Research Statement
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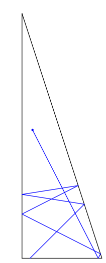
1. Introduction

It is often the case in mathematics that a detailed investigation of a seemingly simple problem leads to deep results and unexpected connections. One instance of this phenomenon is the modern and quickly growing field of *Teichmüller dynamics*. Tying together ideas from algebraic and differential geometry, ergodic theory, Lie groups, and low-dimensional topology, Teichmüller dynamics studies questions that have arisen in trying to understand a problem that even grade schoolers can appreciate: how does a billiard ball bounce around a billiard table? The modern era for work on this problem was initiated in the 1970's and 1980's by Howard Masur [Mas82] [Mas86] and William Veech [Vee78] [Vee89] [Vee86], and has grown by leaps and bounds in this century with major advancements in describing Teichmüller curves in genus two [Cal04] [McM03] [McM05], classifying connected components of strata [KZ03] and computing their volume [EO01], and most recently understanding the structure $\mathrm{SL}(2,\mathbb{R})$-orbit closures [EM18] [EMM14] [MW18].

My current research in this fascinating area largely focuses on two distinct types of problems: dynamics of infinite area translation surfaces, and the isoperiodic foliation of the Hodge bundle. At first blush these two types of problems seem to have nothing to do with billiard balls bouncing around a billiard table, but in the sections below I will briefly discuss how billiards naturally leads to the Hodge bundle over the moduli space of algebraic curves. After establishing some motivation and background, I will describe my current research projects, and finally outline my agenda for future research while emphasizing connections to other areas of mathematics.

2. The Motivating Problem: Polygonal Billiards

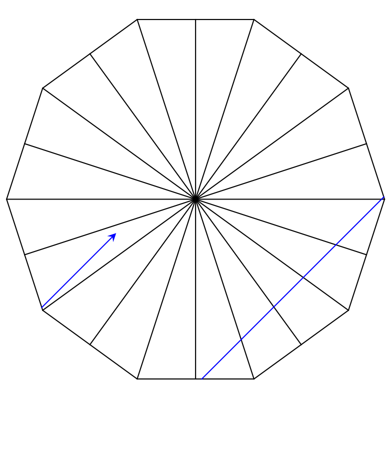
Imagine you have a triangular billiard table, and on this table you fire off a billiard ball in a straight line. In the absence of friction, the ball will move at a constant velocity until reaching a side of the billiard table. Upon hitting a side of the table, the ball is reflected off the side according to the rule that angle of incidence equals angle of reflection. The figure to the right illustrates the path of such a billiard after the first six reflections on sides of the table.



If we suppose the billiard ball is an ideal point-mass which always moves at unit speed and always reflects off sides in the way described above, we may ask simple questions about the trajectory of this ball. For example, will the ball visit every region of the table? If so, will the ball spend equal amounts of time in regions of equal area or will it prefer some regions over others? Are there any periodic trajectories?

Answering these questions requires us to understand how the geometry of the billiard table influences the motion of the ball as it bounces around. We can of course also consider non-triangular billiard tables, and a good theory of mathematical billiards should be expected to apply to a large family of tables. For example, we may wish to understand billiards in general polygons.

We of course need to understand all of the possible directions in which the ball could move after reflecting off the sides of the table. By considering the group generated by reflections in the sides of the table, we can parametrize all possible directions in which the billiard may move. It was noticed by Katok and Zemlyakov [ZK75] that if we consider copies of the billiard table reflected by the elements of this group, those reflected copies could be glued together to form a surface. Furthermore, this surface is equipped with a natural metric coming from the reflected polygons, and geodesics in this geometry correspond to billiard trajectories in the original billiard table. In the example of a triangular billiard table with interior angles $\pi/2$, $2\pi/5$, and $\pi/10$, the group generated by reflections in the sides of the triangle has twenty elements. The relations among these elements tell us how to glue sides of the reflected triangles



together to obtain a genus two surface, represented by the decagon above where opposite sides of the decagon are identified by translation. The blue billiard path in the triangle above becomes the blue geodesic on this decagon surface.

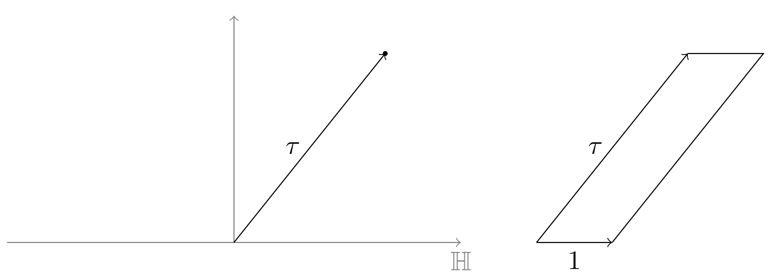
The gluings which identify sides of the reflected polygons are holomorphic, and so the surface is endowed with a natural complex structure. The planar geometry of the surface can be described in a way that's compatible with the complex structure: the flat metric of the surface is encoded by a 1-form which is holomorphic with respect to this complex structure. Thus Katok and Zemlyakov describe a method of converting billiard trajectories in a polygon to geodesic flows on a surface, and we can encode this information as a pair (X, ω) where X is a Riemann surface and ω a holomorphic 1-form.

In general, a pair (X, ω) with X a compact Riemann surface and ω a holomorphic 1-form, whether this surface and 1-form come from the Katok-Zemlyakov construction or not, is called an *abelian differential* or a *translation surface*. All abelian differentials, even those that don't arise from billiards, come with a singular flat metric and so have a natural geodesic flow. By studying general abelian differentials and properties of their geodesic flows, we can gain insight into the behavior of billiards in polygonal billiard tables.

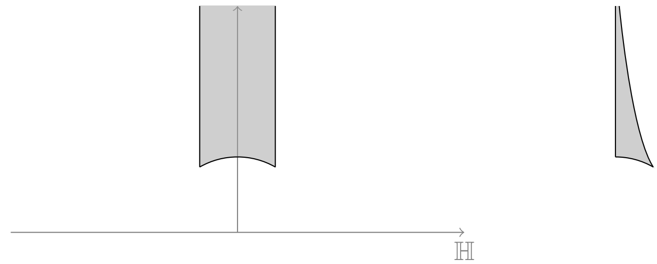
3. Moduli and Teichmüller Space

Before going any further, we need to understand the notion of a *moduli space* and the closely related idea of *Teichmüller space*. Intuitively, a moduli space is a space which parametrizes all objects of a given type. One simple example is the moduli space of all complex tori. A basic fact about tori is that they can be represented as parallelograms with opposite sides identified. Topologically, all of these objects are the same torus, but they are not the same Riemann surface: there does not necessarily exist a biholomorphic map between two complex tori.

Translating a torus' representative parallelogram so that one corner is at the origin in the plane, rotating to make one side horizontal such that the other side is above the x-axis, and then scaling so the horizontal side has length one are all holomorphic operations, and so they do not change the complex torus. Thus we see the only piece of information which determines a complex torus is the non-horizontal side of the parallelogram, which we can think of as a point in the upper half plane. That is, we can parametrize complex tori as points in $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. This is called the *Teichmüller space* of complex tori and it parametrizes not simply tori, but tori with an additional piece of information. The torus corresponding to a point $\tau \in \mathbb{H}$, call it $\mathbb{T}_\tau$, has an obvious choice of generators for its fundamental group: the vectors 1 and τ project to generators of $\pi_1(\mathbb{T}_\tau)$.



If we wish to ignore this extra information about generators of the fundamental group, we need to consider the group of transformations which identifies points in $\mathbb{H}$ representing biholomorphic tori, but with different choices of generators for the fundamental group. This is the *mapping class group* of the torus, and it acts on $\mathbb{H}$ as the matrix group $\text{SL}(2, \mathbb{Z})$ via Möbius transformations. The quotient of $\mathbb{H}$ by this action is the *modular curve*, which can be visualized as the fundamental domain $\{z \in \mathbb{H} \mid |\text{Re}(z)| \leq 1/2, |z| \geq 1\}$, but points on the lines $|\text{Re}(z)| = \pm 1$ are identified if they have the same imaginary parts, and points on the portion of the circle $|z| = 1$ in this region are likewise identified if they have the same imaginary parts.



We now have two spaces which we may consider for parametrizing complex tori: the Teichmüller space, which parametrizes tori together with an additional piece of information, and the moduli space, which parametrizes complex tori ignoring that extra information. While the moduli space may seem like the more natural object to consider, for technical reasons it can be hard to work with. For example, the moduli space is not a smooth manifold. Intuitively, the problem has to do with the "sharp corners" of the modular curve. Note, however, that the Teichmüller space $\mathbb{H}$ is a simply connected manifold. In fact, the Teichmüller space is the universal covering manifold of the moduli space, and $\mathrm{SL}(2,\mathbb{Z})$ is the group of deck transformations.

A similar, though technically more involved, discussion can be carried out for Riemann surfaces of higher genus to produce the Teichmüller and moduli spaces of Riemann surfaces of genus g, respectively denoted $\mathcal{T}_g$ and $\mathcal{M}_g$. These are both extremely important objects in mathematics which can be thought of from several points of view: hyperbolic geometry, algebraic geometry, and complex analysis can all be used to study these spaces.

4. The Hodge Bundle, Strata, and the $\mathrm{SL}(2,\mathbb{R})$ Action

We mentioned above that an abelian differential is a pair (X,ω) with X a Riemann surface and ω a holomorphic 1-form. The collection of holomorphic 1-forms on a genus g Riemann surface forms a g-dimensional complex vector space, and the set of all genus g abelian differentials forms a vector bundle over the moduli space $\mathcal{M}_g$, denoted $\Omega\mathcal{M}_g$. Since some abelian differentials have special symmetries, this space fails to be a manifold and so sometimes it is convenient to work with the corresponding bundle over Teichmüller space, $\Omega\mathcal{T}_g$. These spaces are called the *Hodge bundle* over moduli and Teichmüller space.

Abelian differentials always come with a special set of distinguished points, corresponding to the zeros of the 1-form. The number of zeros, counted by multiplicity, on a genus g abelian differential always equals $2g-2$. By considering different integer partitions of $2g-2$, we can introduce a partition of the Hodge bundle. For example, a holomorphic 1-form on a genus 3 surface must have four zeros, counting multiplicity. Now consider a partition of four, such as $(1,1,2)$. We may use this partition to select a subset of $\Omega\mathcal{M}_4$ (or $\Omega\mathcal{T}_4$), which we denote $\Omega\mathcal{M}_4(1,1,2)$. This set consists of all genus four abelian differentials which have two simple zeros and one double zero.

These sets $\Omega\mathcal{M}_g(m_1,m_2,...,m_n)$ are called *strata* of the Hodge bundle, and are themselves geometric objects: strata for the Hodge bundle over moduli space are complex orbifolds, while strata for the Hodge bundle over Teichmüller space are complex manifolds. In both cases the dimension of the space depends on the genus and the number of distinct zeros.

One of the key features of these strata is that they come with a natural action of $\mathrm{SL}(2,\mathbb{R})$. It can be shown that each abelian differential can be represented as a polgyon whose edges come in parallel pairs identified by translation. (As edges are identified by translation, abelian differentials are sometimes also called *translation surfaces*.) The group $\mathrm{SL}(2,\mathbb{R})$ acts on these polygonal representatives in the "obvious" way. Since the action is linear, parallel pairs of sides are preserved and can be identified to give a new abelian differential.

A wealth of information can be obtained by understanding the $\mathrm{SL}(2,\mathbb{R})$-orbits of points in a stratum. For example, when an abelian differential has a very large group of symmetries, the geodesic flow satisfies a remarkable property called the *Veech dichotomy*: the flow in each direction is either periodic or uniquely ergodic. This remarkable property can be detected by the $\mathrm{SL}(2,\mathbb{R})$

orbit of that abelian differential: if the projection of the orbit to $\mathcal{M}_g$ is an algebraic curve, then the geodesic flow will satisfy the Veech dichotomy.

Relating this back to the original problem of billiards, it can be shown that the abelian differential associated with the triangle with angles $\pi/2$, $2\pi/5$, and $\pi/10$ above has an orbit projecting to an algebraic curve. This implies that for each initial direction for a billiard in this triangular billiard one of two things happens: the billiard either has a closed, periodic trajectory, or it never closes up and instead visits all areas of the triangle, asymptotically spending equal amounts of time in any two regions of equal area. Moreover, this depends *only* on the initial direction and not the choice of starting point!

5. Current Research

Now that some background has been established, I can describe some of my current research projects. This includes some recently completed (but not yet published) work concerning the number and multiplicity of zeros of Abelian differentials, understanding the structure of loci of differentials with prescribed periods, and the non-singular ergodic theory of affine interval exchanges.

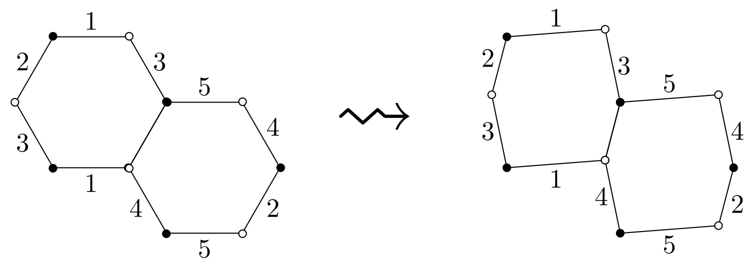
5.1. **Zeros of Abelian differentials.** A smooth, complex-valued 1-form on a surface has an associated collection of complex numbers, called the periods of the 1-form, obtained by integrating the form over closed curves on the surface. If a complex structure on the surface is selected, then we may consider the subset of holomorphic 1-forms. Changing the complex structure changes which forms are holomorphic, and so we are lead to the following question: which collections of complex numbers form the periods of some holomorphic 1-form on some choice of complex structure on that surface?

Independent work of Haupt and Kapovich shows there are two necessary and sufficient conditions on the complex numbers to be the periods of some holomorphic 1-form. The results of Haupt and Kapovich say nothing about the orders of the zeros of forms with preselected periods, however. In recent joint work with Mattin Bainbridge, Chris Judge, and Insung Park, we answer the question of the number and orders of zeros that can appear for a holomorphic 1-form with prescribed periods, and also determine which connected components of strata of the Hodge bundle contain those 1-forms.

Taking advantage of recent work of Calsamiglia, Deroin, and Francaviglia [CDF15], we show that given any collection of allowable periods (i.e., satisfying the Haupt-Kapovich conditions), the collection of Abelian differentials with these periods intersects every connected component of every stratum of the Hodge bundle provided the periods are not discrete. When the periods are discrete, there is an additional condition which must be satisfied for the collection of differentials with those periods to intersect a stratum. Our proof of this result has two parts: in the non-discrete case, we use the *local product structure* near the boundary of strata to show that the set of differentials crosses the boundary to adjacent strata. In the discrete case, we describe an algorithm for explicitly constructing Abelian differentials in each connected component of each stratum satisfying the condition alluded to above.

5.2. **The isoperiodic foliation.** Recently, I have been studying problems related to a particular type of structure in the Hodge bundle called the *isoperiodic foliation*. Every 1-form determines a cohomology class, and the values this cohomology class takes on are called *periods* of the 1-form. In Teichmüller space there is a natural way to compare these cohomology classes on different surfaces, and by considering elements of $\Omega\mathcal{T}_g$ which correspond to forms with the same periods, we have a submanifold of $\Omega\mathcal{T}_g$. The collection of all such submanifolds defines the *isoperiodic foliation*, and each individual manifold is called a *leaf* of the foliation.

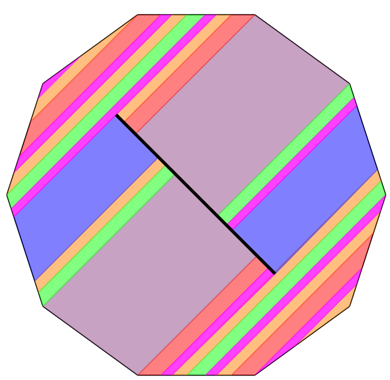
Representing abelian differentials as polygons, elements of the same leaf have a very simple description: they correspond to deformations of the polygons obtained by moving vertices of the polygon relative to one another. In fancier language, this means the absolute cohomology class remains fixed, while the relative class changes.

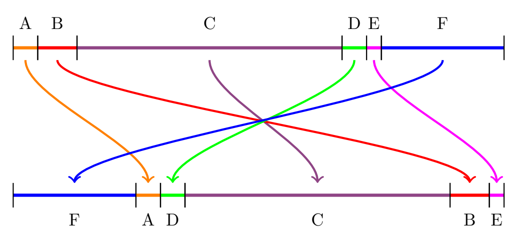


Extremely little is known about the isoperiodic foliation right now, making it an attractive area for research. Numerous obvious questions about the instrinsic topology of leaves, how the leaves are embedded in the space, and dynamics of the foliation are currently unanswered.

One problem related to the isoperiodic foliation that I have been working on with Kevin Pilgrim and Chris Judge (professors at Indiana University, Bloomington) is concerned with how covering maps between surfaces induce covering maps between corresponding leaves of the isoperiodic foliation. It can be shown that special types of branched coverings between topological surfaces induce embeddings from the stratum of the base into the stratum of the cover, and these embeddings preserve leaves of the corresponding isoperiodic foliations. Extending earlier work of Martin Schmoll for torus covers [Sch05], our goal is to show that when projected to the moduli spaces there is a natural covering map relating the leaf of the covering surface to the leaf of the base surface. It is possible that the covering map we are interested in can only be defined in some special cases, but if so the next obvious question to consider involves classifying those special cases.

5.3. **Affine interval exchanges and non-singular ergodic theory.** One way to study the geodesic flow on an abelian differential is to consider the first-return map to a geodesic interval transverse to the direction of the flow. This results in a special type of map called an *interval exchange*. An interval exchange is a bijection from an interval to itself which takes the interval, cuts it into finitely-many pieces, and rearranges the pieces as a piecewise translation. The image below shows the first return map of a flow on the decagon surface and the associated interval exchange. (The flow is towards the North-East and the black segment is the geodesic interval used to define the first-return map.)





Interval exchanges have been extensively studied since the mid-1970's, and the ergodic theory of these maps is now fairly well-understood. One simple family of generalizations which is less understood are the *affine interval exchanges*. Affine interval exchanges are defined similarly to interval exchanges, except individual intervals may be expanded or contracted: they are orientation-preserving piecewise affine maps. This expansion and contraction complicates the ergodic theory of these maps since the Lebesgue measure is no longer preserved.

Though the Lebesgue measure is not preserved, sets of measure zero still map to sets of measure zero. In general, maps that preserve the collection of null sets of a given measure are said to be *non-singular* with respect to that measure. While the ergodic theory of non-singular maps is more complicated than for measure-preserving transformations, questions of ergodicity, mixing, and so on can still be asked.

Motivated by an observation my collaborator Rob Niemeyer and I made while working on [JN17], I have recently started studying a special family of non-singular transformations. By considering the composition of a measure-preserving transformation with an involution exchanging two sets with different measures, we have a non-singular transformation where the measure is distorted in a simple way. Notice the composition of an interval exchange with a piecewise affine involution produces an affine interval exchange in this family.

The most obvious questions to consider about any non-singular transformation concern conservativity and dissipativity. By exploiting a simple observation about how the invariant sets of such a map must intersect the sets exchanged by the involution, I am able to provide necessary and sufficient conditions for both conservativity and dissipativity. Furthermore, I am able to give conditions guaranteeing ergodicity when the original measure-preserving transformation is ergodic. These sufficient conditions are very restrictive, however, and so the last portion of this project concerns weakening those conditions to make it easier to find explicit examples of ergodic affine interval exchanges.

5.4. **Discrete Maharam extensions.** In [Mah64], Dorothy Maharam showed that for every non-singular transformation on a space of finite measure, there is an associated measure-preserving transformation on a space of infinite measure. A modification of this construction gives a natural way to turn a skew product over the maps described above (compositions of a measure-preserving transformation and an involution exchanging sets of different size) into a measure-preserving transformation.

In particular, for the affine interval exchanges given by such a composition, there is an associated infinite interval exchange on the real line. There is a simple way of constructing an abelian differential whose geodesic flow has prescribed dynamics as the suspension of an interval exchange. Applying the same construction to the infinite interval exchange coming from this *discrete Maharam extension* gives an infinite-area translation surface which, for example, will have an ergodic vertical flow if the corresponding infinite interval exchange is ergodic. This is an interesting construction since infinite-area translation surfaces are very poorly understood compared to their finite-area brethren, and in particular, few examples with ergodic flows had been explicitly described before. This project is still in its early stages, but will ultimately provide a tool for explicitly constructing such examples.

6. Plans for Future Research

There are some obvious next steps for my research which I will briefly outline in this section. After stating the next steps, I will discuss my long term goals and describe how my research relates to other areas of mathematics.

6.1. **Next steps.**

6.1.1. *Divergence of isoperiodic leaves.* One project that I have been thinking about, together with Matt Bainbridge and Chris Judge at Indiana University, concerns the divergence of leaves of the isoperiodic foliation. In general, a leaf of a foliation is *divergent* if it can not be contained in any compact set. Leaves of the isoperiodic foliation are always divergent in strata for a simple reason: distinct zeros of a 1-form can collide and give a degenerate (nodal) surface. For this reason we may want to extend our space of Riemann surfaces and holomorphic 1-forms to include these degenerate cases, and we can do this by considering the Deligne-Mumford compactification of moduli space together with stable 1-forms (1-forms which are allowed to have simple poles with opposite residues at a node), denoted $\overline{\Omega\mathcal{M}_g}$.

The results of Martin Schmoll [Sch05] show that isoperiodic leaves of some special torus covers are not divergent in this space of stable 1-forms: torus covers branched over two distinct points are themselves torus covers and are compact surfaces embedded in $\overline{\Omega\mathcal{M}_g}$. A natural question to ask is whether these are the only examples. This question is answered by Calsamiglia, Deroin, and Francaviglia in [CDF15] where they show leaves of the isoperiodic foliation are connected. Their results imply that torus covers are in fact the only non-divergent leaves. A next question to consider is how an isoperiodic leaf accumulates on the boundary of $\Omega\mathcal{M}_g$. Since the absolute periods of abelian differentials on an isoperiodic leaf are constant, moving along a leaf can only pinch a cycle to produce a node if that cycle has zero period. However, leaves may be able to diverge in more complicated ways. Instead of directly pinching a cycle, a cycle may shrink down to its minimum value (determined by the period) moving close to $\partial\Omega\mathcal{M}_g$, then move away from the boundary, but later come back by bringing a different cycle to its minimum value.

The generic abelian differential has arbitrarily short periods, and by [CDF15], these abelian differentials live on dense leaves of the isoperiodic foliation. Having a precise understanding of how these leaves repeatedly move near the boundary, then away from the boundary, then return closer to the boundary, may give insights into the dynamics of the isoperiodic foliation and help answer the biggest open question about the foliation: is it ergodic in every stratum?

6.1.2. *Non-singular ergodic theory and random walks.* I began studying affine interval exchanges and the discrete Maharam extension described above in order to better understand billiards in a "fractal-like" billiard table obtained by repeatedly attaching scaled copies of a particular polygon to itself. The original problem I considered is more complicated than the affine interval exchanges described above for two reasons. In the original problem, there were multiple distinct involutions the interval exchance could be composed with, making the associated discrete Maharam extension more complicated than a simple skew product. Additionally, the original billiard table had a base to which everything was attached and the self-similarity of the table did not extend past this base. The projects described earlier in Sections 5.3 and 5.4 was meant to be a simplification of this problem. As that project nears completion, the natural next step would be to go back and reconsider something closer to the original motivating problem.

In particular, suppose you have a fixed measure-preserving transformation T, together with a finite number of scaling involutions, Φ_1, Φ_2, ..., Φ_n. Now consider a dynamical system where at each time step one of the $\Phi_i T$ is applied, the choice of which is determined by some random variable. This setup can be viewed as a random walk on the semigroup generated by the $\Phi_i T$, where each sample path of the walk determines a sequence of maps to apply. Does such a random walk almost surely give rise to a sequence of transformations with an ergodic action? This is likely a difficult problem in general, but in this particular setup where the maps being applied have a very specific, nice form, the problem may become more tractable.

6.1.3. *Flows on isoperiodic leaves.* One interesting property of leaves of the isoperiodic foliation is that they come with a natural flat geometry. This means we can consider directional flows on the leaf and can ask questions about the dynamics of these flows. In terms of the polygonal representation of an abelian differential, these flows correspond to a continuous deformation of the polygon where, for example, one vertex continually moves in a fixed direction at unit speed. When is this flow periodic? When does it have a dense orbit? Can the flow ever diverge, never coming back near any previously visited point? These sorts of questions are what originally piqued my interest in the isoperiodic foliation, and so I would like to explore these questions more in the near future when some of my current projects are completed.

There is one partial answer to one of these questions which can serve as a starting point for this exploration. In [HW15], Pat Hooper and Barak Weiss show that the flow on the leaf of a very special surface is divergent. The surface they consider, the Arnoux-Yoccoz surface, has some special properties they are able to exploit to show the flow from this surface leaves every compact set in the moduli space. One of the special properties of the Arnoux-Yoccoz surface is that it admits an affine pseudo-Anosov map, a map which stretches the surface in one direction while simultaneously contracting in another. One approach to finding more examples of divergent flows would be to consider other surfaces admitting pseudo-Anosov maps and seeing if the argument in Hooper-Weiss can be carried out for those surfaces. If not, understanding precisely where the argument breaks down may offer clues as to what additional properties are required.

6.2. **Long-term goals.**

6.2.1. *Hyperbolic geometry and variations of Hodge structure.* An abelian differential endows a compact Riemann surface with a flat geometry, and so to each genus g Riemann surface there is a g-dimensional vector space of singular flat metrics. Every Riemann surface also admits a unique hyperbolic metric. Furthermore, endowing a surface with a hyperbolic metric fixes a unique complex structure for which that hyperbolic metric is conformal, and so determines a g-dimensional space of flat metrics. Is there an explicit description of the hyperbolic metric given one of the flat metrics, and an explicit description of all of the flat metrics from the hyperbolic metrics? Put another way, given a hyperbolic surface (e.g., by specifying cuff lengths in a trouser decomposition), can you determine which 1-forms on the surface will be holomorphic? How does this change as the hyperbolic surface is modified?

For any genus g surface X, the first cohomology group with complex coefficients is isomorphic to $\mathbb{C}^{2g}$. However, when a complex structure is placed on the surface, the Hodge decomposition splits this group into two particular subgroups, $H^1(X;\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$. The group $H^{1,0}(X)$ corresponds to cohomology classes which can be represented by holomorphic 1-forms, where as $H^{0,1}(X)$ has antiholomorphic representatives. That is, the complex structure distinguishes certain cohomology classes from others. Describing which cohomology classes correspond to holomorphic forms and how these classes vary as the hyperbolic surface (and hence complex structure) vary amounts to determining the Hodge decomposition of a Riemann surface, and understanding how this structure changes as we move around in the moduli space $\mathcal{M}_g$.

The only result in this direction is [Raf07], where Kasra Rafi shows the hyperbolic length of curves in the thick part of the hyperbolic surface are coarsely related to lengths in the flat metrics. In principle, there should be more to say about the relationship between these metrics since fixing an abelian differential fixes both a flat metric and a hyperbolic metric.

While a complete answer to this problem is likely very difficult, it may be that interpreting the problem in terms of abelian differentials and flat metrics, together with associated tools like the $\mathrm{SL}(2,\mathbb{R})$ action and the isoperiodic foliation, provide a new view to what is essentially a classical problem.

6.2.2. *Ergodic theory of non-singular group actions.* I mentioned above that one natural extension of my current projects is random walks on semigroups of non-singular transformations and understanding the random dynamical systems such a walk induces. Such a question is reminiscient of the classical problem of random products of matrices, a question which leads to the theory of dynamical cocycles and Lyapunov exponents. Are there similar results in the setting of random dynamical systems with non-singular action? For example, is there an analogue of the extremal Lyapunov exponents in this setting which describe the long-term average exponential expansion or contraction of the measure of sets? There may not be enough structure to answer this question on a general measure space, but perhaps in a special setting such as diffeomorphisms on a smooth manifold, or homomorphisms of Lie groups, such questions can be answered.

Along these lines there have recently been several deep results by Benoist-Quint [BQ11] and Eskin-Lindenstrauss [EL18]. These results certainly require the structure of Lie groups, but perhaps similar ideas can be applied to other settings.

6.3. **Relation to other areas.** The next steps and long-term goals above describe the most obvious trajectory for my research. Because my research interests lie at the intersection of dynamics, geometry, and analysis, however, there are several altnerative directions in which my future research could move. In particular, I would be happy to work with colleagues on projects that are tangentially related to my primary interests. Working on projects from other related areas may help me view problems in another light, and this may lead to valuable insights.

For example, my interest in ergodic theory can lead naturally to projects of a more probabilistic nature. As mentioned above, studying random walks on groups seems like a natural next step from some of my current research interests. It's not hard to imagine that ideas from other parts of probability theory, such as stochastic processes or martingale theory, would also be useful to my future research at some point, and so I have an interest in working on projects in these areas if an interesting problem presents itself.

As indicated above, abelian differentials are essentially classical algebro-geometric objects, although Teichmüller dynamics thinks of these objects from a different, more modern point of view. One type of problem I would be interested in concerns using these modern tools to reinterpret classical problems. The Schottky problem, for example, seems to have some kind of close relationship to isoperiodic foliations. Can the Schottky problem be restated in terms of the isoperiodic foliation, or some other object in the Hodge bundle? Can the machinery of Teichmüller dynamics be used to gain new insights into the Schottky problem? I would be interested in working on more problems related to classical complex algebraic geometry to help solidify my understanding and, hopefully, make progress on this interesting question.

As abelian differentials are surfaces together with some extra structure, low-dimensional topology often provides valuable tools for solving problems about abelian differentials. In particular the mapping class group, the curve complex, and the Nielsen-Thurston classification are closely related to my interests. Sometimes problems from Teichmüller dynamics can also motivate more purely topological questions about, for example, special elements of the Torelli group, or lifting properties of mapping class elements. For this reason I am also very interested in working on projects of a more topological flavor to enhance my knowledge about this useful area.

Ultimately, my goal is simply to work on problems I find interesting and make contributions to mathematics. Right now this means working on problems related to Teichmüller dynamics, but because of the number of different closely related areas, I can easily transition into working on projects in other areas.

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