Research Statement

Charles Christopher Johnson

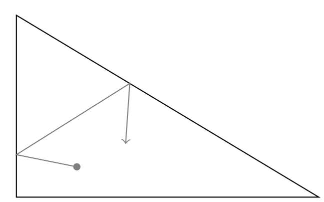
1. Introduction

The geometry of a space often influences the dynamics of maps defined on that space, and my research focuses on understanding this interaction between geometry and dynamics. In particular, I am interested in the long-term, asymptotic behavior of dynamical systems defined on a special class of geometric objects called *translation surfaces*. These objects were originally developed in the 1970's to help understand the dynamics of polygonal billiards, but in the last half century the study of translation surfaces has blossomed into a highly active field with connections to complex analysis, algebraic geometry, low-dimensional topology, and ergodic theory.

My published contributions to this exciting area of mathematics, as well as an overview of my current projects and future research agenda, are described below after a brief, informal overview of the simple problems that led to the development of translation surfaces.

2. Motivation

Imagine you have a triangular billiard table, and on this triangular table you fire off a billiard ball in a straight line. In the absence of friction, the ball will move at a constant velocity until reaching a side of the billiard table. Upon hitting a side of the table, the ball is reflected off the edge according to the rule that angle of incidence equals angle of reflection as illustrated in the figure to the right. If we suppose the billiard ball is an ideal point-mass which always moves at unit speed and always reflects off sides in the way described above, we may ask simple questions about the trajectory of this ball. For example, will the ball visit every region of the table? If so, will the ball spend equal amounts of time in regions of equal area or will it prefer some regions over others? Are there any periodic trajectories?



Answering these innocuous-sounding questions requires us to understand how the geometry of the billiard table influences the motion of the billiard ball as it bounces around the table. We can of course also consider non-triangular billiard tables, and a good theory of mathematical billiards should be expected to apply to a large family of tables. For example, we may wish to understand billiards in general polygons.

An obvious difficulty in understanding billiards in polygons is understanding the possible reflections the billiard ball could make with the sides of the table. By considering the group generated by reflections in the sides of the table, we can parameterize all possible directions in which the billiard may move. It was noticed by Katok and Zemlyakov [ZK75] that if we consider copies of the billiard table reflected by the elements of this group, those reflected copies could be glued together to form a surface. Furthermore, this surface is equipped with a natural geometry coming from the polygons, and geodesics in this geometry correspond to billiard trajectories in the original billiard table. Thus Katok and Zemlyakov describe a method of converting billiard trajectories in a polygon to geodesic flows on a surface.

Generalizing the types of surfaces described by Katok and Zemlyakov's construction in a very minor way gives rise to *translation surfaces*. Translation surfaces are surfaces equipped with a geometry where the notion of direction makes sense at almost every point. That is, there are well-defined notions of vertical and horizontal directions on these surfaces, independent of the coordinate chart you consider. Thus we can ask questions about the dynamics of the geodesic flow in any fixed direction on the surface.

Translation surfaces come in families which have a natural geometric structure, so we can also ask questions about the dynamics of deformations of translation surfaces within these families. Somewhat surprisingly, the dynamics of these deformations have consequences for the dynamics of the geodesic flow on a given surface. In particular, when a certain class of deformations of a given surface has a closed orbit, the geodesic flows on that surface satisfy a dichotomy where in each direction the flow is either periodic or uniformly distributed. Classifying these closed orbits is thus a very important, but also very difficult, problem in the study of translation surfaces.

3. Results to Date

3.1. **Cutting sequences on square-tiled surfaces.** In order to study flows on a surface, it is sometimes useful to model a flow with a simpler object. One simple way of representing a flow is to replace the trajectories of the flow with sequences of symbols describing how the trajectory moves across the surface. For example, if we consider a partition of the surface we may associate to each element of the partition a symbol, and then consider the sequence of symbols obtained as a trajectory cuts across the elements of the partition. These sequences are called *cutting sequences*, and the most obvious question to consider is which sequences of symbols are in fact cutting sequences.

Cutting sequences for geodesic flows on the torus were characterized early in the last century and are precisely Sturmian sequences (see [MH38] and [Ser85]) as well as some special classes of periodic sequences on two letters. In the last few years cutting sequences have been analyzed for various families of translation surfaces: the regular octagon surface [SU11], then later all regular polygon surfaces [Dav13], surfaces which have a polygonal representation that is horizontally and vertically symmetric [Dav14], the L-shaped surface built from three squares [WZ15], and the Bouw-Möller surfaces [Dav14] [DPU15].

In [Joh17a] I characterized cutting sequences on square-tiled surfaces by converting the question *Which sequences are cutting sequences?* into a question about interval exchanges on a square-tiled surface and using the classification of languages of interval exchanges by Ferenczi and Zamboni [FZ08].

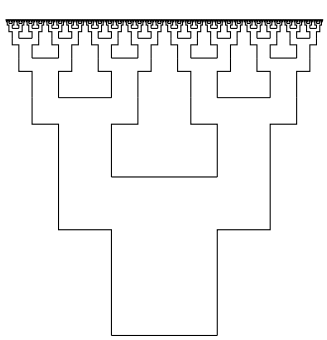
Theorem ([Joh17a])**.** *Let X be a square-tiled surface on d squares determined by a pair of permutations h and v on $\Lambda = \{1, 2, ..., d\}$. Let $E = \{H, V\}$ be symbols for the edges of the unit square torus. Label the left-hand edge of square λ as (λ, H) and the bottom edge as (λ, V). Then a biinfinite sequence $(\lambda_n, \epsilon_n) \in (\Lambda \times E)^{\mathbb{Z}}$ is the cutting sequence of an infinite geodesic on X if and only if the following conditions are satisfied:*

(1) *(λ_n, ϵ_n) is consistent with the gluings of the surface;*
(2) *(λ_n, ϵ_n) is either periodic, or is minimal but not almost symmetric around a bad square of the surface (this is a technical condition that simply rules out certain degenerate sequences); and*
(3) *ϵ_n is the cutting sequence of a geodesic on the square torus.*

This theorem basically states that cutting sequences on square-tiled surfaces can be thought of as Sturmian sequences with some additional information. The method of proof for the above theorem is considerably different from the methods used in previous classifications of cutting sequences which relied on special symmetries of the surface, whereas the proof in [Joh17a] is more dynamical in nature.

Besides describing cutting sequences on a new family of surfaces and using different techniques from previous articles on cutting sequences, the result above is interesting for two reasons. Square-tiled surfaces are branched covers of tori, and the above theorem is the first result relating cutting sequences on a surface to cutting sequences on covers of that surface. Additionally, square-tiled surfaces play a special role in the study of translation surfaces because they form a dense subset of the space of all surfaces.

3.2. **Billiards in the T-fractal.** The Katok-Zemlyakov construction was originally developed to associate a surface to a compact polygonal billiard table, but the construction itself carries over to more exotic families of surfaces. For example, we may use the Katok-Zemlyakov construction to associate a surface to the fractal-like billiard table called the *T-fractal* shown in the figure below.



The motion of a billiard in the T-fractal is equivalent to the geodesic flow on a surface which is extremely strange from the point of view of translation surfaces. Robert Niemeyer (the University of Maine) and I have been studying this surface and recently (March 2017) submitted a paper for publication where we prove a number of results about this surface. Our results can be summarized as follows:

Theorem ([JN17])**.** *The surface associated to the T-fractal is a finite-area, infinite genus translation surface whose metric completion adds a Cantor set of* wild singularities *(following the language of* [BV13]*)* *of Hausdorff dimension one. Each of these singularities has infinitely-many rotational components, and every rotational component of a singularity has zero length. Furthermore, the metric completion of the T-fractal surface is not a surface.*

This theorem essentially says that this surface has a large collection of very poorly behaved singularities, and the existence of such a set of singularities complicates the study of dynamics on the surface. To our knowledge this is the first example of a surface with the properties described in the theorem.

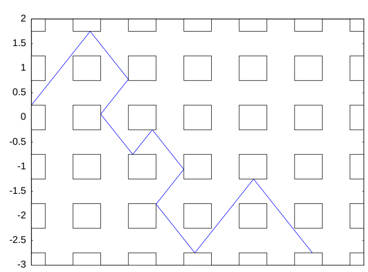
In a forthcoming paper we expand on these results to show how the strange geometry of this surface has undesirable dynamical consequences. In particular, there exists an infinite family of directions on the surface where the flow in one of these directions from any point on the surface intersects a singularity in finite time. This is an extremely peculiar property for translation surfaces, and the first example of a surface with such a property arising from billiards in a finite-area billiard table.

3.3. **Panov planes, twisted differentials, and the wind-tree model.** It had been noted by André Weil [Wei36] that orientable foliations of the torus lift to foliations of the plane with bounded deviation. In [Pan09], Dmitry Panov constructed the first explicit examples of non-orientable foliations whose lifts to the plane had unbounded deviation. Extending the results of Panov, Martin Schmoll and I constructed a family of flat tori admitting a pseudo-Anosov affine diffeomorphism where flows in the eigendirection of the map's derivative lift to dense leaves in the plane. These results can be summarized as follows:

Theorem ([JS14b])**.** *Let T be a flat torus and let φ be a pseudo-Anosov diffeomorphism which is affine in the flat geometry of T. If the action of φ on $H_1(T;\mathbb{Z})$ is elliptic, then the universal cover $\mathbb{C}^T$ of T (that is, the complex plane with the flat structure given by pulling back the flat structure of T) admits a straight-line foliation with dense leaves.*

This result is surprising because of the flat geometry involved: away from a very simple set of measure zero, the local geometry of the universal cover $\mathbb{C}^T$, which we call a *Panov plane*, is the same as that of $\mathbb{C}$ with its usual Euclidean geometry. Despite this, it is possible to construct foliations with dense leaves in $\mathbb{C}^T$.

The dynamics of flows in Panov planes can be used to study billiard trajectories in an infinite billiard table known as the *periodic wind-tree model* which is obtained by placing identical rectangular obstacles at the integer lattice points in the plane as in the figure to the right. We had hoped to use our results for Panov planes to construct an explicit example of a dense billiard trajectory in the wind-tree model, but instead stumbled upon the opposite phenomenon.



Theorem ([Joh14], [JS14b])**.** *If the size of the obstacles in the wind-tree model are such that the L-shaped surface covered by the wind-tree's unfolding is a Veech surface (i.e., satisfies the conditions described in* [McM03]*), and if the flat torus obtained by* twisting *the surface as described in* [JS14a] *admits an affine pseudo-Anosov φ with elliptic action in homology, then the billiard in wind-tree whose initial direction is the expanding or contracting direction of φ is escaping.*

This is a surprising result because it describes a very explicit family of directions which give escaping billiard trajectories on the wind-tree, which is in contrast to the work of Artur Avila, Pascal Hubert [AH], Samuel Lelièvre, and Serge Troubetzkoy [HLT11] which shows that recurrence is the generic property for billiards in the wind-tree. That is, for almost every direction and almost every initial point, a billiard starting from that initial point in the chosen direction will return arbitrarily close its initial point infinitely often.

4. Current Work

Interval exchanges are very simple maps which take an interval, cut it into finitely-many pieces, and rearrange the pieces. The dynamics of interval exchanges are closely related to the dynamics of flows on translation surfaces and have been actively studied since the mid-1970's. One simple generalization is given by affine interval exchanges which are defined similarly to interval exchanges, but allow intervals to be expanded or contracted. This expansion and contraction complicates the ergodic theory of these maps since the Lebesgue measure is no longer preserved.

Even though the natural Lebesgue measure of the interval is not preserved, it is not distorted too much. In particular, the collection of sets of measure zero *is* preserved by an affine interval exchange. In general, maps that preserve the collection of null sets of a given measure are said to be *non-singular* with respect to that measure. While the ergodic theory of non-singular maps is more complicated than for measure-preserving transformations, questions of invariant sets and functions, mixing, etc. can still be asked.

Currently I am working on understanding the non-singular ergodic theory of a special collection of non-singular transformations obtained by composing a measure-preserving transformation with an involution which exchanges two sets of different size. (In the case that the measure-preserving transformation is an interval exchange and the involution is piecewise affine, this gives an affine interval exchange.) Exploiting a simple observation about how the invariant sets of such a map must intersect the sets exchanged by the involution, I am able to provide necessary and sufficient conditions for when such a map is conservative, and when it is totally dissipative.

Theorem ([Joh17b])**.** *Let $T : X \to X$ be a measure-preserving transformation on a probability space $(X, \mathcal{B}, \mu)$, and $\Phi : X \to X$ an involution exchanging two sets $S, B \subseteq X$ with $\mu(B) > \mu(S)$ and $\frac{d\,\mu\Phi}{d\,\mu}$ constant on each of S and B. The map $\widehat{T} = \Phi \circ T$ is a non-singular transformation which does not preserve measure, but the first return map $\widehat{T}_S$ to S is defined for almost every point of S; $\widehat{T}$ will be conservative if and only if $\mu(S \backslash \widehat{T}_S(S)) = 0$; $\widehat{T}$ will be totally dissipative if and only if $\lim_{n\to\infty} \mu(\widehat{T}_S^n(S)) = 0$.*

Understanding conservativity and dissipativity is only the first and most obvious question to consider, and the next question is when $\widehat{T}$ is mixing, ergodic, etc. It would be desirable to know that dynamic properties of T carry over to $\widehat{T}$, but this is not guaranteed. It is in fact very easy to construct explicit examples of uniquely ergodic maps T and choose Φ in such a way that $\widehat{T}$ is totally dissipative. A first step to better understanding precisely how the dynamics of T influences those of $\widehat{T}$ is the following.

Theorem ([Joh17b])**.** *If $\widehat{T}$ is conservative and T^2 is ergodic, then $\widehat{T}$ is ergodic.*

Using a variation of the Maharam extension [Mah69], it is possible to build an infinite measure-preserving transformation which extends $\widehat{T}$. Understanding the dynamics of this transformation boils down to understanding the existence or non-existence of a $\widehat{T}$-invariant measure. This can be expressed in terms of a multiplicative group associated to $\widehat{T}$ called its *ratio set*, and I am currently working to calculate the ratio set of transformations defined in this way, $\widehat{T} = \Phi \circ T$.

5. Future Work

Broadly speaking, my main research goals for the future are concerned with contributing to the understanding of dynamics on infinite translation surfaces (surfaces of infinite genus and/or area), an area where little is currently understood, as well as the dynamics of different types of deformations in the space of translation surfaces. Two particular types of problems related to these topics are described below.

5.1. **Groups of non-singular transformations and infinite interval exchanges.** Infinite interval exchanges naturally arise in studying many non-compact translation surfaces, but there is currently no general theory of infinite interval exchanges. The infinite interval exchange that appears in the T-fractal surface has a special self-similarity, and maps with such a self-similarity should be more amenable to study.

It is possible to produce maps with the desired self-similarity by beginning with a single finite interval exchange, composing it with affine involutions, then considering the Maharam extension of the resulting affine interval exchange. This is a situation I am currently studying, and the next step is to consider compositions not with one single affine involution, but with several. A generalization of the Maharam extension can then be used to produce an infinite interval exchange that is built by taking several copies of the original transformation, and scaling the copies in various ways according to the affine involutions used. Understanding the dynamics of such a transformation boils down to understanding the dynamics of the group generated by compositions of the initial interval exchange with involutions. This is likely a difficult problem if taken in complete generality, so a starting point would be to consider simpler versions of this problem.

5.2. **Dynamics of rel deformations.** The moduli space of compact translation surfaces of a fixed genus is stratified by the number and types of singularities of the surface. In each stratum of surfaces with at least two singularities, there exists a family of deformations which are obtained by modifying the relative distances between the singularities while leaving other distances fixed. (More precisely, the space has charts corresponding to the relative cohomology group $H^1(X, \mathrm{Sing}(X); \mathbb{C})$ and these deformations act in these charts by modifying the coordinates corresponding to the relative cocycles while fixing the coordinates coming from the absolute cocycles.) The collection of all such *rel deformations* forms a foliation of the stratum, and surprisingly little is known about the way the leaves of these foliations are embedded into the stratum. The only major result to date is for the *principal strata*, which correspond to surfaces with the simplest type of singularities. It was recently shown in [Ham15] and [CDF15] that the foliation of rel deformations is ergodic in each principal stratum.

Looking only at those rel deformations where the cohomology class of a differential is modified by a real cocycle gives a subfoliation of *real rel deformations*, and looking only at those deformations which are real multiples of a given cocycle gives a flow in the stratum.

Even less is known about the dynamics of these real rel flows than is known about the foliation of all (complex) rel deformations. It was shown by Minsky and Weiss [MW14], however, that these flows are defined for all time on the full measure set of surfaces which do not contain horizontal saddle connections (horizontal geodesics on the surface which connect pairs of singularities). It has also been shown by Hooper and Weiss [HW15] that there are surfaces for which the flow is divergent. The only example known, however, corresponds to flows through the Arnoux-Yoccoz surface, which is a very special surface with some peculiar properties.

Though very little is currently known about the dynamics of rel deformations, this is a project I find to be more interesting the more I think about it, and a subject I hope to explore in the future. The long term goals would be to understand the ergodic theory of these deformations (e.g., What are the measures preserved by these deformations? Are these deformations ergodic? Mixing?), but a starting point would be to generalize the result of Hooper and Weiss, and also understand the action of rel deformations on the boundary of moduli space in the compactification of [BCG+16].

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