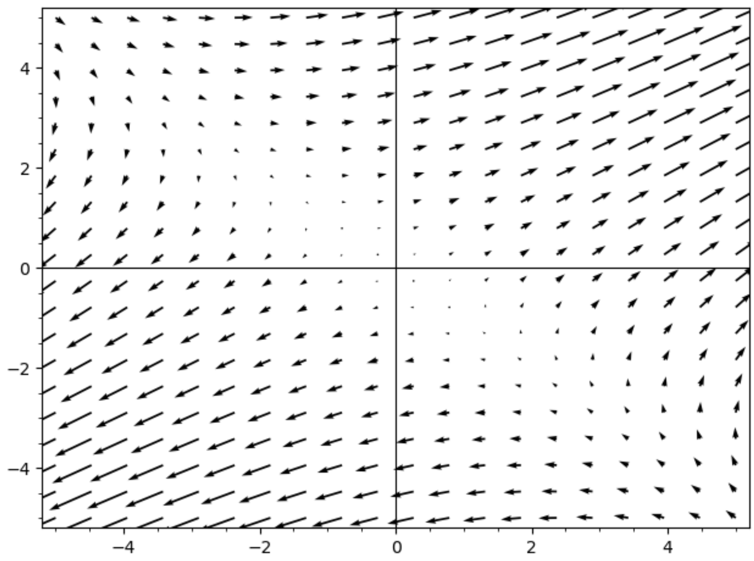
In the case of the system

$$\begin{aligned} x'(t) &= 2x(t) + 3y(t) \\ y'(t) &= 2x(t) + y(t) \end{aligned}$$

our vector field is

$$\mathbf{F}(x, y) = \begin{pmatrix} 2x + 3y \\ 2x + y \end{pmatrix}$$

which we may visualize as



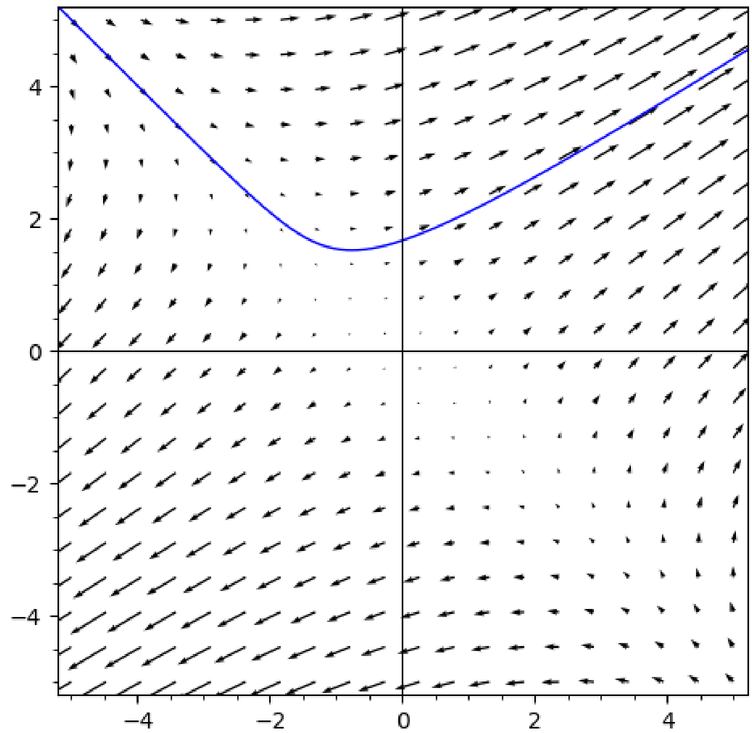
Using the results from the last section, the general solution to this system is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

or, in components,

$$\begin{aligned} x(t) &= c_1 e^{4t} + c_2 e^{-t} \\ y(t) &= c_1 \frac{2}{3} e^{4t} - c_2 e^{-t} \end{aligned}$$

Each choice of c_1 and c_2 gives us a solution to the system together with a parametric curve. For example, plotted below is the curve corresponding to $c_1 = 1$, $c_2 = -1$ plotted against the vector field.



Notice that this curve is tangent to the vector field at every point. This is a general fact: the solutions to a system of differential equations give us an integral curve to the corresponding vector field.

Remark.
The above fact suggests a way to approximate solutions to a vector field: find (approximations to) integral curves. In particular, if we want to find the solution to a system of equations which satisfies some initial condition, say $x(0) = a$ and $y(0) = b$, then we can start by "marking" the point (a, b) on the plane. We then approximate $x(t)$ and $y(t)$ for some small value $t \approx 0$ by evaluating the vector field at (a, b), and drawing a small line segment emanating from (a, b) in the direction determined by the vector field at (a, b), $\mathbf{F}(a, b)$. This gives us a second point on our approximate integral curve, and we repeat the process starting from this new point. Doing this several (hundreds, thousands, perhaps millions of times) for very small increments of t gives us a reasonable approximation of the solution to the system of equations. This technique is called *Euler's method*

and is one of the basic ways solutions to systems of equations are numerically approximated on a computer.

Appendices

A

Linear Algebra

There is hardly any theory which is more elementary [than linear algebra], in spite of the fact that generations of professors and textbook writers have obscured its simplicity by preposterous calculations with matrices.

Jean Dieudonné
Foundations of Modern Analysis, Vol. 1

In this appendix we review some topics from linear algebra which will be helpful for our study of differential equations. We first introduce the notion of a linear transformation and see how it is related to the idea of a matrix. We then spend some time describing products of matrices and vectors, and then matrices with one another. As we will see the odd rule that many people rotely memorize for matrix multiplication is defined in this way precisely so that products of matrices correspond to compositions of linear transformations.

After reviewing these necessary bits of matrix algebra, we review to closely related topics: the Laplace (or cofactor) expansion for computing determinants, and Cramer's rule for solving linear systems of n equations with n variables when the corresponding coefficient matrix is invertible (i.e., has non-zero determinant).

We finally introduce the ideas of eigenvectors and eigenvalues which we will need when we discuss linear systems of differential equations.

A.1 Linear transformations and matrices

Linear transformations

To be a little bit more precise, a ***linear transformation*** is a function T that takes n-dimensional vectors and converts them into m-dimensional vectors (m and n may be the same, but they don't need to be), and such that for all pairs of n-dimensional vectors $\vec{u}$ and $\vec{v}$ and all scalars λ we have the following two identities:

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$
$$T(\lambda\vec{v}) = \lambda T(\vec{v})$$

That is, a linear transformation allows us to split up sums and factor out scalars.

For example, consider the following function that converts three-dimensional vectors into two-dimensional vectors:

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + y \\ z - 5y \end{pmatrix}. \tag{A.1}$$

Here we are specifying T by describing how it uses the components of its input vector (which is three-dimensional) to build its output vector (which is two-dimensional). When this function is applied to the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, for example, the result is

$$T\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 2 \\ 3 - 5 \cdot 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}. \tag{A.2}$$

Sometimes in order to say that T takes n-dimensional vectors and converts them into m dimensional vectors we will write $T : \mathbb{R}^n \to \mathbb{R}^m$.

It is important to notice that for a function T to be linear it *must* satisfy $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(\lambda\vec{v}) = \lambda T(\vec{v})$ *for all* input vectors $\vec{u}$ and $\vec{v}$, and all scalars λ. In order to check that this is the case, we have to leave the components of $\vec{u}$ and $\vec{v}$ as variables and check if the two sides of the equalities mentioned above really are equal.

In the case of the map T described above, we must compute the following:

$$\begin{aligned}
T(\vec{u}+\vec{v}) &= T\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix} \\
&= \begin{pmatrix} 2(u_1+v_1)+u_2+v_2 \\ u_3+v_3-5(u_2+v_2) \end{pmatrix} \\
&= \begin{pmatrix} 2u_1+u_2+2v_1+v_2 \\ u_3-5u_2+v_3-5v_2 \end{pmatrix} \\
&= \begin{pmatrix} 2u_1+u_2 \\ u_3-5u_2 \end{pmatrix} + \begin{pmatrix} 2v_1+v_2 \\ v_3-5v_2 \end{pmatrix} \\
&= T\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + T\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\
&= T(\vec{u}) + T(\vec{v})
\end{aligned}$$

Since we left the components of our vectors as arbitrary values (the variables u_1, u_2, and so on), the calculation above shows that *for all* three-dimensional vectors $\vec{u}$ and $\vec{v}$, we have that $T(\vec{u}+\vec{v})$ equals $T(\vec{u})+T(\vec{v})$. If we had instead plugged in two particular choices for $\vec{u}$ and $\vec{v}$, our calcuation would have only shown that we can break up the sum of those particular vectors, but we need to show we can break up *all* sums of vectors, hence the need to leave the components as variables.

We can similarly check that scalars can be factored out of the function above:

$$\begin{aligned}
T(\lambda\vec{v}) &= T\left(\lambda\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}\right)\\
&= T\begin{pmatrix}\lambda v_1\\\lambda v_2\\\lambda v_3\end{pmatrix}\\
&= \begin{pmatrix}2\lambda v_1+\lambda v_2\\\lambda v_3-5\lambda v_2\end{pmatrix}\\
&= \begin{pmatrix}\lambda(2v_1+v_2)\\\lambda(v_3-5v_2)\end{pmatrix}\\
&= \lambda\begin{pmatrix}2v_1+v_2\\v_3-5v_2\end{pmatrix}\\
&= \lambda T\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}\\
&= \lambda T(\vec{v}).
\end{aligned}$$

Again, because the components of $\vec{v}$ and the scalar λ were left as variables, the above calculation shows that $T(\lambda\vec{v}) = \lambda T(\vec{v})$ *for all* choices of $\vec{v}$ and scalar λ.

Notice that not every function converting three-dimensional vectors into two-dimensional vectors is necessarily a linear transformation. For example, if we replace the function T above with the following,

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x^2+y\\1+z\end{pmatrix},$$

then we will not have a linear transformation. To see this, we just need to find a single example where either of the inequalities $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ or $T(\lambda\vec{v}) = \lambda T(\vec{v})$ fails. Consider, for example, the following:

$$\begin{aligned}
T\left(4\begin{pmatrix}1\\2\\3\end{pmatrix}\right) &= T\begin{pmatrix}4\\8\\12\end{pmatrix}\\
&= \begin{pmatrix}4^2+8\\1+12\end{pmatrix}\\
&= \begin{pmatrix}24\\13\end{pmatrix}
\end{aligned}$$

However, we also have

$$4T\begin{pmatrix}1\\2\\3\end{pmatrix} = 4\begin{pmatrix}1^2+2\\1+3\end{pmatrix}$$
$$= 4\begin{pmatrix}3\\4\end{pmatrix}$$
$$= \begin{pmatrix}12\\16\end{pmatrix}$$

Since this does not equal our earlier calculation, we have

$$T\left(4\begin{pmatrix}1\\2\\3\end{pmatrix}\right) \neq 4T\begin{pmatrix}1\\2\\3\end{pmatrix}.$$

Since we have one instance where $T(\lambda\vec{v}) \neq \lambda T(\vec{v})$, the function T above is not linear.

Remark.
Notice that to be a linear transformation, we must have $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$ and $T(\lambda\vec{v}) = \lambda T(\vec{v})$ *for all* choices of $\vec{u}$, $\vec{v}$, and λ. Once you find one instance where either of these inequalities aren't satisfied, the transformation can not be linear.

One important property of linear transformations is that their composition is a linear transformation. That is, suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation which converts n-dimensional vectors into m-dimensional vectors, and also suppose $S : \mathbb{R}^m \to \mathbb{R}^p$ is a linear transformation converting m-dimensional vectors into p-dimensional vectors. We can then take an n-dimensional vector $\vec{v}$, apply T to obtain an m-dimensional vector $T(\vec{v})$, and then apply S to obtain a p-dimensional vector $S(T(\vec{v}))$. This operation of applying T and then applying S is called ***composition*** and is denoted $S \circ T : \mathbb{R}^n \to \mathbb{R}^p$.

For example, let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation from above

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2x+y\\z-5y\end{pmatrix},$$

and suppose $S : \mathbb{R}^2 \to \mathbb{R}^4$ is the function

$$S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \\ 2x \\ x - 3y \end{pmatrix}. \tag{A.3}$$

Exercise A.1.
Verify that the function S described in Equation A.3 above is in fact a linear transformation. We must show that for any pair of two-dimensional vectors $\vec{u}$ and $\vec{v}$ and any scalar λ, the following two equalities are satisfied:

$$S(\vec{u} + \vec{v}) = S(\vec{u}) + S(\vec{v}) \text{ and } S(\lambda\vec{v}) = \lambda S(\vec{v}).$$

We do this by treaing the scalar and the components of our variables to verify that the equalities are satisfied for all possible choices:

$$\begin{aligned}
S(\vec{u} + \vec{v}) &= S\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \\
&= S\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 + v_1 + u_2 + v_2 \\ u_1 + v_1 - (u_2 + v_2) \\ 2(u_1 + v_1) \\ u_1 + v_1 - 3(u_2 + v_2) \end{pmatrix} \\
&= \begin{pmatrix} u_1 + u_2 + v_1 + v_2 \\ u_1 - u_2 + v_1 - v_2 \\ 2u_1 + 2v_1 \\ u_1 - 3u_2 + v_1 - 3v_2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 + u_2 \\ u_1 - u_2 \\ 2u_1 \\ u_1 - 3u_2 \end{pmatrix} + \begin{pmatrix} v_1 + v_2 \\ v_1 - v_2 \\ 2v_1 \\ v_1 - 3v_2 \end{pmatrix} \\
&= S(\vec{u}) + S(\vec{v})
\end{aligned}$$

$$\begin{aligned} S(\lambda \vec{v}) &= S\left(\lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \\ &= S\begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda v_1 + \lambda v_2 \\ \lambda v_1 - \lambda v_2 \\ 2\lambda v_1 \\ \lambda v_1 - 3\lambda v_2 \end{pmatrix} \\ &= \lambda \begin{pmatrix} v_1 + v_2 \\ v_1 - v_2 \\ 2v_1 \\ v_1 - 3v_2 \end{pmatrix} \\ &= \lambda S(\vec{v}) \end{aligned}$$

The composition $S \circ T : \mathbb{R}^3 \to \mathbb{R}^4$ is then given by

$$\begin{aligned} S \circ T \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= S\left(T\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) \\ &= S\begin{pmatrix} 2x + y \\ z - 5y \end{pmatrix} \\ &= \begin{pmatrix} 2x + y + z - 5y \\ 2x + y - (z - 5y) \\ 2(2x + y) \\ 2x + y - 3(z - 5y) \end{pmatrix} \\ &= \begin{pmatrix} 2x - 4y + z \\ 2x + 6y - z \\ 4x + 2y \\ 2x + 6y - 3z \end{pmatrix} \end{aligned}$$

We would like to see if $S \circ T$ is a linear transformation. We can do this by manually checking that our equalities hold in each example, but this is tedious. Thus we like the following theorem which tells us that provided we already know S and T are both linear transformations, their composition must be a linear transformation as well.

Theorem A.1.
If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ are both linear transformations, then their composition $S \circ T : \mathbb{R}^n \to \mathbb{R}^p$ is also a linear transformation.

Notice that we need to already now S and T are linear transformations for Theorem A.1 to be helpful, but once we've shown this for S and T, we get that $S \circ T$ is linear for free.

Matrices

A ***matrix*** is simply a rectangular array of numbers, such as

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 1 & -3 \end{pmatrix}$$

When referring to a matrix we specify its number of rows and columns by saying the matrix is $m \times n$ (pronounced "m by n") if it has m rows and n columns. The matrices above, for example, are 2×3 and 4×2.

It is often convenient to give matrices a name to save ourselves some writing. For example, if we write

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

then we will refer to this particular 2×3 matrix by A. Once we given a matrix a name, we will sometimes refer to the the individual entries in the matrix by giving the lowercase name of the matrix with subscripts indicating the row and column. For instance, we write a_{13} to mean the entry of matrix A in the first row and third column. For our matrix A above this would be $a_{13} = 0$; and a_{22} would be -5.

In general, when referring to an arbitrary $m \times n$ matrix we will leave the individual entries as variables a_{ij}; these act as a placeholder for the entry in the i-th row and j-th column,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

Matrices are used in many different areas of mathematics and have lots of applications; you could easily spend an entire course discussing matrices and their applications (and if you take Linear Algebra, that's basically what you'll be doing). There are lots of things we could say about matrices, and several different operations, but there are three main things that will be important for us in this course. Two operations we'll describe now, and one we'll discuss later.

The first operation we will consider will allow us to "multiply" a matrix by a vector, the result of which is a new vector. In particular, we will multiply an $m \times n$ matrix by an n-dimensional vector, and the result will be an m-dimensional vector. We are explicitly defining this operation only when the number of columns in the matrix equals the number of entries in the vector. For example, we can multiply a 2×3 matrix by a 3-dimensional vector, but we *can not* multiply a 2×3 matrix by a 4-dimensional vector.

This operation is rather tedious to describe in words, but actually very easy to compute in practice. We will write down the wordy description first, but just bear with it for a moment, and then we will do an example and the example should be easy to follow.

Let's suppose that A is an $m \times n$ matrix and $\vec{v}$ is an n-dimensional vector,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

We will treat each column of the matrix A as an m-dimensional vector,

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \begin{pmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{m3} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix},$$

We will then multiply the first column (thought of as a vector) by the first entry of the vector, v_1, multiply the second column by the second entry in the vector, v_2, multiply the third column by the third entry, v_3, and so on, then add these vectors together. The result is the product of A and $\vec{v}$, denoted $A\vec{v}$:

$$\begin{aligned} A\vec{v} &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \\ &= v_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + v_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + v_3 \begin{pmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{m3} \end{pmatrix} + \cdots + v_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}. \end{aligned}$$

Notice that since we are adding together m-dimensional vectors (each column has m entries since the original matrix A has m rows), the result is an m-dimensional vector.

As a concrete example, let's multiply the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

by the vector

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

We will multiply the first column of the matrix by 1, the second column by 2, and the third column by 3, then add the result together:

$$\begin{aligned} A\vec{v} &= \begin{pmatrix} 2 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= 1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -5 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -10 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -7 \end{pmatrix} \end{aligned}$$

One of the main reasons we care about matrices is that this operation of multiplying a matrix and a vector defines a linear transformation! That is, we claim that matrix multiplication distributes over vector addition,

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v},$$

and commutes with scalar multiplication,

$$A(\lambda\vec{v}) = \lambda(A\vec{v}).$$

Assuming these two properties are true (which is proven in Appendix **??**), this means we can use an $m \times n$ matrix A to define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by defining $T(\vec{v})$ to be the product the product $A\vec{v}$. The two properties mentioned above, and restated in Theorem A.2 below, are exactly what we need to know that $T(\vec{v}) = A\vec{v}$ defines a linear transformation.

Theorem A.2.
If A is an $m \times n$ matrix, $\vec{u}$ and $\vec{v}$ are n-dimensional vectors and λ is a scalar, then

$$\begin{aligned} A(\vec{u} + \vec{v}) &= A\vec{u} + A\vec{v}, \text{ and} \\ A(\lambda\vec{v}) &= \lambda(A\vec{v}). \end{aligned}$$

As a consequence, the function $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{v}) = A\vec{v}$ is a linear transformation.

Thus every matrix determines a linear transformation. Moreover, every linear transformation can be written as matrix multiplication. You may have noticed, for example, that the product we computed above

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}$$

was the same as the result of the linear transformation described in Equation A.2 on page 222. In fact, if we leave the components of the vector as variables, we compute

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 2 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ -5 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2x + y \\ -5y + z \end{pmatrix} = \begin{pmatrix} 2x + y \\ z - 5y \end{pmatrix}$$

gives us the exact same result as the linear transformation described in Equation A.1. This is not a coincidence: all linear transformations are really just multiplication of a matrix and a vector. We can even explicitly compute what the matrix should be for any given linear transformation, as described in Theorem A.3.

Theorem A.3.
If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T has the form

$$T(\vec{v}) = A\vec{v}$$

where A is the $m \times n$ matrix whose first column contains the entries in the vector

$$T\begin{pmatrix}1\\0\\0\\0\\\vdots\\0\end{pmatrix},$$

whose second column contains the entries in the vector

$$T\begin{pmatrix}0\\1\\0\\0\\\vdots\\0\end{pmatrix},$$

whose third column contains the entries in the vector

$$T\begin{pmatrix}0\\0\\1\\0\\\vdots\\0\end{pmatrix},$$

and so on.

In the case of the example described by Equation A.1, notice that we compute

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\cdot 1+0\\0-5\cdot 0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$
$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\cdot 0+1\\0-5\cdot 1\end{pmatrix} = \begin{pmatrix}1\\-5\end{pmatrix}$$
$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\cdot 0+0\\1-5\cdot 0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

and these are the columns of the matrix

$$\begin{pmatrix}2 & 1 & 0\\0 & -5 & 1\end{pmatrix}$$

previously described.

Example A.1.
Determine the matrix associated to the linear transformation S that was described in Equation A.3 on page 226.

Recall $S : \mathbb{R}^2 \to \mathbb{R}^4$ was defined by

$$S\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\x-y\\2x\\x-3y\end{pmatrix}.$$

Since this transformation takes two-dimensional vectors and converts them into four-dimensional vectors, we expect its representative matrix to be 4×2. To find our two columns, we need to apply S to the vectors

$$\begin{pmatrix}1\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix}.$$

We simply compute

$$S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\2\\1\end{pmatrix}$$

$$S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\0\\-3\end{pmatrix}.$$

Putting these together, we see that our matrix is

$$\begin{pmatrix}1 & 1\\1 & -1\\2 & 0\\1 & -3\end{pmatrix}.$$

Just to double-check this really is the correct matrix, we can multiply this 4×2 matrix by some arbitrary two-dimensional vector and verify that the result is the same as the transformation S applied to that vector,

$$\begin{pmatrix}1 & 1\\1 & -1\\2 & 0\\1 & -3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}1\\1\\2\\1\end{pmatrix} + y\begin{pmatrix}1\\-1\\0\\-3\end{pmatrix} = \begin{pmatrix}x+y\\x-y\\2x\\x-3y\end{pmatrix}.$$

Notice this is exactly the value of $S\begin{pmatrix}x\\y\end{pmatrix}$.

Exercise A.2.
Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3x-y+2z\\-x+z\\6y-3z\end{pmatrix}.$$

Compute the 3×3 matrix that represents this transformation. We simply compute the three quantities

$$T\begin{pmatrix}1\\0\\0\end{pmatrix}, T\begin{pmatrix}0\\1\\0\end{pmatrix}, \text{ and } T\begin{pmatrix}0\\0\\1\end{pmatrix},$$

and use these as the columns of our matrix.

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}3\\-1\\0\end{pmatrix}$$
$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\6\end{pmatrix}$$
$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\-3\end{pmatrix}.$$

The matrix representing our transformation is thus

$$\begin{pmatrix}3 & -1 & 2\\-1 & 0 & 1\\0 & 6 & -3\end{pmatrix}.$$

Multiplying matrices

Suppose now that we have two linear transformations which we can compose; say $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$. We had previously seen in Theorem A.1 that their composition $S \circ T$ is a linear transformation which takes n-dimensional vectors, applies T to convert them into m-dimensional vectors, and then applies S to finally convert them into p-dimensional vectors. Since $S \circ T$ is linear, it should be represented by a matrix. Is there an "easy" way for us to compute this matrix if we already have the matrices for S and T?

We have seen that we can multiply a matrix and a vector, and this was tantamount to applying a linear transformation. To describe the composition of two linear transformations we will extend our notion of multiplication to allow us to multiply two matrices together. Our goal here is to define matrix multiplication in such a way that multiplying two matrices is the same as composing linear transformations. We can only compose linear transformations, though, when the dimensions "match up" appropriately. That is, the composition $S \circ T$ is only defined if the output of T has the same dimension as the input of S. In terms of our matrices, the dimension of the output of a linear transformation is the number of columns of the matrix, and the dimension of the number is the number of rows. Thus we will only define matrix multiplication when the number of rows of the right-hand matrix equals the number of columns of the left-hand matrix.

Before describing the general procedure, let's consider a concrete example. We had previously seen linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + y \\ -5y + z \end{pmatrix} \qquad S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \\ 2x \\ x - 3y \end{pmatrix}.$$

The composition of these matrices, $S \circ T$, will take three-dimensional vectors and ultimately convert them into four-dimensional vectors. Thus the matrix representing $S \circ T$ will be 4×3. Our goal will be to determine this 4×3 matrix just from the matrices that represent T and S.

Let's denote the 2×3 matrix representing T by A, and let B denote the 4×2 matrix representing S. We had calculated above that

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 1 & -3 \end{pmatrix}.$$

Now let's imagine that we apply $S \circ T$ to some arbitrary three-dimensional vector $\vec{v}$. Since T is applied first and S is applied second, the matrix A (representing T) should be multiplied with $\vec{v}$ first, and then afterwards we should multiply the result by the matrix B (representing S). That is, we want to compute

$$BA\vec{v} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Multiplying A and $\vec{v}$ together first, leaving B alone, this becomes

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 2x + y \\ -5y + z \end{pmatrix}$$

We can multiply this matrix and vector together to obtain

$$\begin{pmatrix} 2x + y + (-5y + z) \\ 2x + y - (-5y + z) \\ 2(2x + y) \\ 2x + y - 3(-5y + z) \end{pmatrix} = \begin{pmatrix} 2x - 4y + z \\ 2x + 6y - z \\ 4x + 2y \\ 2x + 6y - 3z \end{pmatrix}$$

Notice this is the same as the product

$$\begin{pmatrix} 2 & -4 & 1 \\ 2 & 6 & -1 \\ 4 & 2 & 0 \\ 2 & 6 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

That is, if our definition of matrix multiplication is going to agree with our composition of linear transformations, we will need to define matrix multiplication in such a way that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -4 & 1 \\ 2 & 6 & -1 \\ 4 & 2 & 0 \\ 2 & 6 & -3 \end{pmatrix}$$

To see exactly what's going on, let's replace the numbers in our example above with variables:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}.$$

Now let's again consider the product $BA\vec{v}$,

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We first multiply $A\vec{v}$, leaving B alone for the moment, to obtain

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix} \begin{pmatrix} x\, a_{11} + y\, a_{12} + z\, a_{13} \\ x\, a_{21} + y\, a_{22} + z\, a_{23} \end{pmatrix}$$

Now we multiply the B matrix with this vector which gives us

$$\begin{pmatrix} (x\, a_{11} + y\, a_{12} + z\, a_{13})\, b_{11} + (x\, a_{21} + y\, a_{22} + z\, a_{23})\, b_{12} \\ (x\, a_{11} + y\, a_{12} + z\, a_{13})\, b_{21} + (x\, a_{21} + y\, a_{22} + z\, a_{23})\, b_{22} \\ (x\, a_{11} + y\, a_{12} + z\, a_{13})\, b_{31} + (x\, a_{21} + y\, a_{22} + z\, a_{23})\, b_{32} \\ (x\, a_{11} + y\, a_{12} + z\, a_{13})\, b_{41} + (x\, a_{21} + y\, a_{22} + z\, a_{23})\, b_{42} \end{pmatrix}$$

We can distribute and rewrite this as

$$\begin{pmatrix} x\,(b_{11}a_{11} + b_{12}a_{21}) + y\,(b_{11}a_{12} + b_{12}a_{22}) + z\,(b_{11}a_{13} + b_{12}a_{23}) \\ x\,(b_{21}a_{11} + b_{22}a_{21}) + y\,(b_{21}a_{12} + b_{22}a_{22}) + z\,(b_{21}a_{13} + b_{22}a_{23}) \\ x\,(b_{31}a_{11} + b_{32}a_{21}) + y\,(b_{31}a_{12} + b_{32}a_{22}) + z\,(b_{31}a_{13} + b_{32}a_{23}) \\ x\,(b_{41}a_{11} + b_{42}a_{21}) + y\,(b_{41}a_{12} + b_{42}a_{22}) + z\,(b_{41}a_{13} + b_{42}a_{23}) \end{pmatrix}.$$

This is the same as the following multiplication of a matrix and a vector:

$$\begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} + b_{11}a_{13} + b_{12}a_{23} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} + b_{21}a_{13} + b_{22}a_{23} \\ b_{31}a_{11} + b_{32}a_{21} & b_{31}a_{12} + b_{32}a_{22} + b_{31}a_{13} + b_{32}a_{23} \\ b_{41}a_{11} + b_{42}a_{21} & b_{41}a_{12} + b_{42}a_{22} + b_{41}a_{13} + b_{42}a_{23} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

That is, the product of our two matrices should be defined to be

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} & b_{11}a_{13} + b_{12}a_{23} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} & b_{21}a_{13} + b_{22}a_{23} \\ b_{31}a_{11} + b_{32}a_{21} & b_{31}a_{12} + b_{32}a_{22} & b_{31}a_{13} + b_{32}a_{23} \\ b_{41}a_{11} + b_{42}a_{21} & b_{41}a_{12} + b_{42}a_{22} & b_{41}a_{13} + b_{42}a_{23} \end{pmatrix}$$

This is quite an ugly expression, but it is the "right" expression, the right way to define matrix multiplication, if we want matrix multiplication to be the same thing as composition of linear transformations.

Even though this expression is rather ugly at first glance, if you looked at it for a moment you might realize there are some nice patterns. In particular, you might observe that the first column of our product matrix, is obtained by multiplying the B matrix with the first column of the A

matrix, thought of as a vector:

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} \\ b_{21}a_{11} + b_{22}a_{21} \\ b_{31}a_{11} + b_{32}a_{21} \\ b_{41}a_{11} + b_{42}a_{21} \end{pmatrix}$$

Similarly, the second column of the product matrix is the product of the entire B matrix with the second column of the A matrix; the third column of the product is the same as the product of the entire B matrix with the third column of the A matrix. This, of course, is not simply a coincidence: what's really happening here is that Theorem A.3 is being applied.

We know from Theorem A.3 that the matrix representing $S \circ T$ is given by applying $S \circ T$ to the vector with a 1 as its first component and all zeros otherwise, this gives the first column of the matrix. The second column is obtained by applying $S \circ T$ to the vector with a 1 in its second component and all zeros otherwise, and so on. When we actually do this, however, we apply T first, and we know tha T applied to the vector with first coordinate 1 and all zeros otherwise gives us the first column of of the matrix representing T. When we then apply S, we are thus multiplying the matrix representing S by the first column of the matrix representing T, and this product gives us the first column of the matrix representing $S \circ T$. It's all a bit tedious to write out in detail, but that's all that's happening with matrix multiplication.

Example A.2.
Compute the following product of a 3×2 matrix A with a 2×4 matrix B,

$$A = \begin{pmatrix} 1 & 1 \\ 3 & -5 \\ -1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 3 & 3 & 4 \\ 1 & -4 & 0 & 2 \end{pmatrix}$$

We compute the product AB one column at a time by multiplying A by the first column of B, then multiplying A by the second column of B, and so on. This gives us the following:

$$\begin{pmatrix} 1 & 1 \\ 3 & -5 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -11 \\ 5 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 3 & -5 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ 29 \\ -15 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 3 & -5 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 3 & -5 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix}$$

We then put these four vectors together as the columns of the product matrix AB,

$$AB = \begin{pmatrix} -1 & -1 & 3 & 6 \\ -11 & 29 & 9 & 2 \\ 5 & -15 & -3 & 2 \end{pmatrix}$$

A.2 Determinants

Every $n \times n$ real matrix determines a special type of function from the set of n-dimensional real vectors, denoted $\mathbb{R}^n$, to itself called a *linear transformation*. This is simply a function which takes n-dimensional vectors and converts them into some (normally different) n-dimensional vector subject to two rules: for every pair of vectors u and v, we have $T(u + v) = T(u) + T(v)$; and for every scalar λ and vector v we have $T(\lambda v) = \lambda T(v)$. It turns out that linear transformations and matrices are essentially two sides of the same coin: each matrix determines a linear transformation, and every linear transformation is determined by a matrix. In order to understand what determinants are, it can be helpful to use this linear transformation interpretation of a matrix.

Properties of Determinants

Given an $n \times n$ matrix A, we will associate to A a number called the ***determinant*** of A and denoted $\det(A)$. We will first give some geometric properties of determinants, and then give some algebraic properties.

Geometric properties

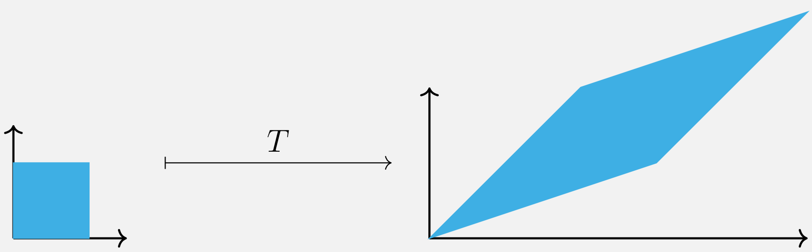
One way to interpret the determinant is as a way of describing the "size" of subsets of $\mathbb{R}^n$, and how a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ changes the size of a subsets.

Example A.3.
In $\mathbb{R}^1$, a linear transformation $T : \mathbb{R}^1 \to \mathbb{R}^1$ is the same thing as multiplication by some fixed number which we'll call A. For example, $T(x) = 3x$ is a linear transformation of $\mathbb{R}^1$. Given any interval, say $I = [2, 4]$ in $\mathbb{R}^1$, the image of I under T is another interval. In this case, $T(I) = [6, 12]$. Notice that the length of $T(I)$ is 6 while the length of I is 2; so applying T stretched out the interval by a factor of 3.

Example A.4.
Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix

$$\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$$

This map takes the unit square and converts it into some parallelogram of area 4.



Notice that the determinant of this matrix (which we had previously defined for 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as $ad - bc$) is 4.

In general, if we have a set $S \subseteq \mathbb{R}^2$ and we apply a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ to each point of S to get a new set $T(S)$. The absolute value of $\det(T)$ is the ratio of the areas of S and $T(S)$:

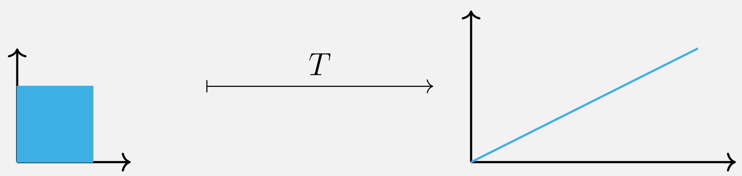
$$\text{area}(T(S)) = |\det(T)| \cdot \text{area}(S).$$

If it happened to be that our linear transformation was not invertible, then intuitively we should expect the linear transformation to "collapse" along certain directions. (If $T(v) = T(w)$, then $T(v - w) = \vec{0}$ and everything parallel to $v - w$ gets sent to the zero vector.) In terms of sizes of regions, this means we can take a set of positive size and send it to something with zero size, and so the determinant of a non-invertible transformation should be zero. In general, if $v \in \ker(T)$, then all multiples of v are in the kernel of T as well, so T collapses all vectors parallel to v.

Example A.5.
Consider the linear transformation in $\mathbb{R}^2$ given by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1/2 \end{pmatrix}$$

Notice that $\begin{pmatrix} 1 & -2 \end{pmatrix}^T$ is in the kernel of this map, and so everything parallel to this vector collapses to zero.



Since the area of the image on the right is zero, the determinant of the linear transformation should be zero.

The same thing holds in higher dimensions: if $S \subseteq \mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$

is a linear transformation, then

$$\text{volume}(T(S)) = |\det(T)| \cdot \text{volume}(S).$$

It is possible for the determinant of a matrix to be negative, so it's important that we use absolute values when we discuss areas or volumes since we don't want to talk about negative area or negative volume.

This idea extends to higher-dimensional spaces as well.

Remark.
Here the "size" of a set depends on what dimension we're talking about. In one dimension, "size" means the arclength of a subset of the real line. In two dimensions, "size" means the area of a subset of the plane. In three dimensions, "size" means volume. To define size in higher dimensions it's helpful if you know some calculus. In particular, we can define the "size" of an set S in $\mathbb{R}^n$ as the integral

$$\iint_S \cdots \iint 1 \, dx_1 \, dx_2 \cdots dx_{n-1} \, dx_n$$

This notion of size is sometimes called *n-dimensional hypervolume*.

Besides telling us how the size of a set changes, we want the determinant to also tell us if a linear transformation "reverses" a set. This is simplest to describe in $\mathbb{R}^1$ and $\mathbb{R}^2$, but the idea extends to higher dimensions.

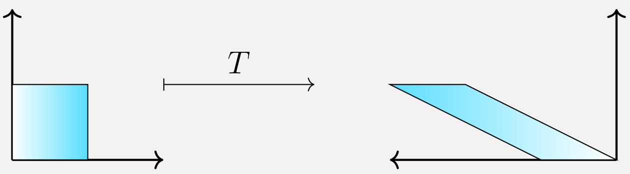
Example A.6.
If $T : \mathbb{R}^1 \to \mathbb{R}^1$ is the linear transformation given by $T(x) = -2x$, then T not only stretches subsets by a factor of two, but it also reverses them. That is, given an interval $I = [a, b]$, its image $T(I) = [-2b, -2a]$ has the "opposite" left- and right-hand sides compared to I: the left-hand side of I became the right-hand side of $T(I)$, and the right-hand side of I became the left-hand side of $T(I)$. In a situation such as this we say that T is ***orientation reversing***. We want the determi-

nant of T to tell us if the map is orientation reversing, $\det(T) < 0$, or ***orientation preserving***, $\det(T) > 0$.

Example A.7.
Consider a map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix

$$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

This map shears sets to the right, which doesn't change the area of the set, but then flips the set over.



Since this map "reverses" a set we again expect its determinant to be negative, which it is: here the map has determinant -1.

Algebraic properties

In order to give some properties that this determinant will satisfy, it will be helpful to think of $\det$ as a function which takes n vectors, all of which are n-dimensional, and converts them into a single real number. That is, we think of $\det$ as a function

$$\det : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$$

When we write $\det(A)$ what we will really mean is

$$\det(a_1, a_2, ..., a_n)$$

where a_1, ..., a_n are the columns of the $n \times n$ matrix A.

By "algebraic properties" of the determinant we mean the rules the determinant should obey when we modify the arguments of the function.

There are only three algebraic properties we need to uniquely determine the determinant:

1. **Linearity**
   Our function $\det$ should be linear in each component. That is, for the i-th argument of $\det$ we should have

$$\det(\text{—}, v + w, \text{—}) = \det(\text{—}, v, \text{—}) + \det(\text{—}, w, \text{—})$$

   and

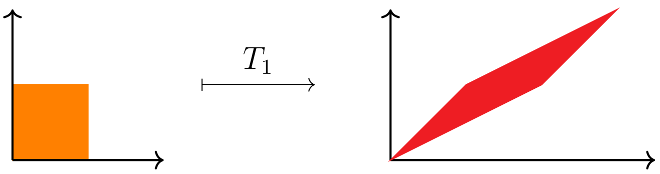
$$\det(\text{—}, \lambda v, \text{—}) = \lambda \det(\text{—}, v, \text{—})$$

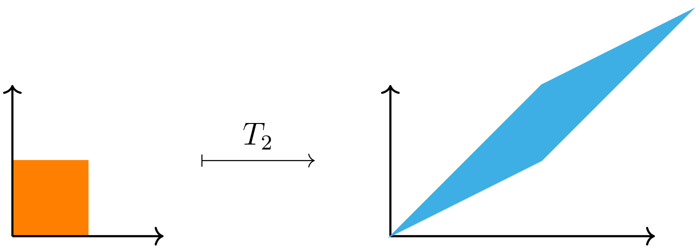
   where the dashes simply mean that the other entries of $\det$ don't matter. (If you want, pretend you've fixed all of the other entries and are only letting the i-th entry change.)

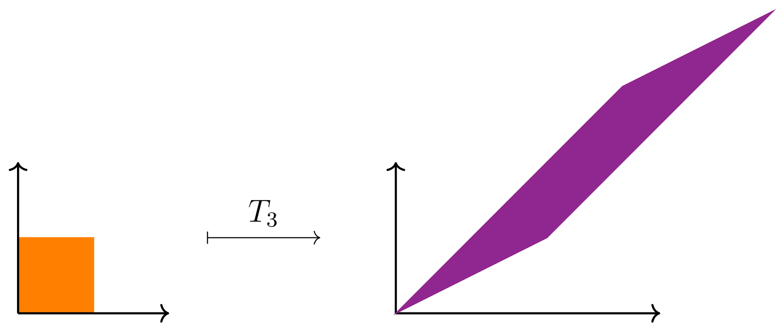
   Geometrically we should expect this property because it's telling us that if we extend a region along some axis, the areas should add. For example, consider three linear transformations $T_1, T_2, T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ with T_i given by the matrix A_i below.

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \quad A_3 = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$$

   Consider how these linear transformation map the unit square to parallelograms:







Notice that the area of the purple parallelogram is the sum of the areas of the red and blue parallelograms.

2. **Alternating**
The determinant is ***alternating*** in each argument. This means that if we swap two arguments, the determinant negates.

$$\det(\text{—}, v, \text{—}, w, \text{—}) = -\det(\text{—}, w, \text{—}, v, \text{—})$$

This is how the determinant is "aware" of whether a linear transformation is orientation preserving or reversing.

3. **Identity**
The identity map id, whose matrix is the identity matrix I, doesn't do anything to sets, and so doesn't change the size of a set or reverse the set, and so we should expect that $\det(\text{id}) = 1$. Interpreting the determinant as a map from $\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}}$ to $\mathbb{R}$, this means we want

$$\det(e_1, e_2, \cdots, e_n) = 1.$$

These three algebraic properties, that the determinant must be linear in each argument, is alternating, and assigns 1 to the identity, are enough to completely specify the determinant. That is, if you were to come up with another map

$$\varphi : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$$

which was linear, alternating, and assigned 1 to the identity, then you would actually have come up with the same map we are about to define. For the sake of completeness we will prove this fact later in these notes, but you can safely ignore this proof if you want.

Computing the Determinant

So far we have described some properties we want our notion of determinant to have, but haven't said how to actually compute the determinant. We will start off by first saying what the determinant of a 1×1 matrix (i.e., a linear transformation $\mathbb{R}^1 \to \mathbb{R}^1$) $A = \begin{pmatrix} a \end{pmatrix}$ is simply

$$\det(A) = a.$$

To compute the determinant of an $n \times n$ matrix we will combine determinants of some $(n-1) \times (n-1)$ submatrices. We will use the following (non-standard) notation. Suppose our matrix A is $n \times n$ with a_{ij} denoting the entry in the i-th row and j-th column,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & & \vdots & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

We will let A_{ij} denote the submatrix of A obtained by deleting the i-th row and j-th column of A (that is, we remove the row and column containing a_{ij}). For example,

$$A_{32} = \begin{pmatrix} a_{11} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{23} & \cdots & a_{2n} \\ a_{41} & a_{43} & \cdots & a_{4n} \\ & & \vdots & \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

The determinant of A_{ij} is called the (i, j) ***minor*** of the matrix and is sometimes denoted

$$M_{ij} = \det(A_{ij}).$$

If we multiply the (i, j) minor by $(-1)^{i+j}$ we have the (i, j) ***cofactor*** of the matrix, sometimes denoted

$$C_{ij} = (-1)^{i+j} M_{ij} = (-1)^{i+j} \det(A_{ij}).$$

The determinant of A is then given by calculating the ***cofactor expansion*** of A along any row or column. The cofactor expansion of A along the i-th row is

$$a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in},$$

and the cofactor expansion of A along the j-th row is

$$a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

(The cofactor expansion is sometimes also referred to as the ***Laplace expansion***.)

Somewhat surprisingly, the cofactor expansion along any row or any column always gives you the same value, and this value is the ***determinant*** of the matrix.

It is important to realize that the cofactors, regardless of what row or column you expand along, alternate between positive and negative. To keep this straight we can rewrite the determinant as

$$\det(A) = \underbrace{\sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det(A_{ij})}_{\text{cofactor expansion using the } i\text{-th row}} = \underbrace{\sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det(A_{ij})}_{\text{cofactor expansion using the } j\text{-th column}}$$

We claim that the function $\det$ defined by cofactor expansion like this satisfies the three algebraic properties above. Before verifying this, let's use the cofactor expansion to evaluate some determinants.

Example A.8.
Consider a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If we do cofactor expansion along the first row we have

$$\begin{aligned}\det(A) =& (-1)^{1+1}a \cdot \det(d) + (-1)^{1+2}b \cdot \det(c) \\ =& ad - bc\end{aligned}$$

Example A.9.

Consider the 3×3 matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

If we perform the cofactor expansion along the third column we have

$$\begin{aligned} &\det(A) \\ =&(-1)^{1+3} \cdot 0 \cdot \det \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \\ &+ (-1)^{2+3} \cdot 2 \cdot \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \\ &+ (-1)^{3+3} \cdot 0 \cdot \det \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \\ =&0 - 2(1-2) + 0 \\ =&2 \end{aligned}$$

Notice that in the previous example we chose to expand along a column that had some zeros in it, and this in turn makes our calculation a little bit simpler: we don't need to bother calculating the determinants that get multiplied by zero!

Exercise A.3.
Show that the determinant of a general 3×3 matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is given by the following formula:

$$\det(A) = aei + bfg + cdh - ceg - bdi - afh.$$

Consider cofactor expansion along the first row:

$$\begin{aligned}
&\det(A)\\
=&a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}\\
=&a(ei - fh) - b(di - fg) + c(dh - eg)\\
=&aei - afh - bdi + bfg + cdh - ceg\\
=&aei + bfg + cdh - ceg - bdi - afh.
\end{aligned}$$

Example A.10.

$$
\begin{aligned}
&\det\begin{pmatrix}1 & 2 & 4 & 1\\ 2 & 1 & 1 & -1\\ 3 & 1 & 2 & 4\\ 1 & -1 & 2 & 2\end{pmatrix}\\
=&1\det\begin{pmatrix}1 & 1 & -1\\ 1 & 2 & 4\\ -1 & 2 & 2\end{pmatrix} - 2\det\begin{pmatrix}2 & 1 & -1\\ 3 & 2 & 4\\ 1 & 2 & 2\end{pmatrix}\\
&+4\det\begin{pmatrix}2 & 1 & -1\\ 3 & 1 & 4\\ 1 & -1 & 2\end{pmatrix} - 1\det\begin{pmatrix}2 & 1 & 1\\ 3 & 1 & 2\\ 1 & -1 & 2\end{pmatrix}\\
=&1\cdot\left(1\cdot\det\begin{pmatrix}2 & 4\\ 2 & 2\end{pmatrix} - 1\cdot\det\begin{pmatrix}1 & 4\\ -1 & 2\end{pmatrix} + (-1)\cdot\det\begin{pmatrix}1 & 2\\ -1 & 2\end{pmatrix}\right)\\
&-2\cdot\left(2\cdot\det\begin{pmatrix}2 & 4\\ 2 & 2\end{pmatrix} - 1\cdot\det\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix} + 1\cdot\det\begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}\right)\\
&+4\cdot\left(2\cdot\det\begin{pmatrix}1 & 4\\ -1 & 2\end{pmatrix} - 1\cdot\det\begin{pmatrix}3 & 4\\ 1 & 2\end{pmatrix} + (-1)\cdot\det\begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}\right)\\
&-1\cdot\left(2\cdot\det\begin{pmatrix}1 & 2\\ -1 & 2\end{pmatrix} - 1\cdot\det\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix} + 1\cdot\det\begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}\right)\\
=&1\cdot(1\cdot(2\cdot 2-4\cdot 2) - 1\cdot(1\cdot 2-4\cdot(-1)) + (-1)\cdot(1\cdot 2-2\cdot(-1)))\\
&-2\cdot(2\cdot(2\cdot 2-4\cdot 2) - 1\cdot(1\cdot 2-4\cdot 1) + (-1)\cdot(3\cdot 2-2\cdot 1))\\
&+4\cdot(2\cdot(1\cdot 2-4\cdot(-1)) - 1\cdot(3\cdot 2-4\cdot 1) + (-1)(3\cdot(-1)-1\cdot 1))\\
&-1\cdot(2\cdot(1\cdot 2-2\cdot(-1)) - 1\cdot(3\cdot 2-2\cdot 1) + 1\cdot(3\cdot(-1)-1\cdot 1))\\
=&1\cdot(1\cdot(-4) - 1\cdot 6 - 1\cdot 4)\\
&-2\cdot(2\cdot(-4) - 1\cdot(-2) - 1\cdot 4)\\
&+4\cdot(2\cdot 6 - 1\cdot 2 - 1\cdot(-4))\\
&-1\cdot(2\cdot 4 - 1\cdot 4 + 1\cdot(-4))\\
=&1\cdot(-14) - 2\cdot(-14) + 4\cdot(14) - 1\cdot 0\\
=&-14+28+56-0\\
=&70
\end{aligned}
$$

A.3 Cramer's Rule

Cramer's rule is a method for describing the solution to a system $Ax = b$, provided A is invertible, in terms of determinants. In order to state Cramer's rule we need one bit of notation. Given an $n \times n$ matrix A and an n-dimensional vector b, let $A_i(b)$ denote the matrix obtained by replacing the i-th column of A with b.

Example A.11.
Let A and b be the following:

$$A = \begin{pmatrix} 6 & 2 & 1 & 1 \\ 3 & 7 & 8 & 2 \\ 4 & 1 & 3 & 3 \\ 2 & 2 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 1 \end{pmatrix}$$

Then

$$A_3(b) = \begin{pmatrix} 6 & 2 & 4 & 1 \\ 3 & 7 & 2 & 2 \\ 4 & 1 & 3 & 3 \\ 2 & 2 & 1 & 1 \end{pmatrix}$$

Theorem A.4 (Cramer's Rule)**.**
If A is an invertible $n \times n$ matrix, and b is an n-dimensional vector, then the unique solution to $Ax = b$ has components

$$x_i = \frac{\det(A_i(b))}{\det(A)}$$

Proof.
Notice that if we consider the matrix obtained by replacing the i-th column of the identity matrix I with x, $I_i(x)$, then the determinant of this matrix is simply x_i. To see this, perform cofactor expansion

along the i-th row of $I_i(x)$: all entries in the i-th row are zero for the entry coming from the column which was replaced by x which has x_i in this location: the i-th row and i-th column. The corresponding (i,i)-minor is simply the $(n-1)\times(n-1)$ identity matrix which has determinant 1, and so we have that the cofactor (and hence the entire determinant) is

$$(-1)^{i+i} x_i \det(I) = x_i.$$

Note too that

$$A \cdot I_i(x) = \begin{pmatrix} Ae_1 & Ae_2 & \cdots & Ax & \cdots & Ae_n \end{pmatrix}$$

So if x solves the equation $Ax = b$ (since A is invertible there is exactly one vector solving the equation) this becomes

$$A \cdot I_i(x) = \begin{pmatrix} Ae_1 & Ae_2 & \cdots & b & \cdots & Ae_n \end{pmatrix}$$

But notice that Ae_j is the j-th column of A, which we'll denote a_j, and so

$$A \cdot I_i(x) = \begin{pmatrix} a_1 & a_2 & \cdots & b & \cdots & a_n \end{pmatrix} = A_i(b)$$

If we take the determinant of both sides of the equation we have

$$\begin{aligned} &\det(A \cdot I_i(x)) = \det(A_i(b) \\ \implies &\det(A) \cdot \det(I_i(x)) = \det(A_i(b)) \\ \implies &\det(A) \cdot x_i = \det(A_i(b)) \\ \implies &x_i = \frac{\det(A_i(b))}{\det(A)} \end{aligned}$$

□

Cramer's rule is generally not a very efficient way to solve large systems of equations, but can sometimes be helpful for theoretical situations (e.g., in proving some theorem it might be helpful to have a way to express the components of a solution, and Cramer's rule allows us to do precisely that).

Example A.12.
Use Cramer's rule to solve the following system

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 4 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

First note that the determinant of our matrix is

$$\det \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 4 \\ 2 & 0 & 1 \end{pmatrix} = 21$$

By Cramer's rule,

$$x = \frac{1}{21} \det \begin{pmatrix} 0 & 3 & 2 \\ 0 & 1 & 4 \\ 2 & 0 & 1 \end{pmatrix} = \frac{20}{21}$$

$$y = \frac{1}{21} \det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 2 & 1 \end{pmatrix} = \frac{-8}{21}$$

$$z = \frac{1}{21} \det \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix} = \frac{2}{21}$$

Thus our system is solved by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 20/21 \\ -8/21 \\ 2/21 \end{pmatrix}$$

A.4 Eigenvectors and Eigenvalues

Introduction

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, we should typically expect that T will move the vectors around in $\mathbb{R}^n$ in a somewhat complicated way. However, it may happen that some vectors only get stretched out (multiplied by a scalar). For example, consider the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix}.$$

It's not immediately obvious, but it's easy to check that this matrix leaves the vector

$$u = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

alone; that is, $Au = u$:

$$Au = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = u.$$

Similarly, the vector

$$v = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

is simply negated; that is, $Av = -v$:

$$Av = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = -v$$

And finally, the vector

$$w = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix}$$

gets multiplied by a factor of 4:

$$\begin{aligned}
Aw =& \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} \\
=& 5 \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \\
=& \begin{pmatrix} 20 \\ 16 \\ 4 \end{pmatrix} \\
=& 4 \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} \\
=& 4w.
\end{aligned}$$

At first glance this may not seem like the most useful observation, but notice that these vectors u, v, and w form a basis for $\mathbb{R}^3$. Thus every vector can be written as some linear combination of these vectors,

$$\alpha u + \beta v + \gamma w.$$

It is now extremely easy to describe how our linear transformation acts on these vectors:

$$A(\alpha u + \beta v + \gamma w) = \alpha u - \beta v + 4\gamma w.$$

That is, if we write our vector as

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_{\mathcal{B}}$$

then

$$T \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} \alpha \\ -\beta \\ 4\gamma \end{pmatrix}_{\mathcal{B}}.$$

Thus having vectors which are simply stretched out by the linear transformation can make it extremely easy to describe the linear transformation, and this can make studying linear transformations we may have interest in considerably easier.

Eigenvectors and Eigenvalues

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, we say that a vector $v \neq 0$ is an ***eigenvector*** of T with ***eigenvalue*** λ if v and λ satisfy the following equation:

$$T(v) = \lambda v.$$

That is, eigenvectors are precisely the vectors that simply get stretched out by T and the eigenvalue tells us how much the vector gets stretched out.

Exercise A.4.
Notice that eigenvectors are by definition never zero, but eigenvalues *are* allowed to be zero. Show that a linear transformation will have zero as an eigenvalue if and only if the linear transformation is not injective. Notice that non-zero $v \in \ker(T)$ is an eigenvector with eigenvalue 0 as $T(v) = 0 = 0 \cdot v$. Thus if T is not injective, and so there are non-zero elements of the kernel, then T has 0 as an eigenvalue. If T has eigenvalue 0, then by definition this means there exists a non-zero vector v such that $T(v) = 0 \cdot v = 0$, so $\ker(T)$ contains non-zero elements, and T is not injective.

Remark.
The word *eigen* is an adjective in German that means something like "owned by." So the eigenvectors and eigenvalues are the vectors and scalars "owned" by the linear transformation.

The vectors u, v, and w from the example in the introduction are thus eigenvectors of the linear transformation with matrix

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

while 1, -1, and 4 are the respective eigenvalues.

Perhaps that first thing to notice about eigenvectors and eigenvalues is that they are somewhat special: not every linear transformation will have eigenvector and eigenvalues.

Example A.13.
The linear transformation in $\mathbb{R}^2$ with matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

does not have any eigenvectors/eigenvalues. The easiest way to see this is to think geometrically: this matrix acts on the plane $\mathbb{R}^2$ by $90°$ rotations, and no non-zero vector in the plane is simply stretched out by a $90°$ rotation.

The other thing to notice about eigenvectors is that they come in families. For example, the vector

$$w = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix}$$

from before is an eigenvector with eigenvalue 4 of the matrix

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix}.$$

Notice that any scalar multiple λw is also an eigenvector with eigenvalue 4: As $Aw = 4w$ we have $A\lambda w = \lambda Aw = \lambda 4w = 4\lambda w$. The collection of all eigenvectors of a given eigenvalue forms a subspace of $\mathbb{R}^n$ called the ***eigenspace*** of T with the given eigenvalue.

Lemma A.5.
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and suppose λ is an eigenvalue of T. The set of all eigenvectors of T with eigenvalue λ is a subspace of $\mathbb{R}^n$, provided this collection of eigenvectors is not empty (i.e., that λ really is an

eigenvalue).

Proof.
The set of all eigenvectors with eigenvalue λ is precisely the set of all vectors $v \in \mathbb{R}^n$ satisfying $T(v) = \lambda v$. Let's momentarily denote this set U:

$$U = \left\{ v \in \mathbb{R}^n \,\middle|\, T(v) = \lambda v \right\}.$$

Since we're already assuming U is non-empty, we need to show that U closed under vector addition, and closed under scalar multiplication. Suppose $u, u' \in U$ and μ is any scalar.

Checking that U is closed under vector addition is easy:

$$T(u + u') = T(u) + T(u') = \lambda u + \lambda u' = \lambda(u + u'),$$

as is checking that U is closed under scalar multiplication:

$$T(\mu u) = \mu T(u) = \mu \lambda u = \lambda \cdot (\mu u).$$

Thus U, the eigenspace of vectors in $\mathbb{R}^n$ which are eigenvectors of T with eigenvalue λ, is a subspace of $\mathbb{R}^n$. □

Computing Eigenvectors and Eigenvalues

The question now is how do we go about finding the eigenvectors and eigenvalues of a matrix. This is a two-step process: first we have to find the eigenvalues λ, and then for each eigenvalue we need to find the associated eigenvectors.

If we want to find eigenvalues of T, then we need to find the scalars λ for which there is a solution v to the equation $T(v) = \lambda v$. For simplicity, let's suppose our linear transformation has domain $\mathbb{R}^n$ and codomain $\mathbb{R}^n$ so that we can represent T by an $n \times n$ matrix A. We then want to find the scalars λ for which there is a solution to

$$Av = \lambda v.$$

Equivalently, we want to find the λ's for which there is a solution to

$$Av - \lambda v = 0.$$

We can rewrite $Av - \lambda v$ as $(A - \lambda I)v$: just distribute the v and notice that λI is the matrix with all zeros except for λ's on the diagonal, thus $(\lambda I)v = \lambda v$. So we want to find the λ's for which there is a non-zero v solving

$$(A - \lambda I)v = 0.$$

Since $A - \lambda I$ is an $n \times n$ matrix, this equation has a non-zero solution precisely when $A - \lambda I$ is not invertible: i.e., there is a non-zero solution exactly when $\det(A - \lambda I) = 0$. Long story short, we have the following:

Proposition A.6.
A scalar λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Example A.14.
Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$$

We want to find the values of λ for which

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\left(\begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix}\right)$$

is zero.

$$\begin{aligned}
\det\left(\begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix}\right) &= 0 \\
\implies (2-\lambda)(1-\lambda) - 6 &= 0 \\
\implies 2 - 3\lambda + \lambda^2 - 6 &= 0 \\
\implies \lambda^2 - 3\lambda - 4 &= 0 \\
\implies (\lambda-4)(\lambda+1) &= 0
\end{aligned}$$

Thus our eigenvalues are $\lambda = 4$ and $\lambda = -1$.

Once the eigenvalues of A are known, we can then search for the eigenvectors.

Example A.15.
Find the eigenvectors associated with eigenvalue $\lambda = 4$ for the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}.$$

We are trying to find the solutions to $Av = 4v$, or equivalently $(A - 4I)v = 0$. That is, we want to find the solutions to the homogeneous system

$$\begin{pmatrix} 2-4 & 3 \\ 2 & 1-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Putting the coefficient matrix into RREF we have

$$\begin{pmatrix} 1 & -3/2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus the eigenspace of this matrix, associated with the eigenvector 4, is

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \, x - \frac{3}{2}y = 0 \right\}$$

Exercise A.5.
Find the eigenvectors associated with eigenvalue $\lambda = -1$ for the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}.$$

We want to find the vectors v solving $Av = -v$ which we can rewrite as $(A + I)v = 0$. This means we are trying to solve the system

$$(A + I)v = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Putting the matrix in RREF gives

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

and so the system is solve when $x = -y$. Hence the space of eigenvectors for this matrix with eigenvalue -1 is

$$\left\{ \begin{pmatrix} y \\ -y \end{pmatrix} \mid y \in \mathbb{R} \right\}$$

Perhaps unsurprisingly (since eigenvalues are related to determinants), eigenvalues for triangular matrices are very easy to compute.

Theorem A.7.
If A is a triangular matrix, then the eigenvalues of A are the entries on the diagonal.

Proof.
Suppose that A is a triangular matrix with diagonal entries a_{11}, a_{22}, ..., a_{nn}. Then $A - \lambda I$ is also a triangular matrix, but with diagonal

entries $a_{11} - \lambda$, $a_{22} - \lambda$, ..., $a_{nn} - \lambda$. Thus

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda).$$

This will only be zero when one of the factors is zero, that is when λ equals a_{ii} for some diagonal entry a_{ii}. □

Example A.16.
The eigenvalues of

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

are 3, -1, 0, 5, and 3.

Example A.17.
Find the eigenvalues and associated eigenvectors of the following matrix:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thanks to the previous theorem we can easily determine that the eigenvalues are 2 and 1. Now we simply need to find the associated eigenvectors.

For the eigenvalue $\lambda = 2$ we need to solve the equation $(A - 2I)v = 0$,

$$\begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Putting the matrix into RREF, this system is equivalent to

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus the eigenspace associated to 2 is

$$\left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$$

For the eigenvalue $\lambda = 1$ we need to solve the equation $(A - I)v = 0$,

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This is equivalent to solving the system

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and so the eigenspace associated to 1 is

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x = -2z, y = -z, z \in \mathbb{R} \right\}$$

In our motivating example at the start of the lecture, notice that the eigenvectors for

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

formed a basis for $\mathbb{R}^3$. Even though this doesn't happen in Example A.16, notice that the eigenvectors associated to different eigenvalues are linearly independent. This is true in general.

Theorem A.8.
If v_1, v_2, ..., v_m are eigenvectors associated with distinct eigenvalues λ_1, λ_2, ..., λ_m of A, then $\{v_1, v_2, ..., v_m\}$ is a linearly independent set.

Proof.
Suppose instead that $\{v_1, v_2, ..., v_m\}$ is linearly dependent. Rearranging the order of eigenvectors and eigenvalues if necessary, we may assume that $\{v_1, v_2, ..., v_r\}$ is linearly independent (it could be that $r = 1$) and $\{v_1, v_2, ..., v_r, v_{r+1}\}$ is linearly dependent. That is, there exist scalars μ_1 through μ_r such that

$$v_{r+1} = \mu_1 v_1 + \cdots + \mu_r v_r.$$

If we apply A to both sides of the equation we have

$$\begin{aligned} A v_{r+1} &= A(\mu_1 v_1 + \cdots + \mu_r v_r) \\ \implies \lambda_{r+1} v_{r+1} &= \mu_1 \lambda_1 v_1 + \cdots + \mu_r \lambda_r v_r \end{aligned}$$

But notice that if we multiply both sides of

$$v_{r+1} = \mu_1 v_1 + \cdots + \mu_r v_r$$

by λ_{r+1} we have

$$\lambda_{r+1} v_{r+1} = \mu_1 \lambda_{r+1} v_1 + \cdots + \mu_r \lambda_{r+1} v_r.$$

Thus

$$\mu_1 \lambda_1 v_1 + \cdots + \mu_r \lambda_r v_r = \mu_1 \lambda_{r+1} v_1 + \cdots + \mu_r \lambda_{r+1} v_r.$$

Subtracting the right-hand side from the left-hand side gives

$$\mu_1(\lambda_1 - \lambda_{r+1}) v_1 + \cdots + \mu_r(\lambda_r - \lambda_{r+1}) v_r = 0.$$

But this is a contradiction since $\{v_1, ..., v_r\}$ is a linearly independent set. □

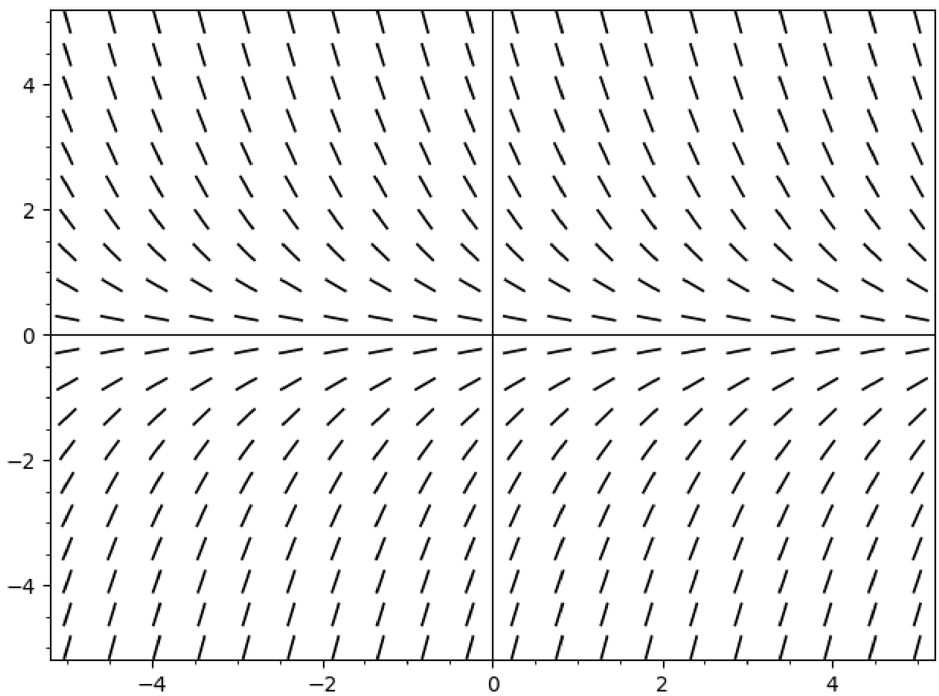
B

Solutions to Practice Problems

B.1 Chapter 1

1.1 (a)

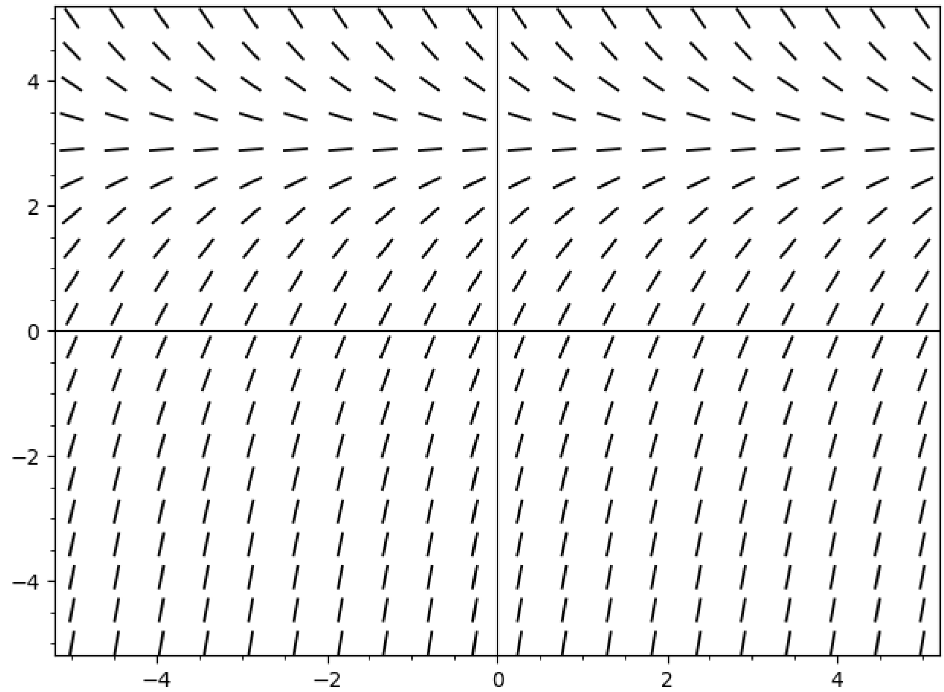
```
x,y = var('x,y')
plot_slope_field(-y, (x, -5, 5), (y, -5, 5))
```



As x goes to infinity, all solutions approach $y = 0$.

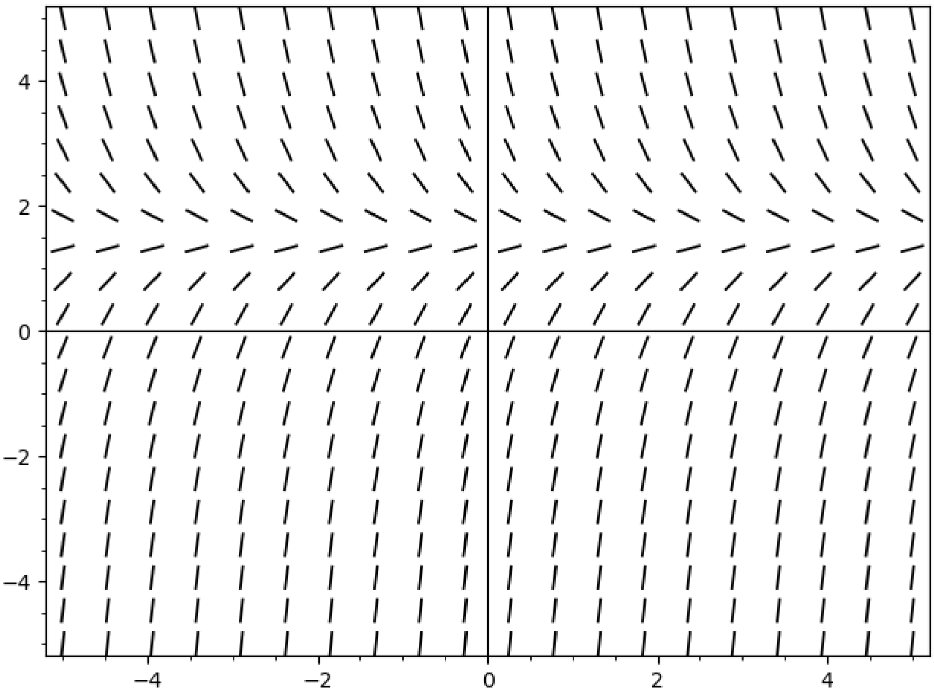
(b)

```
x,y = var('x,y')
plot_slope_field(-y + 3, (x, -5, 5), (y, -5, 5))
```



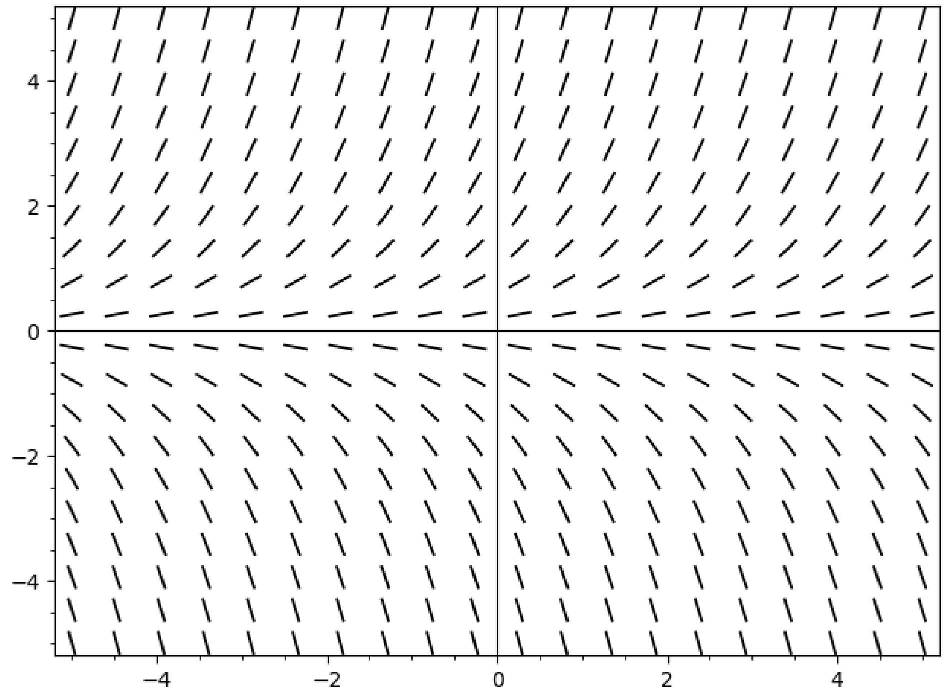
As x goes to infinity, all solutions approach $y = 3$.

(c)
```
x,y = var('x,y')
plot_slope_field(-2*y + 3, (x, -5, 5), (y, -5, 5))
```



As x goes to infinity, all solutions approach $y = {}^{3}/_{2}$.

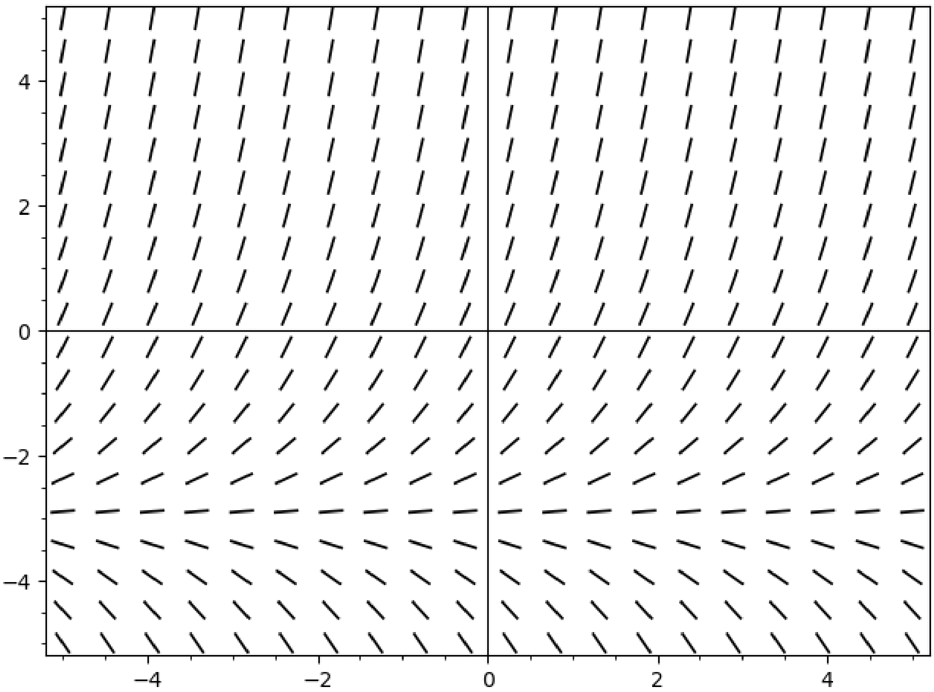
(d)
```
x,y = var('x,y')
plot_slope_field(y, (x, -5, 5), (y, -5, 5))
```



Here there are three possibilities, although one is slightly subtle. It is easy to see that as if a solution passes through a point above the x-axis, then that solution will go to infinity as x goes to infinity. If a solution passed through a point below the x-axis, then that solution will go to negative infinity as x goes to infinity. The third possibility is that if the solution passes through the x-axis, then that solution will necessarily be zero for all values of x. (I.e., the constant function $y = 0$ is a solution to the equation, as is easily checked.)

(e)

```
x,y = var('x,y')
plot_slope_field(y + 3, (x, -5, 5), (y, -5, 5))
```



Here solutions which pass through a point above $y = -3$ will move to infinity as x increases; solutions passing through a point below $y = -3$ will approach negative infinity as x increases; and $y = -3$ is also a solution.

It may be surprising that $y = -3$ is the solution instead of $y = 3$, but let's notice this solution is special. This is an *equilibrium* solution because it is always the same value, it's always constant. If we have an equilibrium solution, then that means the rate of change (derivative) is zero. As $\frac{dy}{dx} = y + 3$, if we have $\frac{dy}{dx} = 0$, then that means $y + 3 = 0$ and so $y = -3$.

1.2 (a) Notice that in particular $y = 2$ would be a solution to this differential equation. This would necessarily be an equilibrium solution, and so we would require that $my+b$ satisfy $my+b = 0 \implies y = 2$. There are two ways we could choose m and b for this to happen: we could have $m = 1$ and $b = -2$ and so the equation becomes $\frac{dy}{dx} = y-2$, or we could have $m = -1$ and $b = 2$ and the equation is $\frac{dy}{dx} = -y+2$. One of these will correspond to our solutions converging towards 2, and the other will correspond to our solutions diverging away from 2. For the purposes of this lab we can distinguish these just by plotting the slope fields and seeing which is which. A slightly more analytic approach would be to notice we need for solutions about the line $y = 2$ to decrease, so their derivatives are negative, and this will tell us we want the equation $\frac{dy}{dx} = -y + 2$.

(b) This is essentially the same as part (a), but with $\frac{3}{4}$ in place of 2. Our solution is thus $\frac{dy}{dx} = -y + 3/4$. (We could also use $\frac{dy}{dx} = -4y + 3$.)

(c) By the same logic as in part (a), the solution is $\frac{dy}{dx} = y + \frac{2}{5}$.

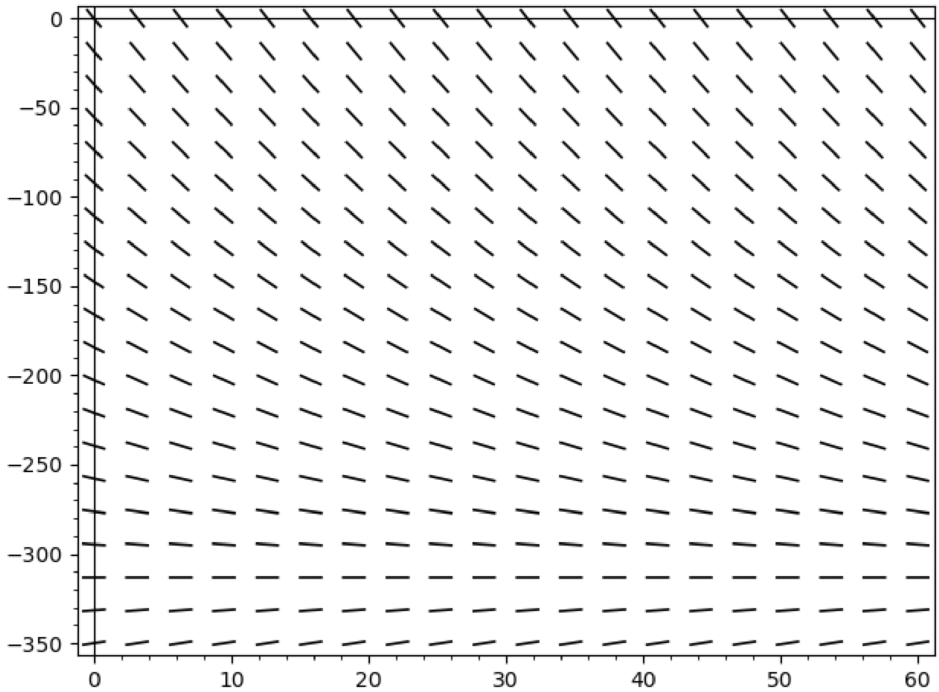
1.3 (a) The forces acting on the object are its weight pulling it down, $W = -9.8 \cdot 15$ (this is the object's mass times acceleration due to gravity), and the drag which resists the direction of motion, $D = -0.47v$. Notice the negative sign here is so that drag will point in the opposite direction of the velocity (when v is negative, D will be positive). By Newton we know force equals mass times acceleration, and since acceleration is the derivative of velocity we have

$$15\frac{dv}{dt} = -9.8 \cdot 15 - 0.47v$$

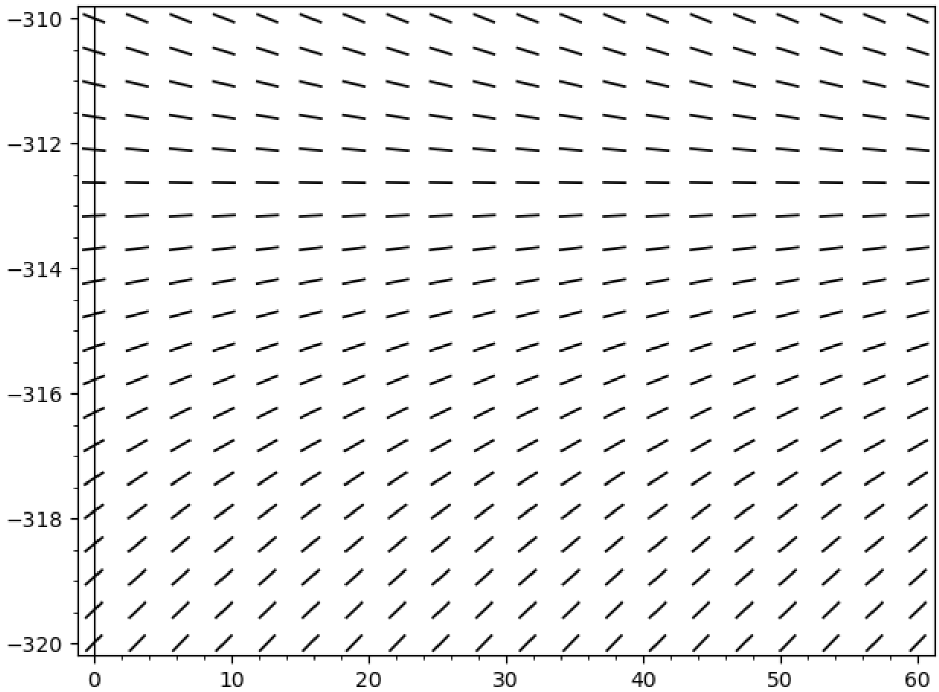
or

$$\frac{dv}{dt} = -9.8 - \frac{0.47}{15}v.$$

(b) Our slope field is



To get a better estimate of the terminal velocity we can "zoom in" on the slope field by altering out window to obtain



and from this we may guess that the terminal velocity is roughly $-312.5\frac{\mathrm{m}}{\mathrm{s}}$.

1.4 (a) This is a linear first-order equation. Notice that even though an x^2 term appears, both y and $\frac{dy}{dx}$ are only raised to the first power, and they are not multiplied together, which is why this is considered linear. The equation is first order since only a first derivative appears.

(b) This is a non-linear second-order equations. Because y and $\frac{dy}{dx}$ are multiplied together, the equation is not linear. It is second order since a second order derivative appears.

(c) This is a non-linear second-order equation. As in part (b) this is non-linear since y and $\frac{dy}{dx}$ are multiplied together. It is second-order since second derivatives appear.

(d) This is a linear third-order equation. It is third order since we have a third derivative, and it is linear because y and its derivatives are not raised to powers or multiplied together.

(e) This is a linear second-order equation. This is linear because y and its derivatives are not multiplied together or raised to powers, and it is second order because a second derivative is involved.

1.5 (a) There are a few different ways to solve this equation, but for now we are mainly just focusing on the following fact:

$$\frac{dy}{dx} = my + b$$
$$\implies y = Ce^{mx} - \frac{b}{m}$$

In this case $m = -1$ and $b = 5$ so we have

$$y = Ce^{-x} + 5.$$

(b) As in part (a) we are simply using the fact that we have seen that $y = Ce^{mx} - \frac{b}{m}$ solves $\frac{dy}{dx} = my + b$. Here $m = 2$ and $b = -3$ and so this differential equation is solved by

$$y = Ce^{2x} + \frac{3}{2}.$$

(c) As in parts (a) and (b) we are simply applying our formula for these particularly simple differential equations. For this one, though, we need to do the tiniest bit of algebra and rewrite the equation before we can use our formula. Subtracting $3y$ from both sides of the equation and multiplying through by -1 the equation becomes

$$3y - \frac{dy}{dx} = 4y + 1$$
$$\implies -\frac{dy}{dx} = y + 1$$
$$\implies \frac{dy}{dx} = -y - 1$$

and so our equation is solved by

$$y = Ce^{-x} - 1.$$

B.2 Chapter 2

2.1 (a) This is a linear differential equation and so we may solve it using the method of integrating factors. In particular, we seek a function $\mu(x)$ so that after multiplying both sides of the equation by $\mu(x)$, our equation becomes

$$\mu(x)\frac{dy}{dx} + \mu(x)x^2 y = 0$$

where the left-hand side will be given by a product rule,

$$\frac{d}{dx}\mu(x)y = \mu(x)\frac{dy}{dx} + \mu(x)x^2y.$$

When we apply the product rule to differentiate $\mu(x)y$, however, we obtain

$$\frac{d}{dx}\mu(x)y = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y.$$

Equating the right-hand sides of the two equations above (since they both equal $\frac{d}{dx}\mu(x)y$, we have

$$\begin{aligned}
&\mu(x)\frac{dy}{dx} + \mu(x)x^2y = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y \\
\implies &\mu(x)x^2y = \frac{d\mu}{dx}y \\
\implies &\mu(x)x^2 = \frac{d\mu}{dx}.
\end{aligned}$$

This equation we can now solve as follows:

$$\begin{aligned}
&\frac{d\mu}{dx} = x^2\mu \\
\implies &\frac{1}{\mu}\frac{d\mu}{dx} = x^2 \\
\implies &\int \frac{1}{\mu}\frac{d\mu}{dx}dx = \int x^2\,dx \\
\implies &\ln|\mu| = \frac{x^3}{3} + C \\
\implies &|\mu| = e^{x^3/3+C} = Ce^{x^3/3} \\
\implies &\mu = \pm Ce^{x^3/3} = Ce^{x^3/3}.
\end{aligned}$$

Any choice of C will give us a solution to our equation $\frac{d\mu}{dx} = x^2\mu$, so we will take $C = 1$. Thus we want to multiply our original equation through by $e^{x^3/3}$. This gives us the following:

$$\begin{aligned}
&\frac{dy}{dx} + x^2 y = 0 \\
\Longrightarrow\ & e^{x^3/3}\frac{dy}{dx} + e^{x^3/3}x^2 y = 0 \\
\Longrightarrow\ & \frac{d}{dx} e^{x^3/3} y = 0 \\
\Longrightarrow\ & \int \frac{d}{dx} e^{x^3/3} y \, dx = \int 0 \, dx \\
\Longrightarrow\ & e^{x^3/3} y = C \\
\Longrightarrow\ & y = Ce^{-x^3/3}
\end{aligned}$$

(b) Once again we will need to multiply through by an integrating factor, which in this problem will be

$$\mu = e^{\int (-2)\, dx} = e^{-2x}.$$

After multiplying *both* sides of the equation by μ we have

$$e^{-2x}\frac{dy}{dx} - 2e^{-2x} y = 4e^{-2x} - xe^{-2x}.$$

By the product rule we may rewrite this as

$$\frac{d}{dx} e^{-2x} y = 4e^{-2x} - xe^{-2x}.$$

Now we integrate both sides of the equation with respect to x:

$$\int \frac{d}{dx} e^{-2x} y \, dx = \int \left(4e^{-2x} - xe^{-2x}\right) \, dx.$$

On the left-hand side we are finding the antiderivative of a derivative, so those operations cancel out and the left-hand side is just $e^{-2x}y$. For the right-hand side we break the integral in two:

$$\int \left(4e^{-2x} - xe^{-2x}\right) \, dx = \int 4e^{-2x} \, dx - \int xe^{-2x} \, dx.$$

The first term is easily computed using u-substitution: using $u = -2x$, $du = -2dx$, we have $\int 4e^{-2x} \, dx = -2e^{-2x}$. (We will suppress the "$+C$" in our integrals since they all get combined into a single C at the end.)

The second integral requires integration by parts. Using

$$u = x \qquad dv = e^{-2x}\,dx$$
$$du = dx \qquad v = \frac{-1}{2}e^{-2x}$$

the integral becomes

$$\frac{-xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\,dx = \frac{-xe^{-2x}}{2} - \frac{1}{4}e^{-2x} = \frac{-e^{-2x}}{4}(2x+1)$$

Plugging these integrals into the right-hand side of our calculation above gives us

$$e^{-2x}y = -2e^{-2x} + \frac{e^{-2x}}{4}(2x+1) + C,$$

and solving for y we have

$$y = -2 + \frac{2x+1}{4} + Ce^{2x} = \frac{2x-7}{4} + Ce^{2x}$$

2.2 (a) Since this equation has the form $\frac{dy}{dx} = my + b$, with $m = 3$ and $b = 0$, we know it is solved by

$$y = Ce^{mx} - \frac{b}{m} = Ce^{3x}.$$

To determine the value of C we must use the initial condition $y(0) = 2$. This tells us

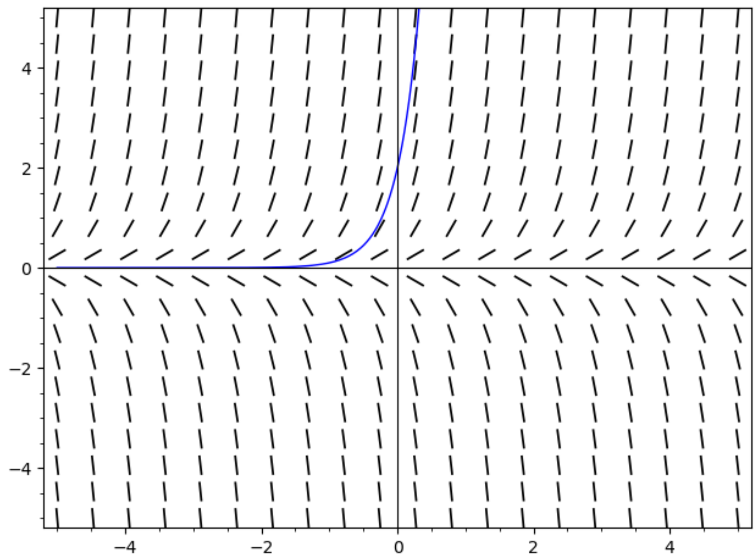
$$2 = y(0) = Ce^{3\cdot 0} = C$$

and so $C = 2$ and our initial value problem is solved by $y = 2e^{3x}$.

We can plot the slope field together with the curve $y = 2e^{3x}$ in Sage with the following commands.

```
x,y = var('x,y')
img = plot_slope_field(3*y, (x, -5, 5), (y, -5, 5))
img += plot(2*e^(3*x), (x, -5, 5))
img.show(ymax = 5)
```

(The `ymax = 5` argument in the last line just makes sure our plot gets cut off at $y = 5$, which is convenient to use because e^{3x} grows so quickly it makes the slope field hard to see.) This produces the following plot:



(b) The general solution to this equation is given by

$$y = Ce^{-x} + 7.$$

We use the fact $y(1) = -2$ to determine C:

$$-2 = y(1) = Ce^{-1} + 7 \implies -9 = Ce^{-1} \implies C = -9e$$

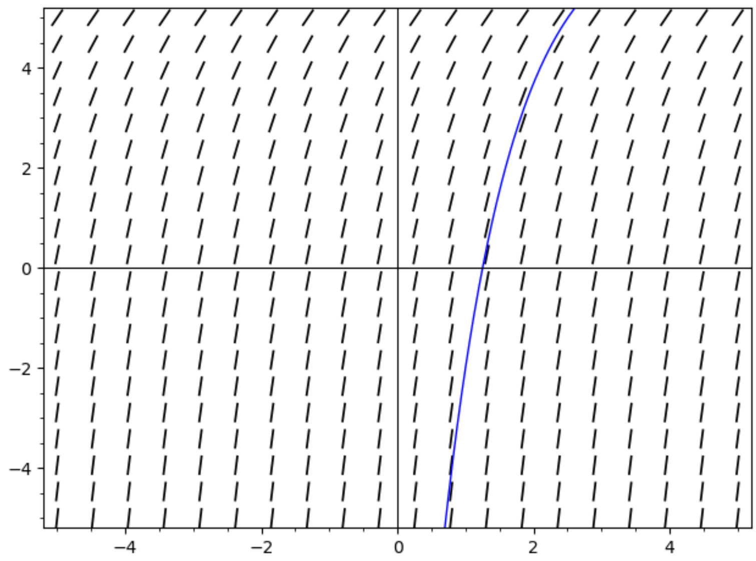
and so our IVP is solved by

$$-9e^{1-x} + 7$$

Our slope field together with this solution can be plotted in Sage with

```
x,y = var('x,y')
img = plot_slope_field(-y + 7, (x, -5, 5), (y, -5, 5))
img += plot(-9*e^(1-x)+7, (x, -5, 5))
img.show(ymin = -5, ymax=5)
```

This produces the following plot:



2.3 Our differential equation is solved by

$$b(t) = Ce^{t/2} + 900$$

where $b(0) = b_0 = C + 900$ and so $C = b_0 - 900$ and we may write

$$b(t) = (b_0 - 900)e^{t/2} + 900.$$

We want to know when $b(t) = 0$:

$$\begin{aligned} & b(t) = 0 \\ \implies & (b_0 - 900)e^{t/2} + 900 = 0 \\ \implies & e^{t/2} = \frac{-900}{b_0 - 900} = \frac{900}{900 - b_0} \\ \implies & t/2 = \ln(900) - \ln(900 - b_0) \\ \implies & t = 2\left(\ln(900) - \ln(900 - b_0)\right) \end{aligned}$$

2.4 (a) We may rewrite this as

$$y\frac{dy}{dx} = x^2.$$

Integrating both sides gives us

$$\begin{aligned} & \int y\,dy = \int x^2\,dx \\ \implies & \frac{y^2}{2} = \frac{x^3}{3} + C \\ \implies & y^2 = \frac{2x^3}{3} + C. \end{aligned}$$

To obtain an explicit solution we would need to make a choice of positive or negative square root, but we can't make such a choice without an initial condition.

(b) We write the equation as

$$(3+2y)\frac{dy}{dx} = 3x^2 - 1$$

and integrate both sides:

$$\begin{aligned} \int (3+2y)\,dy &= \int (3x^2-1)\,dx \\ \implies 3y + y^2 &= x^3 - x + C \end{aligned}$$

(c)

$$\begin{aligned} \frac{dy}{dx} &= \frac{4-2x}{3y^2-5} \\ \implies (3y^2-5)\frac{dy}{dx} &= 4-2x \\ \implies \int (3y^2-5)\,dy &= \int (4-2x)\,dx \\ \implies y^3 - 5y &= 4x - x^2 + C \end{aligned}$$

2.5 (a) Dividing the y to the other side of the equation and integrating both sides gives us

$$\ln|y| = x^2 + C$$

and so $y = Ce^{x^2}$. Plugging in our initial conditions this becomes $2e = C$ and so the IVP is solved by

$$y = 2e \cdot e^x = 2e^{x+1}.$$

(b) We first write the equation as

$$\frac{dy}{dx} = 2y - 1$$

which we may further write as

$$\frac{1}{2y-1}\frac{dy}{dx} = 1.$$

Now we integrate both sides. The right-hand side is simply $x + C$, and the left-hand side becomes $\frac{1}{2}\ln|2y-1|$. To get an explicit solution let us rewrite this as follows:

$$\begin{aligned}
&\frac{1}{2}\ln|2y-1| = x + C \\
\implies &\ln|2y-1| = 2x + C \\
\implies &|2y-1| = e^{2x+C} = Ce^{2x} \\
\implies &2y - 1 = \pm Ce^{2x} = Ce^{2x} \\
\implies &2y = Ce^{2x} + 1 \\
\implies &y = Ce^{2x} + \frac{1}{2}.
\end{aligned}$$

When $x = 1$ and $y = 1$ this becomes

$$1 = Ce^2 + \frac{1}{2} \implies Ce^2 = \frac{1}{2} \implies C = \frac{1}{2e^2}$$

and the differential equation is explicitly solved by

$$y = \frac{1}{2e^2}e^{2x} + \frac{1}{2}.$$

(c) Notice $e^{2x-y} = e^{2x}e^{-y}$ and so we may rewrite the equation as

$$e^y\frac{dy}{dx} = 6e^{2x}.$$

Integrating both sides yields

$$e^y = 3e^{2x} + C$$

Plugging in $x = y = 0$ this becomes $1 = 3 + C$ and so $C = -2$ and our solution is

$$y = \ln(3e^{2x} - 2)$$

2.6 Notice that if the current concentration of saltwater is C, then the amount of salt being removed from the tank at each minute is

$$2\frac{\text{gal}}{\text{min}} \cdot C\frac{\text{lb}}{\text{gal}} = 2C\frac{\text{lb}}{\text{min}}.$$

Thus the concentration of saltwater is decreasing at a rate of

$$\frac{2C\,\text{lb}/\text{min}}{100\text{gal}} = \frac{C}{50}\frac{\text{lb}/\text{gal}}{\text{min}}.$$

However, salt is being introduced to the tank at a rate of

$$2\frac{\text{gal}}{\text{min}} \cdot \frac{1}{2}\frac{\text{lb}}{\text{gal}} = 1\frac{\text{lb}}{\text{min}}$$

and so the concentration is also increasing at a rate of

$$\frac{1}{100}\frac{{}^{\text{lb}}/_{\text{gal}}}{\text{min}}$$

Putting these together, the rate of change of the concentration is

$$\frac{dC}{dt} = \frac{1}{100} - \frac{C}{50} = \frac{1-2C}{100}.$$

Solving this differential equation gives us the following (we use k as the constant of integration here since C is being used to represent concentration):

$$\begin{aligned}
&\frac{dC}{dt} = \frac{1-2C}{100} \\
\implies &\frac{100}{1-2C}\frac{dC}{dt} = 1 \\
\implies &\int \frac{100}{1-2C}dC = \int dt \\
\implies &-50\ln|1-2C| = t+k \\
\implies &\ln|1-2C| = \frac{-t}{50} + k \\
\implies &1-2C = ke^{-t/50} \\
\implies &C = \frac{1}{2} - ke^{-t/50}
\end{aligned}$$

Since $C(0) = 0$, we have $k = 1/2$ and so

$$C(t) = \frac{1-e^{-t/50}}{2}.$$

Thus after 10 minutes the concentration is

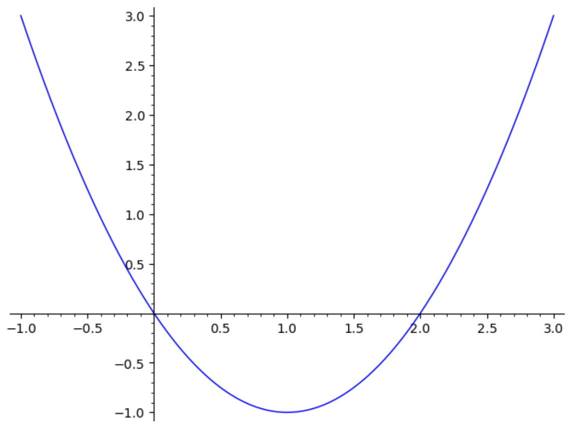
$$C(10) = \frac{1-e^{-1/5}}{2} \approx 0.0906\frac{\text{lb}}{\text{gal}}$$

2.7 (a) Notice the coefficient on y is only continuous in $(-2, 2)$, and the right-hand side is continuous everywhere. This tells us the solution to the differential equation only exists in $(-2, 2)$.

(b) As in part (a), the coefficient of y is continuous in $(-2, 2)$. The right-hand side, however, is continuous in $(-\infty, 0)$ and $(0, \infty)$. We need the largest interval where *both* of these are continuous which contains our initial point $x = 1$, and so the solution exists in $(0, 2)$.

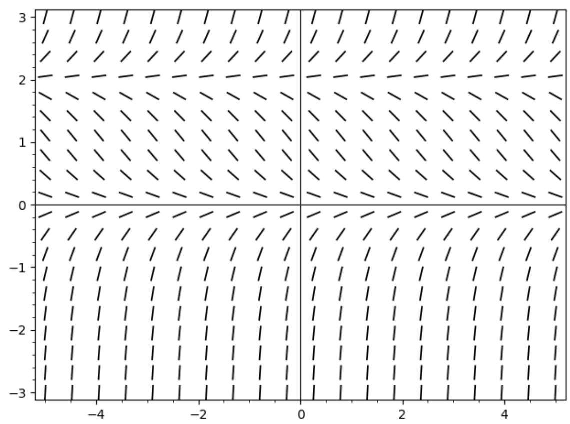
(c) Here the coefficient on y and the right-hand side are both continuous on the entire real line, so the solution exists for all real numbers, $(-\infty, \infty)$.

2.8 (a) The graph of $\frac{dy}{dt}$ is

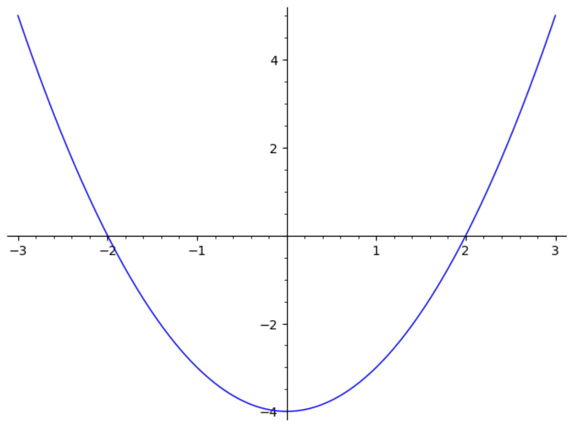


The equilibria occur at $y = 0$ and $y = 2$. Since $\frac{dy}{dt} > 0$ to the left of $y = 0$ and negative to the right of $y = 0$, it is increasing to the left and decreasing to the right, so $y = 0$ is asymptotically stable. The equilibrium $y = 2$, however, is asymptotically unstable since the derivative is negative to the left of 2 and positive to the right of 2, and so nearby solutions move away from the equilibrium.

The slope field is

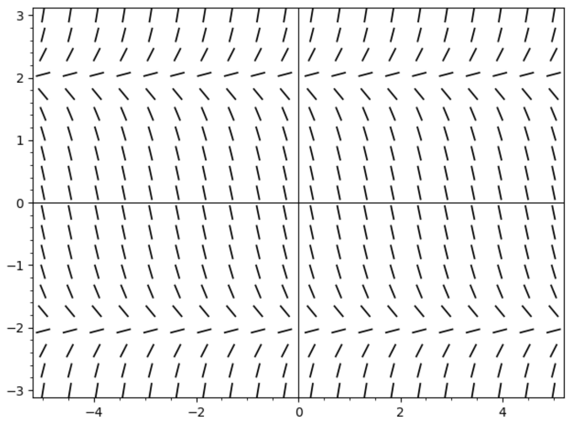


(b) The graph of $\frac{dy}{dt}$ is

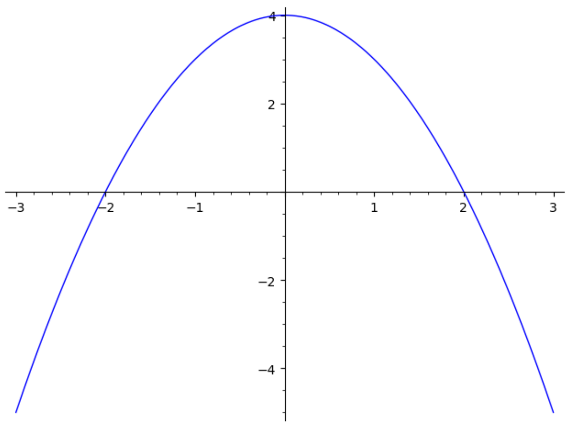


The equilibria are $y = -2$ and $y = 2$. From the signs of the derivative, we see that solutions to the left of $y = -2$ increase towards -2 and solutions to the right decrease towards -2 as well, so $y = -2$ is asymptotically stable. However, solutions to the left of 2 decrease and solutions to the right of 2 increase, and so $y = 2$ is asymptotically unstable.

The slope field is

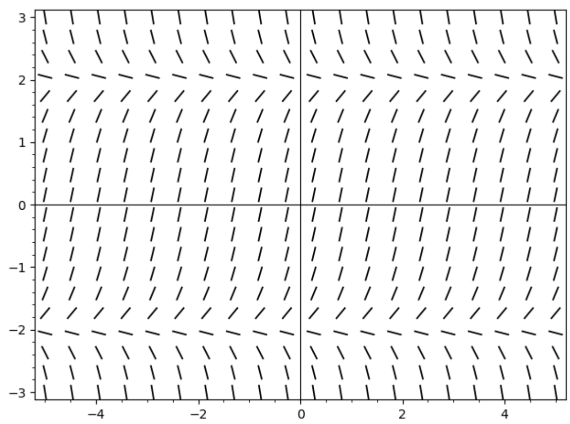


(c) The graph of $\frac{dy}{dt}$ is



This differential equation is the negative of the one in part (b), and so the roles of "increasing" and "decreasing" have reversed. This means $y = -2$ is not unstable whereas $y = 2$ is stable.

The slope field is



2.9 (a)

$$\int \left(x^2y - y + 3x\right)\, dx = \frac{x^3y}{3} - xy + \frac{3x^2}{2} + g(y)$$
$$\int \left(x^2y - y + 3x\right)\, dy = \frac{x^2y^2}{2} - \frac{y^2}{2} + 3xy + h(x)$$

(b)

$$\int \frac{x}{y^2+1}\, dx = \frac{x^2}{2y^2+2} + g(y)$$
$$\int \frac{x}{y^2+1}\, dy = x \arctan(y) + h(x)$$

2.10 (a) Notice that

$$\frac{\partial}{\partial y} e^x \sin(y) = e^x \cos(y) = \frac{\partial}{\partial x} e^x \cos(y)$$

and so a ψ with the desired derivatives must exist. We can compute it by integrating either of its partial derivatives:

$$\begin{aligned}\psi &= \int \psi_x x\, dx \\ &= \int e^x \sin(y)\, dx \\ &= e^x \sin(y) + g(y)\end{aligned}$$

Now we must choose $g(y)$ so that our other partial derivative is satisfied:

$$e^x \cos(y) = \psi_y = \frac{\partial}{\partial y} e^x \sin(y) + g(y) = e^x \cos(y) + g'(y)$$

we may thus take $g'(y) = 0$ and so $g(y)$ is a constant, C. Our function ψ thus has the form

$$\psi(x, y) = e^x \sin(y) + C$$

(b) Notice

$$\frac{\partial}{\partial y} x^2 = 0 = \frac{\partial}{\partial x} y^2$$

and so a ψ with the desired derivatives exists. We can compute it by integrating either of its partial derivatives:

$$\begin{aligned}\psi &= \int \psi_x \, dx \\ &= \int x^2 \, dx \\ &= \frac{x^3}{3} + g(y)\end{aligned}$$

Now we must choose $g(y)$ so that our other partial derivative is satisfied:

$$\begin{aligned} y^2 = \psi_y &= \frac{\partial}{\partial y}\left(\frac{x^3}{3} + g(y)\right) = g'(y) \\ \implies g(y) &= \int y^2 \, dy = \frac{y^3}{3} + C \end{aligned}$$

and so our function ψ is

$$\psi(x, y) = \frac{x^3}{3} + \frac{y^3}{3} + C.$$

(c) Notice

$$\frac{\partial}{\partial y} 3x^2 y = 3x^2 \neq 3y^2 = \frac{\partial}{\partial x} 3xy^2$$

and so there does not exist a function ψ so that $\psi_x = 3x^2 y$ and $\psi_y = 3xy^2$.

2.11 1. First we check to see if this is an exact equation or not:

$$\frac{\partial}{\partial y}(y + 2x) = 1 = \frac{\partial}{\partial x}\left(x + 3y^2\right)$$

and so there must exist a function ψ with these antiderivatives, and the equation is exact. We compute ψ as follows:

$$\begin{aligned}\psi &= \int \psi_y \, dy \\ &= \int \left(x + 3y^2\right) \, dy \\ &= xy + y^3 + h(x)\end{aligned}$$

Now we must choose $h(x)$ so that our partial with respect to x equals $y + 2x$

$$y + 2x = \frac{\partial}{\partial x}\left(xy + y^3 + h(x)\right) = y + h'(x)$$

Thus $h'(x) = 2x$ and so $h(x) = x^2$ and our function ψ is

$$\psi(x, y) = xy + y^3 + x^2.$$

The differential equation is solved implicitly by

$$xy + y^3 + x^2 = C.$$

To solve the initial value problem we simply plug in $x = 2$ and $y = 3$ to determine $C = 6 + 27 + 4 = 37$, and so the IVP is solved implicitly by

$$xy + y^3 + x^2 = 37.$$

2. Let's first rewrite the equation as

$$\frac{2x}{1+x^2} - y\sin(xy) + \left(2 + 3y^2 - x\sin(xy)\right)\frac{dy}{dx} = 0.$$

Note this is an exact equation as

$$\frac{\partial}{\partial y}\left(\frac{2x}{1+x^2} - y\sin(xy)\right) = -\sin(xy) - xy\cos(xy) = \frac{\partial}{\partial x}\left(2 + 3y^2 - x\sin(xy)\right).$$

Now we compute ψ:

$$\begin{aligned}\psi &= \int \psi_x \, dx \\ &= \int \left(\frac{2x}{1+x^2} - y\sin(xy)\right) \, dx \\ &= \ln(1 + x^2) + \cos(xy) + g(y)\end{aligned}$$

If we differentiate with respect to y we obtain

$$-x\sin(xy) + g'(y)$$

but we know this should equal $2 + 3y^2 - x\sin(xy)$, and so $g'(y) = 2 + 3y^2$ and $g(y) = 2y + y^3$. Thus our differential equation is solved implicitly by

$$-x\sin(xy) + 2y + y^3 = C.$$

Seeting $x = y = 0$, we see our particular IVP is solved by

$$-x\sin(xy) + 2y + y^3 = 0.$$