HYPERELLIPTIC TRANSLATION SURFACES AND FOLDED TORI

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Abstract. Following constructions of McMullen [McM06] and Vasilyev [Vas05] we list strict quadratic differentials on (marked) tori defined by *twists* of one-forms on hyperelliptic surfaces. We count the total number of twists in each genus and show that the quadratic differentials on (marked) tori are strict. The examples establish constructions of (non-arithmetic) lattice Panov planes [Pan09] from genus two lattice surfaces.

1. Introduction and statement of main results

Take a Riemann surface X with an orientation preserving involution $\sigma : X \to X$ and consider a twofold cover $\pi : \widetilde{X} \to X$ which carries a lifted involution $\widetilde{\sigma} : \widetilde{X} \to \widetilde{X}$ and deck transformation $\delta : \widetilde{X} \to \widetilde{X}$. Adapting the terminology of McMullen [McM06], we call the surface $\widetilde{X}/\widetilde{\sigma}$ a *twist* of the surface (X, σ) and the procedure *twisting*. In this paper we study *torus twists*, i.e. twists $\widetilde{X}/\widetilde{\sigma}$ having genus one.

$$(X,\sigma) \xleftarrow{\ \pi_\delta\ } (\widetilde{X}, \widetilde{\sigma}, \delta) \xrightarrow{\ \pi_{\widetilde{\sigma}}\ } \widetilde{X}/\widetilde{\sigma}$$

Twisting preserves certain structures from its source, for instance if X has a polygonal presentation where edges are identified by translations, then $\widetilde{X}/\widetilde{\sigma}$ (see Figure 1 below) has a polygonal presentation where edges are identified by half-translations, i.e. translations, or translations combined with a half-turn. To begin our analysis we notice that every twist

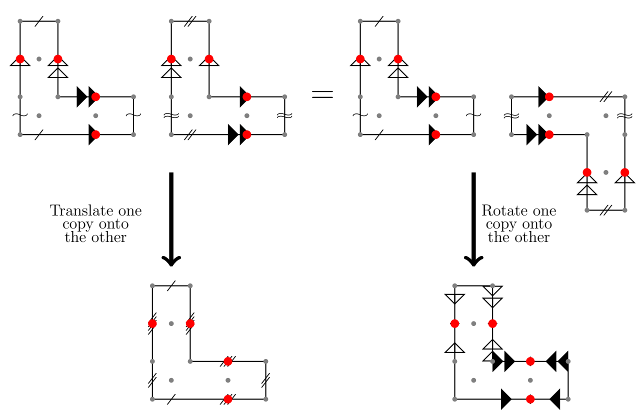


Figure 1. Twist of a genus two L-shaped surface.

Date: October 3, 2013.

surface can be obtained by a quotient-cover relation, instead of a cover-quotient construction. Consequently the existence of torus twists is limited to particular surfaces.

Proposition 1. *Let (X, σ) be a Riemann surface with involution σ. Then any twist $\widetilde{X}/\widetilde{\sigma}$ of (X, σ) is a two-cover of X/σ. In particular, torus twists only exist if X/σ is already a torus or a sphere.*

If X/σ is already a torus, we have what we want. This is in itself a considerable case and there are half-translation tori appearing as quotients of genus 3 surfaces by an involution, which are not covers of genus 2 surfaces [McM06]. Moreover some of those surfaces have the *lattice property* [McM06]. In the other case, (X, σ) is a hyperelliptic surface. For those surfaces covering topology implies:

Theorem 2. *Let (X, σ) be a hyperelliptic surface of genus g, with hyperelliptic involution σ. Further let $\pi(\mathcal{W})$ denote the image of the Weierstrass points $\mathcal{W} = Fix(\sigma) \subset X$ with respect to the canonical projection $\pi : X \to X/\sigma \cong \mathbb{CP}^1$. Then there are $\binom{2g+2}{2\widehat{g}+2}$ genus $\widehat{g}$ twists $\widehat{X}$ of X. Altogether these are the $2(4^g - 1)$ regular two-covers of $\mathbb{CP}^1 \backslash \pi(\mathcal{W})$ branched over a proper subset of $\pi(\mathcal{W})$.*

Particularly for tori we keep in mind:

Corollary 3. *For a hyperelliptic surface (X, σ) there are*

$$\binom{2g+2}{4} = \frac{1}{6} g(g+1)(4g^2 - 1)$$

torus twists. Those are regular double covers $\widehat{X} \to \mathbb{CP}^1 \backslash \pi(\mathcal{W})$ branched over four points $\pi(\mathcal{W}) \subset X/\sigma$.

Half-translation surfaces. We restrict our considerations to surfaces (X, σ) carrying an involution σ. We further assume that X is presented by polygons in the complex plane, see for example the surfaces in Figure 1, with edge identifications of the shape $z \mapsto \pm z + c$, $c \in \mathbb{C}$. If all edge identifications are *translations*, i.e. of the shape $z \mapsto z + c$, the surfaces are called *translation surfaces*, in the other case one talks about *half-translation surfaces*. The surface on the right in Figure 1, for example, is a half-translation surface, the other two are translation surfaces. Both kinds of surfaces are well-studied, standard references are [MT02], [Zor06] and [HS06]. The quadratic differential $(dz)^2 \in \mathcal{Q}(\mathbb{C})$ is invariant under the edge identifications and therefore defines a (meromorphic) quadratic differential $q \in \mathcal{Q}(X)$ on the surface X. A quadratic differential has simple singularities in a finite number of points, corresponding to points of total angle π. Singular points are typically considered marked points (or punctures) on the surface; we don't make this distinction, since they are cone points of the polygonal representation. If and only if all edge transformations are translations the one-form $dz \in \Omega(\mathbb{C})$ defines a (non null) holomorphic one-form $\omega \in \Omega(X)$. That way we obtain presentations $(X, q) = (P, (dz)^2)/\sim$, or $(X, \omega) = (P, dz)/\sim$ in the translation case, for a suitable polygon P with edge identifications $(\sim)$. Given a quadratic differential (X, q) local coordinates $\zeta(p) = \int_{p_0}^{p} \sqrt{q}$ can be used to construct a polygonal presentation of X. In particular the differential inherits a flat metric, direction foliations, linear flows etc. These structures make (half-)translation surfaces the suitable to study dynamics on rational polygonal billiards [MT02]. Selecting a one-form on a surface does not affect the previous topological results, since any one-form twists into a quadratic differential:

A one-form ω on X pulls back to a one-form $\widetilde{\omega} = \pi^*\omega$ on any cover $\pi : \widetilde{X} \to X$. Since the (lifted) involution $\widetilde{\sigma} : \widetilde{X} \to \widetilde{X}$ acts on the space of one-forms having eigenvalues ± 1, it always preserves the quadratic differential $\widetilde{\omega}^2$, i.e. $\widetilde{\sigma}^*\widetilde{\omega}^2 = \widetilde{\omega}^2$ and so descends to the quotient $\widetilde{X}/\widetilde{\sigma}$. The construction works analogously if we take a quotient first and then the cover.

If (X, σ) is hyperelliptic with involution σ and quotient $\pi : X \to \mathbb{CP}^1$, denote by $\mathcal{B}_\sigma = \pi(\mathrm{Fix}(\sigma)) \subset \mathbb{CP}^1$ the branch points of the cover. Any nontrivial homology class $\alpha \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ defines a two-cover $\pi_\alpha : X_\alpha \to \mathbb{CP}^1$, branched over $\partial\alpha \subset \mathcal{B}$. If $\omega \in \Omega(X)$ is a nontrivial one-form on X denote the twist of (X, ω) defined by the two-covers $X \overset{\pi}{\to} \mathbb{CP}^1 \overset{\pi_\alpha}{\leftarrow} X_\alpha$ by (X_α, q_α). We call (X_α, q_α) a *g-twist*, if the genus of X_α is g. We are mainly interested in 1-twists, which we also call *torus-twists*.

Following [McM06], we call a quadratic differential *strict*, if it is not the square of a one-form.

Theorem 4. *Torus-twists of a hyperelliptic one-form (X, ω) of genus $g(X) > 1$ are strict.*

Lattice surfaces. The twist construction is useful to produce quadratic differentials on (marked) tori having the *lattice property.*

We call an orientation preserving homeomorphism $\phi : X \to X$ of a polygonal surface *affine*, if ϕ is induced by an affine map of $\mathbb{C}$. Affine homeomorphisms define a group of are preserving maps with constant derivative located in $\mathrm{PSL}_2(\mathbb{R})$, if X is has finite area. One says a quadratic differential is a *lattice (differential)*, if the derivatives of its affine maps define a lattice in $\mathrm{PSL}_2(\mathbb{R})$. The same notion applies to (half)-translation surfaces, where it is common to say simply *lattice surface* suppressing the relevance of the half-translation structure. The lattice property is preserved under quotients of involutions [McM06] and it is also stable under pullback along branched covering maps, as long as branching appears only over periodic points with respect to the action of the affine group. Consequently the lattice property prevails under twisting: *The twist of a lattice surface is a lattice surface.*

In fact, one can construct strict lattice quadratic differentials on tori from many lattice surfaces described in [McM06]. Here we exclusively consider hyperelliptic surfaces of genus 2 and in this genus there are many lattice surfaces, see [McM06] and the references therein.

Panov planes. Since the universal cover of any torus is a plane, a strict quadratic differential on a (marked) torus induces a strict quadratic differential on a (singular) complex plane. Following a recent study of such planes by Panov [Pan09], we call these *Panov planes.* In fact in [Pan09] Panov describes directions having dense leaves on certain Panov planes by explicitly presenting such leaves in eigendirections of pseudo-Anosov maps. Motivated by this result we study eigenfoliations of pseudo-Anosov maps on (lattice) Panov planes in [JS13b]. In the same paper we also describe a natural way to convert the dynamics on the doubly periodic wind-tree model using a pair of Panov planes. In the article [JS13a] we exploit that relation and generally study properties of directional foliations on Panov planes. For this and related applications it is helpful to know constructions of lattice Panov planes from low genus translation surfaces, such as genus two and three.

Studying Panov planes may be viewed as a (further) stepping stone towards understanding the dynamics of $\mathbb{Z}^d$-covers (of half-translation surfaces). While the dynamics on $\mathbb{Z}$-covers is reasonably well understood [HW12], $\mathbb{Z}^2$-covers show more complex behavior, e.g. see [HLT11] and [Pan09].

Related results. Compared to the heavy research output concerning translation surfaces, particularly during last two decades, the number of results on half-translation surfaces are moderate. Out of those we would like to mention the research of Boissy and Lanneau [BL09] and two recent papers by Athreya, Eskin and Zorich [AEZ12] and [AEZC12] (with an attachment of J. Chaika).

Closely related to our results are the preprints of Vasilyev [Vas05] and the paper of McMullen [McM06]. In fact Vasilyev [Vas05] uses the twist construction in Figure 1 backwards. In [McM06, §6] McMullen defines twists and applies them to construct lattice quadratic differentials from lattice one-forms, both having the same genus.

Acknowledgements. We thank the organizers of the Teichmüller Dynamics Workshop at the Hausdorff Institute, Bonn for inviting us and in particular Ferran Valdez, who motivated this research with an inspiring presentation. We are more than grateful to the referee for his careful reading and for pointing out a gap in the initial version of Theorem 4.

2. Covering theory

Covers with commuting involutions. We show our initial claim, that surface twists may also be constructed by a quotient followed by a cover. This transformation was already used by Vasilyev [Vas05].

Take a Riemann surface X with involution $\sigma : X \to X$ and consider any (branched) regular two-cover $\pi_\delta : (\widetilde{X}, \widetilde{\sigma}, \delta) \to (X, \sigma)$, with deck exchange $\delta : \widetilde{X} \to \widetilde{X}$ having a lifted involution $\widetilde{\sigma} \neq \delta$, i.e. an involution such that $\pi_\delta \circ \widetilde{\sigma} = \sigma \circ \pi_\delta$. The deck exchange map δ is itself an involution commuting with $\widetilde{\sigma}$. So the automorphism group generated by $\widetilde{\sigma}$ and δ is the Klein group $\mathcal{K} := \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\delta \circ \widetilde{\sigma} : \widetilde{X} \to \widetilde{X}$ is an involution as well. We see, the fourfold cover $\widetilde{X} \to X/\sigma$ is a $\mathcal{K}$-cover and it factors over both covers $\pi_{\widetilde{\sigma}} : \widetilde{X} \to \widetilde{X}/\widetilde{\sigma}$, or $\pi_{\delta\circ\widetilde{\sigma}} : \widetilde{X} \to \widetilde{X}/\delta \circ \widetilde{\sigma}$. Thus the respective base surface of these factors are two-covers of X/σ. In fact the involution δ descends to a non-trivial involution on both quotients $\widetilde{X}/\widetilde{\sigma}$ and $\widetilde{X}/\delta \circ \widetilde{\sigma}$. Denote those involutions by $\delta^+ : \widetilde{X}/\widetilde{\sigma} \to \widetilde{X}/\widetilde{\sigma}$ and $\delta^- : \widetilde{X}/\delta \circ \widetilde{\sigma} \to \widetilde{X}/\delta \circ \widetilde{\sigma}$. All covers, quotients and maps fit in a commutative *diamond diagram*, where the spaces in the middle row are two-covers of X/σ.

$$\begin{array}{ccccc} & & (\widetilde{X},\widetilde{\sigma},\delta) & & \\ & \swarrow^{\pi_{\widetilde{\sigma}}} & \downarrow^{\pi_\delta} & \searrow^{\pi_{\delta\circ\widetilde{\sigma}}} & \\ (\widetilde{X}/\widetilde{\sigma},\delta^+) & & (\widetilde{X}/\delta,\sigma)=(X,\sigma) & & (\widetilde{X}/\delta\circ\widetilde{\sigma},\delta^-) \\ & \searrow_{\pi_+} & \downarrow & \swarrow_{\pi_-} & \\ & & X/\sigma & & \end{array} \tag{1}$$

Standard Riemann surface theory (see also [Vas05]) implies:

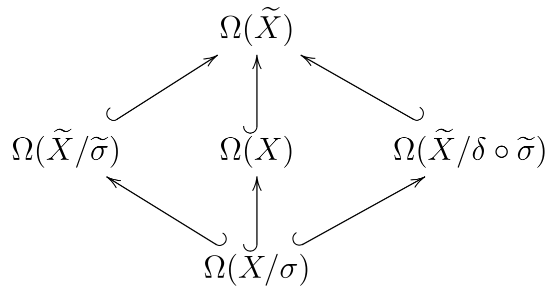
Proposition 5. *The genus g and the Euler characteristic χ of the surfaces in Diagram 1 fulfill the relations:*

$$g(\widetilde{X}) + 2g(X/\sigma) = g(\widetilde{X}/\widetilde{\sigma}) + g(X) + g(\widetilde{X}/\delta \circ \widetilde{\sigma})$$

and

$$\chi(\widetilde{X}) + 2\chi(X/\sigma) = \chi(\widetilde{X}/\widetilde{\sigma}) + \chi(X) + \chi(\widetilde{X}/\delta \circ \widetilde{\sigma}).$$

Proof. Commutativity of Diagram 1 implies the derived diagram on the vector space of one-forms is a commutative diagram of injective linear maps.



Since δ and $\widetilde{\sigma}$ commute, the induced maps δ^* and $\widetilde{\sigma}^*$ on the vector space of forms $\Omega(\widetilde{X})$ can be simultaneously diagonalized. So the vector space decomposes in 1-dimensional subspaces with eigenvalue 1 or -1. If δ^* and $\widetilde{\sigma}^*$ are both acting trivially on a subspace, then any form in that subspace is in the image of $\Omega(X/\sigma)$. A form in the eigenspace where both involutions act by -1 is the image of a form on $\widetilde{X}/\delta \circ \widetilde{\sigma}$. If δ^* act as -1 and $\widetilde{\sigma}^*$ as 1, the eigenspace is in the image of $\Omega(\widetilde{X}/\widetilde{\sigma})$, but not in the image of $\Omega(X/\sigma)$. If it is the other way round, the respective forms are images from $\Omega(X)$, which are not already images of $\Omega(X/\sigma)$. Considering the dimensions of the respective vector spaces of holomorphic one-forms implies both stated identities. □

Corollary 6. *If $\widetilde{X}/\widetilde{\sigma}$, or $\widetilde{X}/\delta \circ \widetilde{\sigma}$ has genus* 1*, then either $g(X/\sigma) = 0$, or $g(X/\sigma) = 1$. In particular X/σ is (biholomorphic to) a Riemann sphere $\mathbb{CP}^1$, or it is a certain torus $\mathbb{C}/\Lambda$.*

Proof. For any covering map $f : X \to Y$ of Riemann surfaces $g_X \geq g_Y$. □

Proposition 1 on page 2 follows from this Corollary. That conveniently reduces our problem to $\mathcal{K}$-covers and $\mathbb{Z}_2$-covers of $\mathbb{CP}^1 \backslash \{p_0, ..., p_n\}$.

Riemann-Hurwitz formula. The Riemann-Hurwitz formula for branched covers $f : X \to \mathbb{CP}^1$ of degree n, branched over $\mathcal{B} := \{p_0, ..., p_{2g+1}\} \subset \mathbb{CP}^1$ states

$$\chi(X) = 2n - \sum_{p \in f^{-1}(\mathcal{B})} (e_p - 1).$$

If the degree of the cover is 2, then $e_p = 2$ for all points $p \in f^{-1}(\mathcal{B})$ and since there are $|\mathcal{B}|$ of those points:

$$\chi(X) = 4 - |\mathcal{B}| = 4 - (2g + 2) = 2(1 - g).$$

That is $f : X \to \mathbb{CP}^1$ is a (branched) two-cover and X a hyperelliptic surface. If $f : X \to \mathbb{CP}^1$ has degree 4 and all ramification points have ramification index $e_p = 2$, one has

$$\chi(X) = 8 - 2|\mathcal{B}| = 2(4 - (2g + 2)) = 4(1 - g).$$

Topological representation of hyperelliptic covers. The following is built on the grounds of standard covering topology as provided by [Ful95]. A supporting background on Riemann surfaces is in Farkas & Kra [FK92] and Miranda [Mir95].

To define connected branched covers $\pi : X \to \mathbb{CP}^1$, branched in the finite set $\mathcal{B} = \{p_0, ..., p_{n-1}\} \subset \mathbb{CP}^1$ one looks at the unbranched cover $\pi : \mathring{X} := X \backslash \pi^{-1}(\mathcal{B}) \to \mathring{\mathbb{CP}}^1 :=$

$\mathbb{CP}^1 \backslash \mathcal{B}$. The *monodromy* characterization of covers states that any degree d cover (up to isomorphy) is given by a homomorphism

$$\varphi \in \mathrm{Hom}(\pi_1(\mathring{\mathbb{CP}}^1, x); S_d).$$

Here S_d is the symmetric group of d elements. A G-cover is a cover with deck transformation group G and those are characterized by elements in

$$\varphi \in \mathrm{Hom}(\pi_1(\mathring{\mathbb{CP}}^1, x); G). \tag{2}$$

If the group G is commutative any homomorphism has the commutator subgroup of the fundamental group in its kernel, so we can simplify

$$\mathrm{Hom}(H_1(\mathring{\mathbb{CP}}^1); G) \cong H^1(\mathring{\mathbb{CP}}^1; G) \tag{3}$$

Because we work on a sphere with n punctures $\mathring{\mathbb{CP}}^1$, its the simple loops around the punctures $[\gamma_i] = [\gamma(p_i)]$, $(i = 0, ..., n-1)$ define generate the fundamental group subdue to the only relation

$$[\gamma_0] \cdots [\gamma_{n-1}] = 1,$$

we know a G-cover is defined by $n-1$ elements $g_i = \varphi([\gamma_i]) \in G$ with $g_1 \cdots g_{n-1} = 1$. Thus to give a G-cover of $\mathbb{CP}^1$ branched over the set $\mathcal{B} = \{p_0, ..., p_{n-1}\}$ one can freely choose $n-1$ such elements $g_1, ..., g_{n-1} \in G$ and take $g_0 := (g_1 \cdots g_{n-1})^{-1}$. The following abstract isomorphism includes that observation:

$$\mathrm{Hom}(H_1(\mathring{\mathbb{CP}}^1; \mathbb{Z}), G) \cong H^1(\mathring{\mathbb{CP}}^1; G) \cong \oplus^{n-1} G.$$

While the monodromy description characterizes covers, we want to add a related method which is very natural to construct polygonal covers by means of the *slit construction*.

Recall that intersection defines a (non degenerate) pairing of vector spaces:

$$H_1(\mathbb{CP}^1, \mathcal{B}; G) \times H_1(\mathring{\mathbb{CP}}^1; \mathbb{Z}) \to G$$

Thus regular G-covers of $\mathring{\mathbb{CP}}^1$ (branched in $\mathcal{B}$) are defined by non-trivial classes in $H_1(\mathbb{CP}^1, \mathcal{B}; G)$. In order to define a cover pick a basis of relative homology $\{[\![\gamma_i]\!] \in H_1(\mathbb{CP}^1, \mathcal{B}; G) : i = 1, ..., 2g\}$ with $\partial \gamma_i := g_i p_i - g_i p_0$.

From now on we drop the punctures when writing down $\mathbb{CP}^1$ covers, i.e. we simply write $X \to \mathbb{CP}^1$ instead of $\mathring{X} \to \mathring{\mathbb{CP}}^1$. If important, the branching locus is specified in the context.

Twofold and fourfold covers. A regular twofold cover $X \to \mathbb{CP}^1$ with deck transformation group $\mathbb{Z}_2$ is given by a non-trivial class

$$[\![\gamma]\!] \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2).$$

As before we may take a basis $[\![\gamma_i]\!] \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$, $i = 0, ..., n$ with $\partial [\![\gamma_i]\!] = p_i - p_0 = p_i + p_0$. Because of $\mathbb{Z}_2$-coefficients we may interpret the boundary of a cycle as an unordered tuple in $\mathcal{B} = \{p_0, ..., p_{n-1}\}$.

Given a subset $\mathcal{B} \subset \mathbb{CP}^1$, denote the $\mathbb{Z}_2$ vector space generated by the elements of $\mathcal{B}$ by $\mathbb{Z}_2[\mathcal{B}]$. Because the projective space has no absolute homology taking the boundary $\partial\alpha$ of a relative homology class $\alpha \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ defines an injective homomorphism

$$\partial : H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2) \to \mathbb{Z}_2[\mathcal{B}].$$

The following is a simple consequence of the definitions:

Proposition 7. *Suppose $\pi : X \to \mathbb{CP}^1$ is a nontrivial branched $\mathbb{Z}_2$-cover with involution δ. Then the ramification points of π are the fix-points* $\mathrm{Fix}(\delta)$ *of δ. If $\alpha \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ is the relative homology class defining the $\mathbb{CP}^1$-cover π, then $\pi(\mathrm{Fix}(\delta)) = \partial\alpha$. In particular $\pi(\mathrm{Fix}(\delta)) \subseteq \mathcal{B}$.*

As a Riemann surface double cover of $\mathbb{CP}^1$ of positive genus is hyperelliptic with deck exchange map being the hyperelliptic involution.

Corollary 8. *Let $\mathcal{B} \subset \mathbb{CP}^1$, be a finite set of order $|\mathcal{B}| = n \geq 2$ and $\alpha \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}/2)$ with $\partial\alpha = \mathcal{B}$. If $\beta \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}/2)$ is a class with $|\partial\beta| = 2k < n$, then $|\partial(\alpha + \beta)| = n - 2k$. The genera of the two-covers of $\mathbb{CP}^1$ defined by the three classes α, β and $\alpha + \beta$ fulfill $g_\beta + g_{(\alpha+\beta)} = g_\alpha - 1$.*

Proof. The genus of a two-cover relates to the number of branch-points of the cover by $2g + 2 = |\text{branch points}|$. □

Twists of two-covers. Let $X_\alpha \to \mathbb{CP}^1$ be the branched two-cover given by the class $\alpha \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}/2)$ where $|\mathcal{B}| = n \geq 2$. Then the branch points of the cover are given by $\partial\alpha \in \mathcal{B}$ and therefore its genus is $g = |\partial\alpha|/2 - 1 \in \{0, ..., \lfloor n/2 \rfloor - 1\}$.

Proposition 9. *For any fixed genus $g \in \{0, ..., \lfloor n/2 \rfloor - 1\}$ there are $\binom{n}{2g+2}$ covers of genus g, including $\binom{n}{2} = \frac{1}{2}n(n-1)$ covers of genus $g = 0$ and a unique cover of maximal genus $g_M = \lfloor n/2 \rfloor - 1$.*

Proof. Any two-cover is determined by the subset of its branch points in $\mathcal{B}$ and the covers of fixed genus $g \geq 0$ have $2 \leq 2g + 2 \leq n$ ramification points. There are $\binom{n}{2g+2}$ possible choices of subsets of $2g + 2$ elements. □

Note that the number of elements in $H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ is given by $2^{|\mathcal{B}|-1} = 2^{n-1}$. The trivial cycle gives the trivial, i.e. the non-connected, two-cover. In order to parameterize covers one typically uses the projective space $\mathbb{P}H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2) = H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2) \setminus \{0\}$ which in the case of two-covers only removes the trivial cover.

Fix a class $\alpha \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ and $g \in \{0, ..., \lfloor |\mathcal{B}|/2 \rfloor - 1\}$. Refining our initial definition of twists we call the cover $\mathring{X}_\beta \to \mathring{\mathbb{CP}}^1$ a *g-twist* of $\mathring{X}_\alpha \to \mathring{\mathbb{CP}}^1$, if the genus of X_β equals g and $\beta \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$. The g-twists of the two-cover $X_\alpha \to \mathbb{CP}^1$, branched in a subset of $\mathcal{B} \subset \mathbb{CP}^1$, are simply all genus g two-covers branched over a subset of $\mathcal{B}$.

Corollary 10. *Any two-cover $X_\alpha \to \mathbb{CP}^1$, with $\alpha \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ admits $\binom{n}{2g+2}$ g-twists, with $g \in \{0, ..., \lfloor |\mathcal{B}|/2 \rfloor - 1\}$.*

Proof. Any genus g two-cover $X_\beta \to \mathbb{CP}^1$ with branching in $\mathcal{B}$ is given by a class $\beta \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ with $|\partial\beta| = 2g + 2$. □

Slit construction of $\mathcal{K}$-covers. The slit construction of covers provides a natural reason to use (relative) homology classes instead of cohomology classes to define (half-)translation covers.

Because

$$H_1(\mathbb{CP}^1, \mathcal{B}; \mathcal{K}) \cong H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2) \oplus H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$$

any $\mathcal{K}$-cover is given by a pair of classes in $H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$. In order to construct the $\mathcal{K}$-cover associated to a pair $(\alpha, \beta) \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2) \oplus H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ we use a variant of the *slit construction*, which is cutting and pasting along geodesic representations of α and β. The geodesics are taken with respect to the polygonal structure, or equivalently with respect to the metric given by a quadratic differential (away from its singularities). Homology

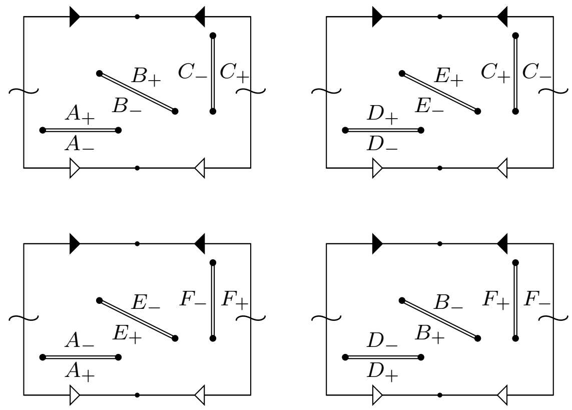


FIGURE 2. Slit construction of a $\mathcal{K}$-cover

classes can be represented by concatenations of line segments in the flat metric induced by a quadratic differential. These segments are called *saddle connections*. Saddle connections start and terminate at singular points of the metric, but do not contain singular points otherwise. Alternatively in a polygonal representation of a surface, a saddle connection is a line segment from a cone point to a cone point.

Suppose we have a half-translation structure on $\mathbb{CP}^1$ with cone points contained in the (finite) set $\mathcal{B}$. Saddle connections have endpoints in $\mathcal{B}$. Take a basis of $H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ defined by saddle connections.

To construct a $\mathcal{K}$-cover cut $\mathbb{CP}^1$ along the saddle connections (relative cycles) representing by relative homology classes $\alpha, \beta \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$. Then take the closure of the cutted surface along the strands of the cuts and denote the resulting surface with boundary by $\mathbb{CP}^1_{\alpha,\beta}$. Consider four copies ${\mathbb{CP}^1}^{\pm\pm}_{\alpha,\beta}$ of $\mathbb{CP}^1_{\alpha,\beta}$ abbreviated symbolically as $(\pm, \pm)$ and organized as in Figure 2. Identify the copies along the strands by using the following rule: If a strand belongs to a cut along a saddle connection in α, or β which is not in β, or α, then identify the same strand on copy $(\pm, -)$, or $(-, \pm)$ with the adjacent strand on copy $(\pm, +)$, or $(+, \pm)$, respectively. If a strand belongs to a cut along a saddle connection contained in both cycles α and β, then identify the respective strand on copies $(-, -)$ and $(-, +)$ with the adjacent strand on copy $(+, +)$, or $(+, -)$, respectively. Figure 2 shows the identifications in a suggestive way: Horizontal strands are identified between vertically adjacent copies, vertical strands are identified between horizontally adjacent copies and diagonal strands of diagonally adjacent copies are identified.

Subcovers and associated homology classes. The slit construction gives a glimpse what properties are to be expected from covers defined by particular homology classes. Take for example the $\mathcal{K}$-cover $\pi : X_{\alpha,\beta} \to \mathbb{CP}^1$ obtained from the slits represented by the classes $\alpha, \beta \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$.

The deck group is generated by two non-trivial involutions: ϕ_h exchanges the decks ${\mathbb{CP}^1}^{\pm+}_{\alpha,\beta}$ with ${\mathbb{CP}^1}^{\pm-}_{\alpha,\beta}$ and ϕ_v exchanges ${\mathbb{CP}^1}^{+\pm}_{\alpha,\beta}$ with ${\mathbb{CP}^1}^{-\pm}_{\alpha,\beta}$. The sub-cover $\pi_v : X_{\alpha,\beta}/\phi_v \to \mathbb{CP}^1$ is a two-cover carrying an involution ϕ_h^* induced by either ϕ_h, or $\phi_v \circ \phi_h$. This two-cover is defined by the class $\alpha \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$. Indeed, the two-cover π_v is branched over the points

$$\pi_v(\mathrm{Fix}(\phi_h^*)) = \pi(\mathrm{Fix}(\phi_h)) + \pi(\mathrm{Fix}(\phi_h \circ \phi_v))$$

and by the slit construction these points define the boundary $\partial\alpha$ of α. We now make those observations precise.

Elementary properties of $\mathcal{K}$-covers. We start with a useful fact:

Proposition 11. *All branch points of a $\mathcal{K}$-cover $X \to \mathbb{CP}^1$ have order 2.*

Proof. Consider the defining homomorphism $\phi \in \mathrm{Hom}(\pi_1(\mathring{\mathbb{CP}}^1), \mathcal{K})$ for the monodromy representation. A branch point of the cover corresponds to a loop with nontrivial image in $\mathcal{K}$. Because all nontrivial elements in $\mathcal{K}$ have order 2, the monodromy around any branch point of the cover has order 2.
Alternatively one may argue as follows: Assume the cover $X_{\alpha,\beta} \to \mathbb{CP}^1$ is defined by the pair of classes $\alpha, \beta \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ and consider a loop around any given branch point. Tracing the loop two times it will intersect all cycles exactly two times and since all cycles have order 2 that brings us back to our initial point (or deck) on the cover. □

Recall that $\mathcal{K}$ is the group of deck-transformations of a given $\mathcal{K}$-cover $X \to \mathbb{CP}^1$. Each of the three non-trivial elements in $\mathcal{K}$ defines an idempotent map. Put $\mathcal{K}^* := \mathcal{K}\backslash\{\mathrm{Id}\}$ and take two elements $\delta, \psi \in \mathcal{K}^*$. From the group structure of $\mathcal{K}$ either $\delta = \psi$, or if not $\delta\cdot\psi = \psi\cdot\delta \neq \mathrm{Id}$ is the third involution in $\mathcal{K}^*$. Let us now state a version of Proposition 7 for $\mathcal{K}$-covers:

Lemma 12. *Let $X \to \mathbb{CP}^1$ be a $\mathcal{K}$-cover, $\delta \in \mathcal{K}^*$ and $\pi_\delta : X \to X/\delta$ the associated double cover. Then the quotient $\pi_{\delta^*} : X/\delta \to \mathbb{CP}^1$ is a $\mathbb{Z}_2$-cover and there is a deck exchange map $\delta^* : X/\delta \to X/\delta$. The cover π_{δ^*} is defined by a class $\alpha \in H_1(\mathbb{CP}^1, \pi_{\delta^*}(\mathrm{Fix}(\delta^*)); \mathbb{Z}_2)$ whose branching locus is:*

$$\partial\alpha = \pi_{\delta^*}(\mathrm{Fix}(\delta^*)) = \pi(\mathrm{Fix}(\psi)) + \pi(\mathrm{Fix}(\delta\cdot\psi)) \in \mathbb{Z}_2[\mathcal{B}] \quad \text{for} \quad \psi \in \mathcal{K}^*\backslash\{\delta\}.$$

Moreover $\delta^* \circ \pi_\delta = \pi_\delta \circ \psi = \pi_\delta \circ (\psi\cdot\delta)$, so both involutions ψ and $\psi\cdot\delta$ are lifts of δ^*.

Proof. Only the last two statements are new. To show the claim on the lifts consider preimages $\pi_\delta^{-1}(y) = \{x, \delta(x)\}$ of a given point $y \in X/\delta$. For $\psi \in \mathcal{K}^*\backslash\{\delta\}$:

$$\psi(\pi_\delta^{-1}(y)) = \{\psi(x), \psi\circ\delta(x)\} = \{\psi\circ\delta^2(x), \psi\circ\delta(x)\} = (\psi\cdot\delta)(\pi_\delta^{-1}(y)) = (\delta\cdot\psi)(\pi_\delta^{-1}(y)).$$

The last identity follows from commutativity. That shows $\psi(\pi_\delta^{-1}(y)) = \delta(\psi(\pi_\delta^{-1}(y))$. Therefore both involutions ψ and $\psi\cdot\delta$ descend to the same involution, say δ^*, on X/δ and we have $\delta^* \circ \pi_\delta = \pi_\delta \circ \psi = \pi_\delta \circ (\psi\cdot\delta)$ as desired. In fact, δ^* is the sheet exchange of the cover $X/\delta \to \mathbb{CP}^1$, because for any $z \in \mathbb{CP}^1$ the preimage $\pi^{-1}(z)$ is the ψ-orbit of two δ-orbits:

$$\pi^{-1}(z) = \{x, \delta(x), \psi(x), \delta\cdot\psi(x)\} = \{x, \delta(x)\} \cup \psi(\{x, \delta(x)\}).$$

Since π_{δ^*} is a $\mathbb{Z}_2$-cover of $\mathbb{CP}^1$ we have $\partial\alpha = \pi_{\delta^*}(\mathrm{Fix}(\delta^*))$. Because $\delta^*\pi_\delta = \pi_\delta\psi$, $\delta^*(y) = y$ implies $\{\psi(x), \psi\cdot\delta(x)\} = \{x, \delta(x)\}$ for the preimages of y. Therefore a preimage of a fix-point of δ^* is either a fix-point of $\delta\cdot\psi$, or a fix-point of ψ, but not of both. Otherwise it would be fix-point of all transformations in $\mathcal{K}^*$ and therefore a ramification point of order four. The identity for $\partial\alpha$ in $\mathbb{Z}_2[\mathcal{B}]$ follows from the definition of the addition in $\mathbb{Z}_2[\mathcal{B}]$. □

Thus each $\mathcal{K}$-cover $X \to \mathbb{CP}^1$ has three sub-covers which are represented by three homology classes, or three sets of branch locations. If we add the trivial two-cover $\mathbb{CP}^1 \cup \mathbb{CP}^1 \to \mathbb{CP}^1$, there is an obvious action of $\mathcal{K}$ on this set of four sub-covers. The next proposition tells us how this group action looks on homology and on branch-points.

Proposition 13. *Let $\pi : X \to \mathbb{CP}^1$ be a $\mathcal{K}$-cover with branching in $\mathcal{B} \subset \mathbb{CP}^1$. Then the group structure on sub-covers defines a homomorphism to homology $H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ and therefore also to the branch module $\mathbb{Z}_2[\mathcal{B}]$*

Proof. It is enough to look at two distinct elements $\psi, \delta \in \mathcal{K}$, with associated $\mathbb{Z}_2$-covers $\pi_{\delta^*} : X/\delta \to \mathbb{CP}^1$, $\pi_{\psi^*} : X/\psi \to \mathbb{CP}^1$ defined by the homology classes $\alpha_\delta, \alpha_\psi \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$. By the previous Lemma $\partial\alpha_\delta = \pi(\mathrm{Fix}(\psi \cdot \delta)) + \pi(\mathrm{Fix}(\psi))$, $\partial\alpha_\psi = \pi(\mathrm{Fix}(\psi \cdot \delta)) + \pi(\mathrm{Fix}(\delta))$ and $\partial\alpha_{\psi\cdot\delta} = \pi(\mathrm{Fix}(\psi)) + \pi(\mathrm{Fix}(\delta))$. Thus

$$\partial\alpha_{\psi\cdot\delta} = \partial\alpha_\delta + \partial\alpha_\psi = \partial(\alpha_\delta + \alpha_\psi),$$

and because the boundary homomorphism is injective we also have

$$\alpha_{\psi\cdot\delta} = \alpha_\delta + \alpha_\psi.$$

□

As a consequence of the lemma and the proposition we can describe $\mathcal{K}$-covers of $\mathring{\mathbb{CP}}^1$ in a convenient way.

Corollary 14. *Giving a $\mathcal{K}$-cover $\pi : X \to \mathbb{CP}^1$ with branching in $\mathcal{B} \subset \mathbb{CP}^1$ is equivalent to*

- *a subgroup $H \trianglelefteq H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ isomorphic to $\mathcal{K}$.*
- *a subgroup $H \trianglelefteq \partial H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2) \subset \mathbb{Z}_2[\mathcal{B}]$ isomorphic to $\mathcal{K}$.*

Proof. By the previous proposition any $\mathcal{K}$-cover defines a subgroup generated by the relative homology classes of its sub-covers. This subgroup is isomorphic to $\mathcal{K}$. On the other hand, given a subgroup $\mathcal{K} \cong H \trianglelefteq H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ we can take any two elements, say $\alpha, \beta \in H\backslash\{0\}$. Those define a $\mathcal{K}$-cover $X_{\alpha,\beta} \to \mathbb{CP}^1$ (slit construction). This cover in turn has sub-covers defined by the non-trivial elements in H. The second equivalence follows because the boundary homomorphism is injective. □

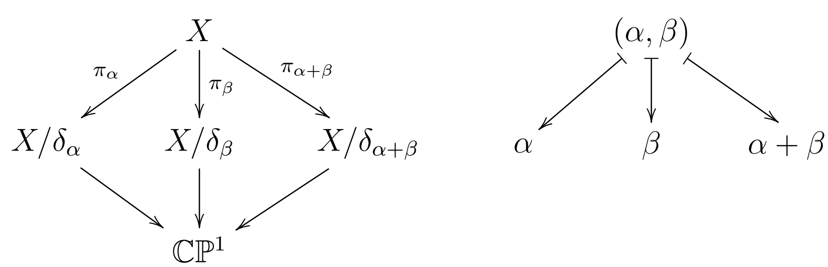
The information contained in the previous Corollary can be stated as follows:

Corollary 15. *A $\mathcal{K}$-cover $\pi : X \to \mathbb{CP}^1$ branching in $\mathcal{B} \subset \mathbb{CP}^1$ is determined by*

- *any two of the its three subcovers $X/\delta \to \mathbb{CP}^1$, $\delta \in \mathcal{K}^*$*
- *the pair of relative homology classes defining any two of its subcovers*
- *the branch points of any two of its subcovers*

□

The following diagrams illustrate the results on $\mathcal{K}$-covers and subcovers in an informal, schematic way. Let $\delta_\alpha, \delta_\beta, \delta_{\alpha+\beta}$ be the three non-trivial involutions of a $\mathcal{K}$-cover $X_{\alpha,\beta} \to \mathbb{CP}^1$ defined by two homology classes $\alpha, \beta \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$.



Quadratic Differentials. We show our main result: Torus-twists of the hyperelliptic translation surface are strictly quadratic differentials.

If a holomorphic one form is locally given by $z^n dz = \frac{1}{n+1} dz^{n+1}$ then the order of the zero in the point 0 is n and the total angle measured around the cone point is $2(n+1)\pi$. Half-translation surfaces are characterized by integrable quadratic differentials which are given locally by $z^n(dz)^2$, $n \geq -1$. In this representation the total angle around the cone point is $(n+2)\pi$, and the order of its zero is n. The origin in the above local representation is a cone point of order $n \geq -1$, which we call *singular (point)*, if $n = -1$, *regular (point)*, if $n = 0$, and *zero*, if $n > 0$.

We need the following observation:

Proposition 16. *Let (X, ω, σ) be an involutive translation surface, such that the affine involution σ has non-trivial derivative, i.e. $D\sigma = -\operatorname{Id}$, then the involution invariant points are of* even order*, equivalently the total cone angle of an invariant point is an* odd *multiple of 2π.*

Proof. Because $D\sigma = -\operatorname{Id}$, in a neighborhood of a fixed point σ acts as a point reflection, i.e. a rotation of order 2. In particular the set of $n+1$ horizontal lines through a conical fix-point of order n is mapped to itself orientation reversed. Thus the total angle of the rotation is an odd multiple of π, say $(2k+1)\pi$. Since σ has order 2, we have $n+1 = 2k+1$ as claimed. □

Corollary 17. *On hyperelliptic translation surfaces only cone points of even order are fix-points of the hyperelliptic involution.*

In particular there is no translation surface in genus 2 having two cone points which are fix-points of the hyperelliptic involution.

Proof. The previous proposition applies. A genus two surface has either one cone point of even order, or two cone points of odd order. □

For a hyperelliptic involution of genus g we conclude, that if a cone point is a fixed point, it has at least order 2. Since the Euler characteristic of a genus g surface is $2(1-g)$, we cannot have more than $g-1$ cone points which are fixed points of the involution.

Proof of Theorem 4. Take the quotient map $\pi : (X, \omega) \to \mathbb{CP}^1$ with respect to the hyperelliptic involution and consider the induced quadratic differential $\pi_* \omega^2 \in \mathcal{Q}(\mathbb{CP}^1)$. By commutativity of the *diamond diagram*, it is enough to consider tori which are two-covers of $(\mathbb{CP}^1, \pi_*\omega^2)$ branched over $\mathcal{B} \subset \mathbb{CP}^1$ furnished with the pullback of $\pi_*\omega^2$.

Evaluating the Euler characteristic using the zeros and singularities of a torus quadratic differential shows, that the total orders of cone points and singular points match:

$$\sum_{\text{singular points}} 1 = \sum_{p \text{ zero}} o(p).$$

Here $o(p)$ denotes the order of the zero in p. Hence, if the pullback quadratic differential on the torus has cone points, it has singular points. Since the order of a zero can only grow under pullbacks it suffices to show, that the induced quadratic differential $q = \pi_*\omega^2$ on $\mathbb{CP}^1$ has zeros. By assumption $g(X) > 1$ and therefore ω has a zero.

First suppose ω has a *fixed point* which is a non-trivial zero. Then this zero has even order, say $n > 1$, and descends to a zero of the quadratic differential $q = \pi_*\omega^2$ having the same order $n > 1$.

Now suppose ω does *not have involution invariant zeros*, then there is a pair of zeros of order $n > 0$ exchanged by the involution. This pair descends to one zero of order $2n$ for q.

In either case the differential q has zeros (and not just singular points). $\square$

Veech's polygonal representation of hyperelliptic surfaces. The following polygonal representation for hyperelliptic surfaces gives one-forms with exactly one, or exactly two cone points. In general there are one-forms with more cone-points on a hyperelliptic surface, but we can use the convenient representation of homology classes by line segments with respect to the given polygonal representation. The polygons also have a central symmetry, i.e. define a Weierstrass point in the center. That allows us to decompose the quotients obtained by the cover-quotient construction into two groups.

Recall, that a polygon P is simple, if it has no intersections and symmetric, if $P = -P$. Since a simple polygon P bounds a disk, symmetry implies $0 \in P$, i.e. the origin is a fixed point of the polygon symmetry $-\mathrm{Id}$. Denote the vertices of the simple $2n$-gon $P = P_{2n}$ by $v_0, ..., v_{2n-1}$ ordered counterclockwise along P, the edges by $e_i = [v_{i-1}, v_i]$ for $i = 1, ..., 2n-1$, and $e_{2n} = [v_{2n-1}, v_0]$.

The central symmetry implies $-e_i = e_{n+i \mod 2n}$, in particular $e_{n+i \mod 2n}$ is parallel to e_i and we may identify both by a translation. After these identifications are done we have a translation surface

$$(S, \omega, \sigma) = (P, dz, -\mathrm{Id})/\sim$$

with involution σ.

With respect to the translation identifications a vertex class $[v_i]$ contains precisely the vertices v_i whose index lies in the orbit $i \mapsto i+n-1 \mod 2n$. The orbit contains all vertices, if n is even. There are two orbits, represented by v_0 and v_n, if n is odd. In the odd case the central symmetry exchanges the two classes $[v_0]$ and $[v_n]$. The holomorphic one-form $\omega \in \Omega(S)$ has one zero of order $n-2$, if n is even and two zeros of order $\frac{n-3}{2}$ if n is odd. In particular the genus of S is $g = g_S = \lfloor \frac{n}{2} \rfloor$, because $2g - 2 = n - 2$, if n is even and $2g - 2 = n - 3$ for n odd.

Modulo identifications the central symmetry σ has $2g + 2$ fixed points given by $[0]$, $[(v_i + v_{i+1})/2]$, for $i = 1, ..., n$ and the vertex class, if n is even. It follows that S is hyperelliptic with respect to the involution σ. The main result in [Vee95] is:

Theorem 18 (Veech 95)**.** *Each hyperelliptic surface (S, σ) branched over $2\lfloor n/2 \rfloor + 2$ points can be realized by a simple centrally symmetric planar $2n$-gon P_S with opposite sides glued by translation. For an open set of full measure in the parameter space of hyperelliptic surfaces, P_S can be taken convex.*

Represent the hyperelliptic surface (S, σ) underlying a simple $2n$-gon P as above. Because P is simply connected the images of the vertices of P on S define a σ invariant basis of homology. More precisely, if n is even every vertex defines a loop containing two fix-points. If n is odd, the union of all those two vertices bounding one edge define a basis of homology. Invariance under σ follows because the center point of an edge is a fixed point for σ.

Corollary 19. *Let (S, ω, σ) be a hyperelliptic translation surface of genus $g \geq 2$, represented by the polygon P, then the edges of P define a homology basis which does not contain the central fixed point $[0] \in S$.*

The cover-quotient construction. We construct twists by taking a cover followed by a quotient (and not the other way round).

To start take a hyperelliptic translation surface (X, ω, σ) and consider the set of double covers of X to which the hyperelliptic the involution lifts. Suppose we allow two-covers branched away from the set of Weierstrass points $\mathcal{W}$ of X, i.e. branching may occur over a set $\mathcal{B} \subset X \backslash \mathcal{W}$. Any such two-cover is given by a σ-invariant homology class in $H_1(X, \mathcal{B}; \mathbb{Z}_2)$. Recall, there is always an absolute homology basis of X given by σ-invariant homology classes. In Veech's polygonal representation of hyperelliptic translation surfaces the center point of the symmetric polygon is not contained in any of the saddle connections (geodesic segments) representing a (σ-invariant) homology basis. Consequently such a point does not play a role in the construction of the two-cover. It does play a role in the subsequent construction of the quotient and in fact we use the "center point" to expose a symmetry in the set of twists (or quotients).

Take a hyperelliptic surface (X, σ) with hyperelliptic covering $\pi : X \to \mathbb{CP}^1$ branched over $\{p_0, ..., p_{2g+1}\} \subset \mathbb{CP}^1$ and sheet exchange map σ. Recall that (X, σ) has $2g + 2$ Weierstrass points $\mathcal{W} = \mathrm{Fix}(\sigma) = \pi^{-1}(\{p_0, ..., p_{2g+1}\})$. Let

$$\mathcal{I} \subset X \backslash \mathcal{W}$$

be a finite, possibly empty, σ-invariant set, i.e. $\sigma(\mathcal{I}) = \mathcal{I}$. Set $\mathrm{Inv}(\sigma) := \mathcal{W} \cup \mathcal{I}$ and let $\mathcal{B} := \pi(\mathrm{Inv}(\sigma)) = \{p_0, ..., p_{2g+1}, ..., p_{2g+n+1}\} \subset \mathbb{CP}^1$.

Take relative classes $[\![\gamma_i]\!] \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$ such that $\partial[\![\gamma_i]\!] = p_{i+1} + p_0$, $i = 1, ..., 2g + n$ which are represented by leaves $\gamma_i \subset \mathbb{CP}^1 \backslash \mathcal{B}$ of foliations of the induced quadratic differential on $\mathbb{CP}^1$. For $i \in \{1, ..., 2g + n\}$ define the curve $\widetilde{\gamma_i}$ on X by concatenating the two lifts of γ_i (properly oriented). The set of those curves is a σ-invariant relative homology basis

$$H_1^\sigma(X, \mathrm{Inv}(\sigma); \mathbb{Z}_2) := \langle [\![\widetilde{\gamma_1}]\!], ..., [\![\widetilde{\gamma}_{2g+n}]\!] \rangle \subset H_1(X, \mathrm{Inv}(\sigma); \mathbb{Z}_2),$$

where the first $2g$ classes are a σ-invariant basis of $H_1(X; \mathbb{Z}_2) \subset H_1^\sigma(X, \mathrm{Inv}(\sigma); \mathbb{Z}_2)$.

For hyperelliptic translation surfaces with two cone points Veech's polygonal representation provides a homology basis consisting of saddle connections avoiding the center point. For a general one-form on a hyperelliptic surface, we do not show that this is true and therefore we will use only topological facts in the following Lemma.

Lemma 20. *Given a hyperelliptic surface (X, σ) and a σ-invariant set $\mathcal{I} \subset X \backslash \mathcal{W}$ of cardinality $|\mathcal{I}| = 2n$, where $\mathcal{W}$ is the set of Weierstrass points. Suppose $\widetilde{X} \to X$ is a two-cover with branching in $\mathcal{I}$ and σ has a lift $\widetilde{\sigma} : \widetilde{X} \to \widetilde{X}$ fixing each preimage of a Weierstrass point $p_0 \in X$. In that case, if δ denotes the deck-exchange automorphism of $\widetilde{X} \to X$, the σ lift $\widetilde{\sigma} \circ \delta$ exchanges the preimages of p_0.*

Then for fixed genus $\widehat{g} \in \{0, ..., g + \lfloor n/2 \rfloor\}$ there are $\binom{2g+1+n}{2\widehat{g}+2}$ quotient quadratic differentials of the shape $\widetilde{X}/\widetilde{\sigma}$. The number of genus $\widehat{g}$ quotients of the shape $\widetilde{X}/\delta \circ \widetilde{\sigma}$ is given by $\binom{2g+1+n}{2\widehat{g}+1}$.

Proof. Because the involution lifts, the combined covers $\widetilde{X} \to X \to \mathbb{CP}^1$ are $\mathcal{K}$-covers and is defined by a pair of homology classes $(\alpha, \beta) \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2) \oplus H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$. Moreover α is the hyperelliptic class, i.e. $\partial\alpha = \sum_{i=0}^{2g+1} p_i$ and $\beta = \sum_{i=1}^{2g+n} c_i [\![\gamma_i]\!]$ is such that $p_0 \notin \partial\beta$, which means p_0 is not a branch point of the 2-cover $X_\beta \to \mathbb{CP}^1$ defined by β. This 2-cover is by definition of $\widetilde{\sigma}$ isomorphic to the (sub-)cover $X/\widetilde{\sigma} \to \mathbb{CP}^1$. Moreover, the sub-cover $X/\delta \circ \widetilde{\sigma} \to \mathbb{CP}^1$ is given by the class $\alpha + \beta \in H_1(\mathbb{CP}^1, \mathcal{B}; \mathbb{Z}_2)$. Since $p_0 \in \partial\alpha$ we have

$p_0 \in \partial(\alpha+\beta)$ and since β can be any class with $p_0 \notin \partial\beta$ the classes of the shape $\alpha+\beta$ are all classes containing p_0 in its boundary. The rest is elementary subset counting. □

If we restrict the above construction to covers defined by absolute homology classes $\widetilde{\beta} \in H_1(X;\mathbb{Z}_2)$, we get Theorem 2. We reformulate it here without the choice of a one-form on X.

Corollary 21. *Given a hyperelliptic surface $\pi : X \to \mathbb{CP}^1$ of genus g with hyperelliptic involution σ and let $\mathcal{W} = \mathrm{Fix}(\sigma)$ denote the Weierstrass points and $\mathcal{B} = \pi(\mathcal{W})$ their images in $\mathbb{CP}^1$. Then the surfaces obtained from the cover-quotient construction are all regular two-covers of $\mathbb{CP}^1\backslash\mathcal{B}$, branched over at most $2g$ points. Fix $p_0 \in \mathcal{B}$ and $\widehat{g} \in \{0,...,g-1\}$.*

For fixed genus $\widehat{g}$ the number of double covers not branched over p_0 is $\binom{2g+1}{2\widehat{g}+2}$ while the number of covers branched in p_0 is given by $\binom{2g+1}{2\widehat{g}+1}$.

Proof. The hyperelliptic cover $\pi : X \to \mathbb{CP}^1 = X/\sigma$ has $2g+2$ branch points, the fixed points of σ. □

Genus two and genus three tables. The following tables show numerical topological data for regular two-covers of $\mathbb{CP}^1\backslash\mathcal{B}$ generated by a hyperelliptic surface (X,σ) of genus $g_X = 2$, or of genus $g_X = 3$. In the tables the symbols $+(-)$ denote covers with(out) *central branching.*

$g_X = 2$				
branch points	2	2	4	4
genus	0	0	1	1
central branching	$-$	$+$	$-$	$+$
covers	5	10	10	5

$g_X = 3$						
branch points	2	2	4	4	6	6
genus	0	0	1	1	2	2
central branching	$-$	$+$	$-$	$+$	$-$	$+$
covers	7	21	35	35	21	7

Numerical data for quotients in genus two and genus three

List of polygonal representations in genus two. In this section we explicitly list the covers and quotients of surfaces using two standard polygonal representations of genus two surfaces, the swiss cross and the decagon. In fact the moduli space of genus two translation surfaces is stratified by the stratum parameterizing surfaces with a single order 2 cone point and a stratum containing the surfaces with two cone points of order 1 (for more see [Zor06]):

$$\Omega\mathcal{M}_2 = \Omega\mathcal{M}(2) \sqcup \Omega\mathcal{M}(1,1).$$

All genus 2 translation surfaces are hyperelliptic. We will represent the one cone point surfaces by a *swiss cross* where the cone point is represented by the (four) cross corners of angle $3\pi/2$. Based on Veech's theorem, we represent surfaces with two cone points (of order 1) by a centrally symmetric decagon. Another typical representation of one cone point surfaces uses L-shaped polygons (see Figure 1 on page 1).

Swiss cross representation of twists – the $\Omega\mathcal{M}(2)$ case. We start with the list of Swiss cross twists. The hyperelliptic involution acts on the cross by central reflection, or equivalently by 180-degree rotation. The six Weierstrass points occur at the corners of the figure, the midpoints of the horizontal and vertical edges furthest from the center, and the center of the figure. Let (X_S,σ) denote the hyperelliptic translation surface defined by the swiss cross S with hyperelliptic involution σ. To construct $\mathbb{Z}_2$-covers of X, with lift of

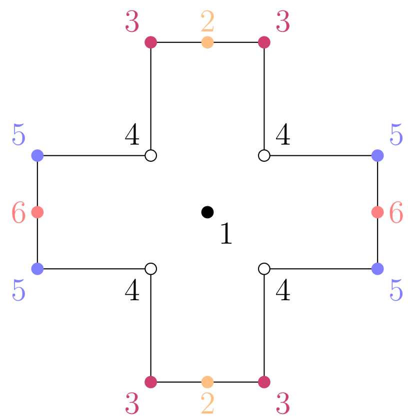
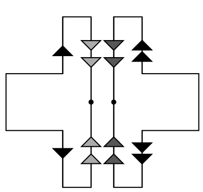
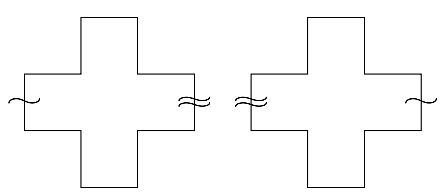
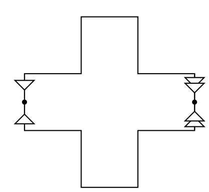


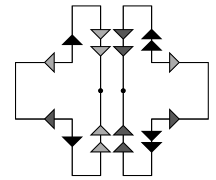
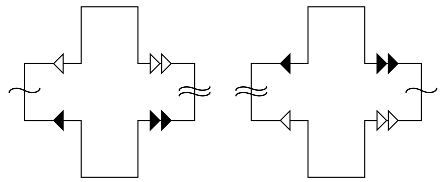
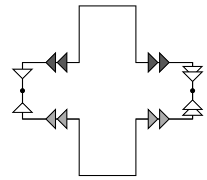
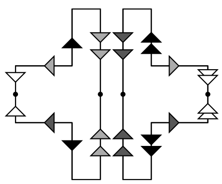
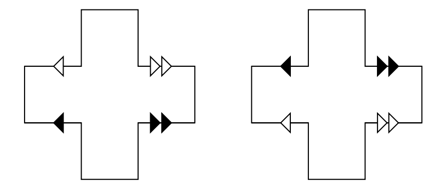
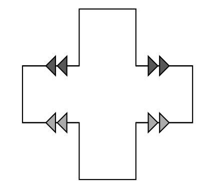
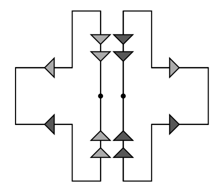
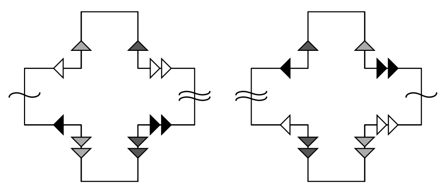
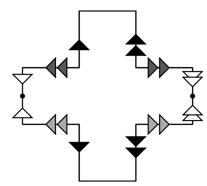
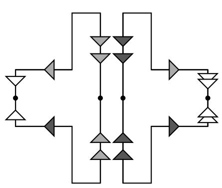
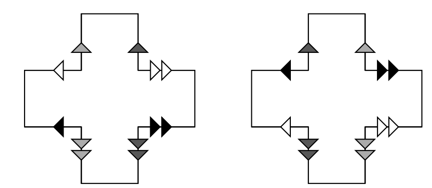
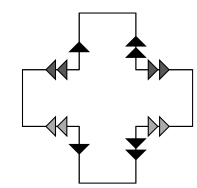
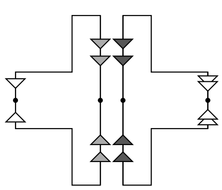
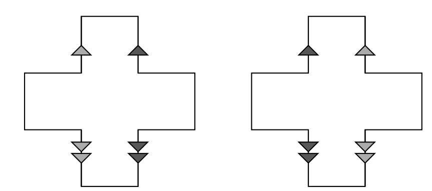
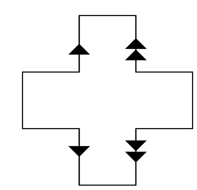
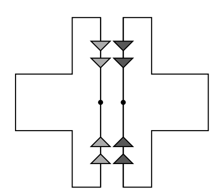
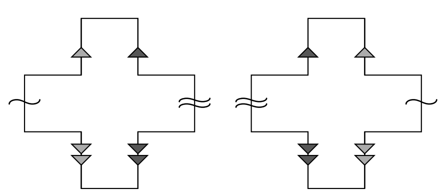
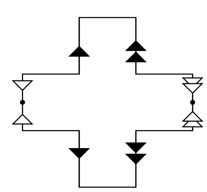
FIGURE 3. The Swiss cross with marked Weierstrass points.

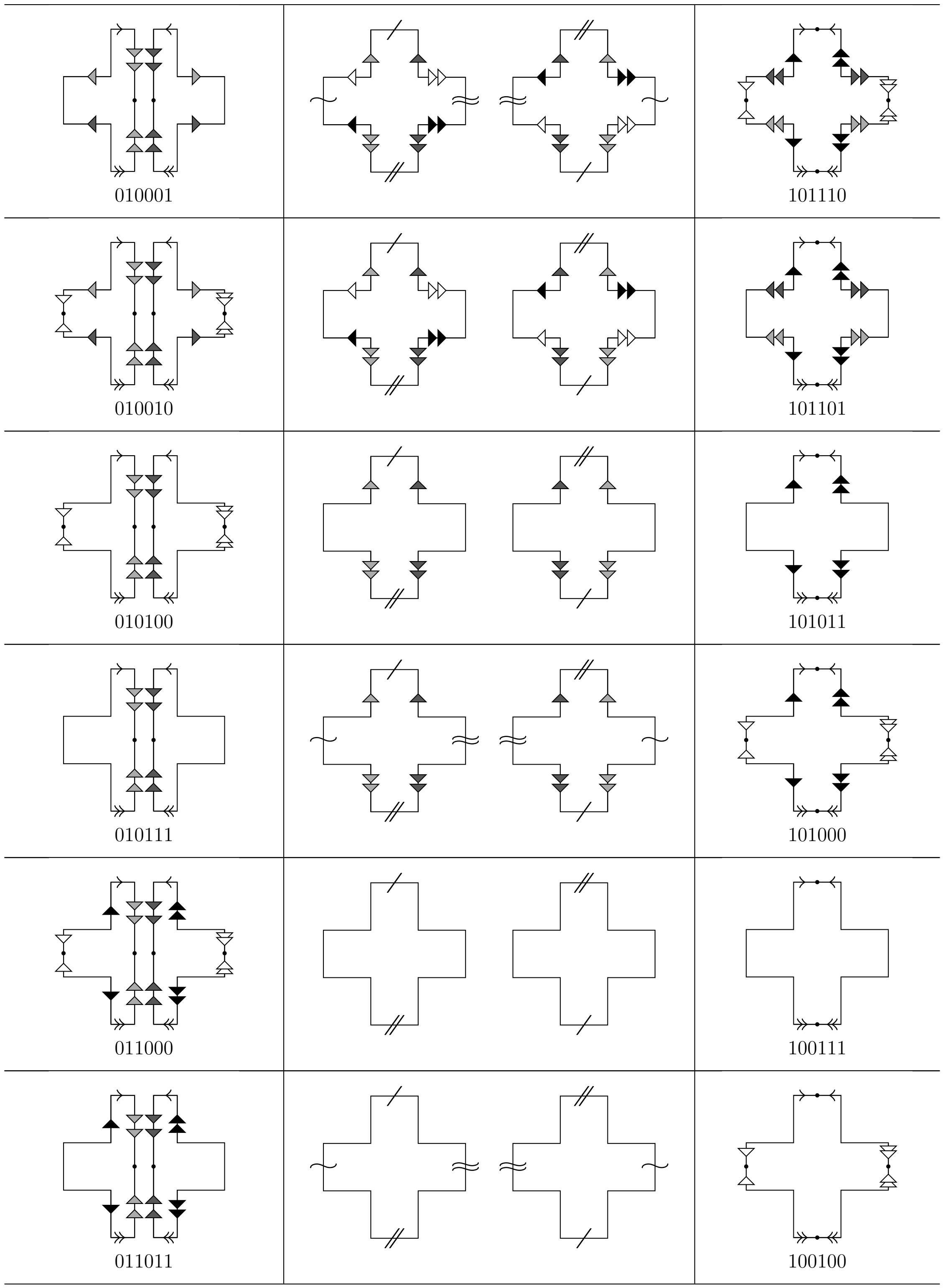
the hyperelliptic involution, notice an involution invariant homology basis of $H_1(X;\mathbb{Z}/2)$ is presented by reflection invariant pairs of edges on S. A double cover of the Swiss cross is given by taking two copies and performing the slit construction along the edges representing symmetric homology classes. These double covers are depicted in the *middle column* of the table below.

The column to the right shows the covers with central ramification (as $\mathbb{CP}^1$-covers). It is the quotient of the middle surface in the respective row under the hyperelliptic involution composed with the sheet exchange. Geometrically take one copy of the cross and rotate onto the other copy. The edges connecting the two crosses are turned onto themselves, in other words they are folded in half over their center points, creating a singular point. The left column shows the quotients obtained modulo the involution rotating the crosses in the middle column.

The *folded edges* in each figure are represented by pairs of edges having two arrow symbols pointing in different directions. The appearance of folded edges *implies* that the *quadratic differential* defined by the polygon *is strict*, i.e. it has singular points. The *ramification vector* $(r_1,\ldots.,r_6)\in\mathbb{Z}_2^6$ below each $\mathbb{CP}^1$-cover labels ramification points with respect to the order shown in Figure 3. In both of the following tables the torus twists are the ones with four branch points, that is branching vectors with exactly four 1's.

Unbranched Center	**Cover**	**Branched Center**
 000011		 111100

 000101		 111010
 000110		 111001
 001001		 110110
 001010		 110101
 001100		 110011
 001111		 110000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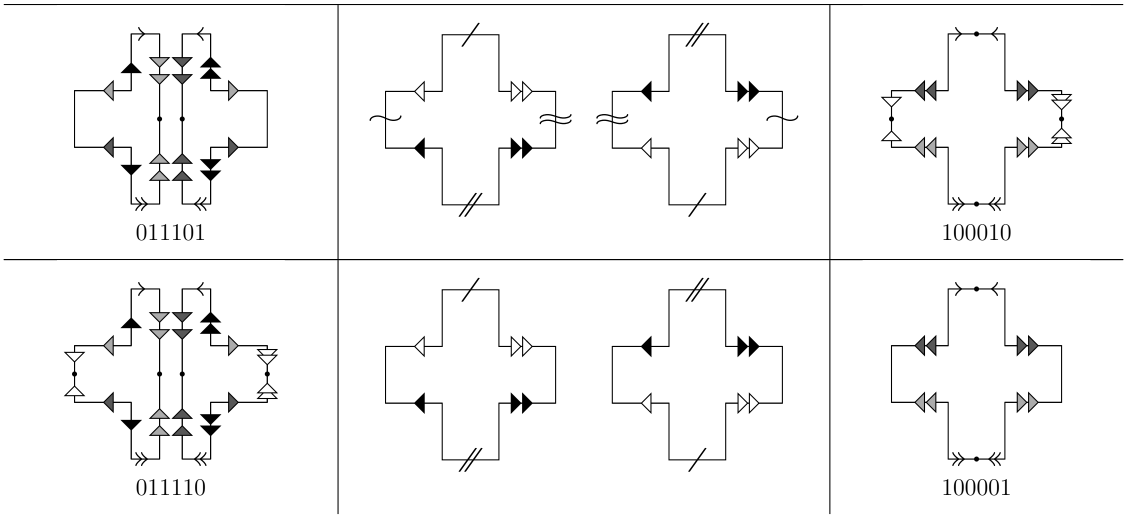


Decagon representation of twists – the $\Omega\mathcal{M}(1,1)$ case. Here is the list of decagon-surface twists. The decagon surface has two cone points, both of total order 1. The meaning of the figures in each column is the same as for the swiss cross, see before those.

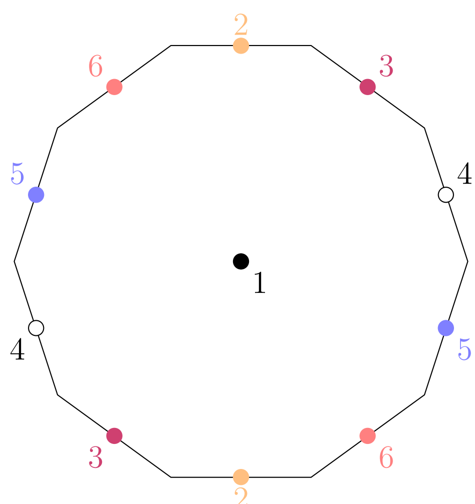
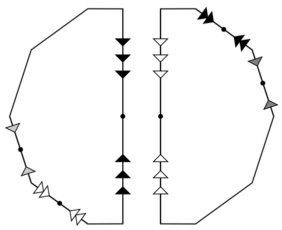
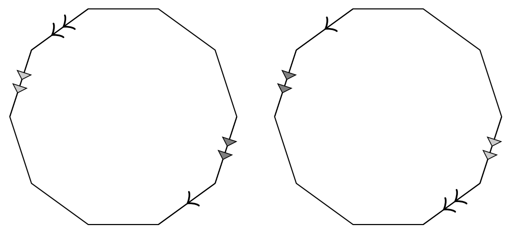
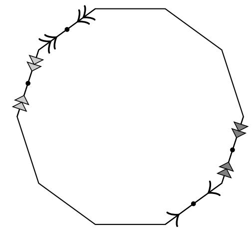
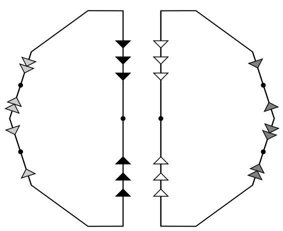
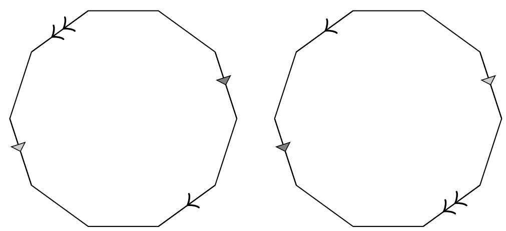
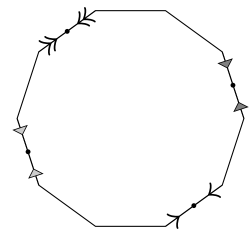
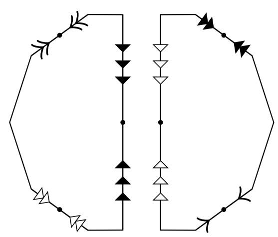
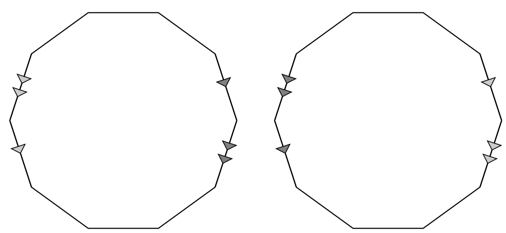
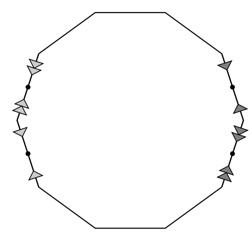
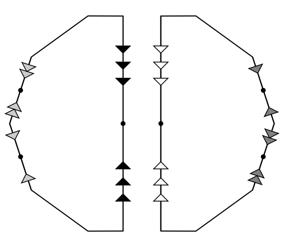
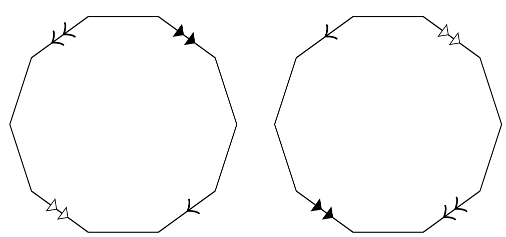
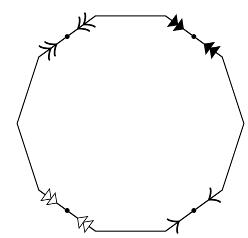
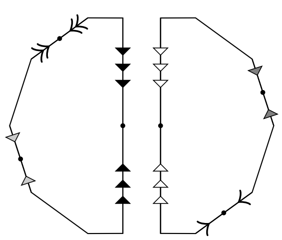
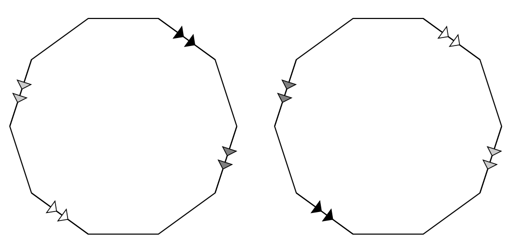
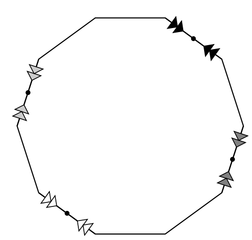
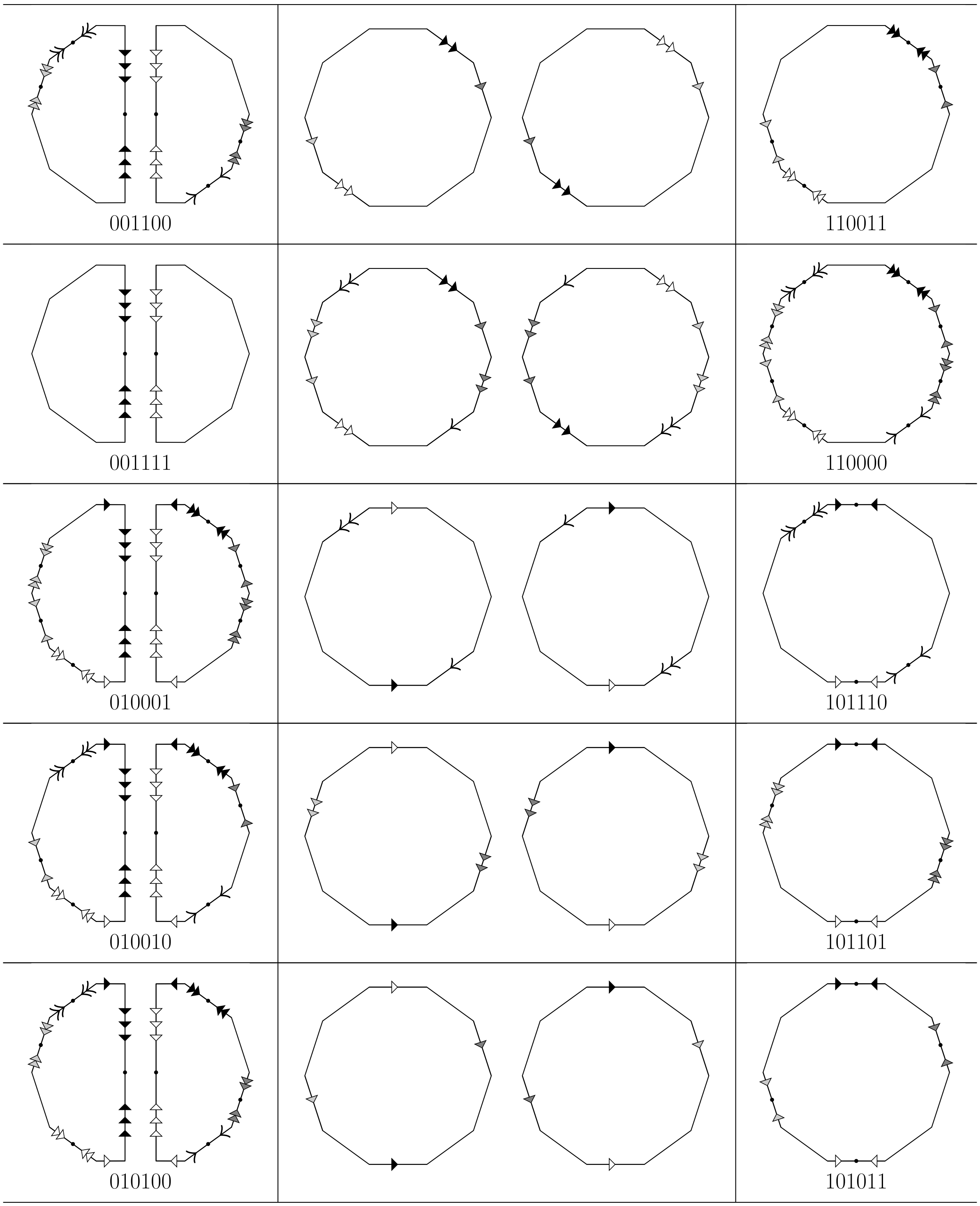


Figure 4. The decagon with opposite sides identified is a surface in $\Omega\mathcal{M}(1,1)$.

If the n-th entry of the branching vector is a 1, branching appears at the (images of the) n-th Weierstrass point, ordering as shown in Figure 4.

Unbranched Center	**Cover**	**Branched Center**
 000011		 111100
 000101		 111010
 000110		 111001
 001001		 110110
 001010		 110101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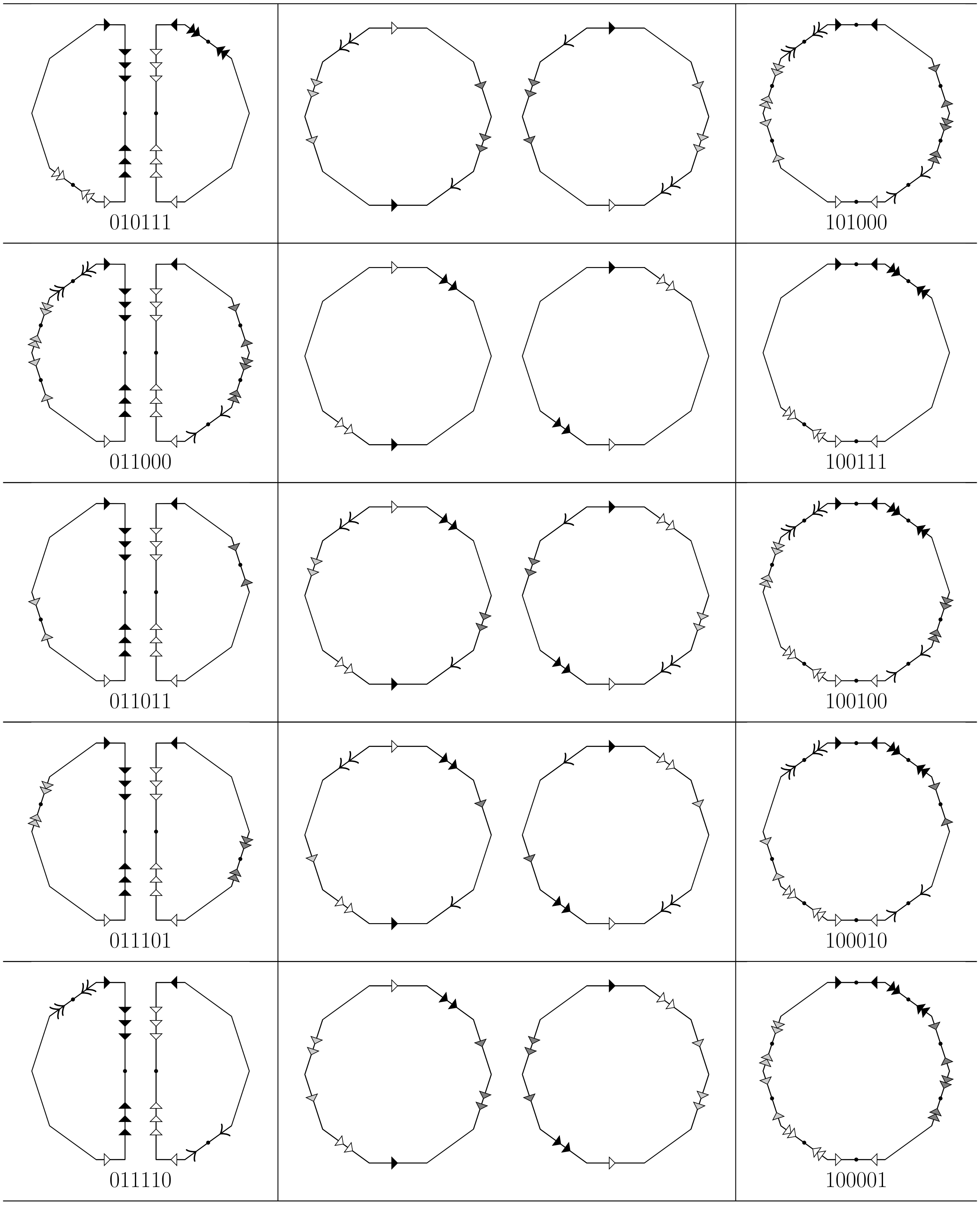

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